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Control & Dynamical Systems

Cylindrical Manifolds and Tube Dynamics in the Restricted Three-Body Problem

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April 7, 2004

Acknowledgements

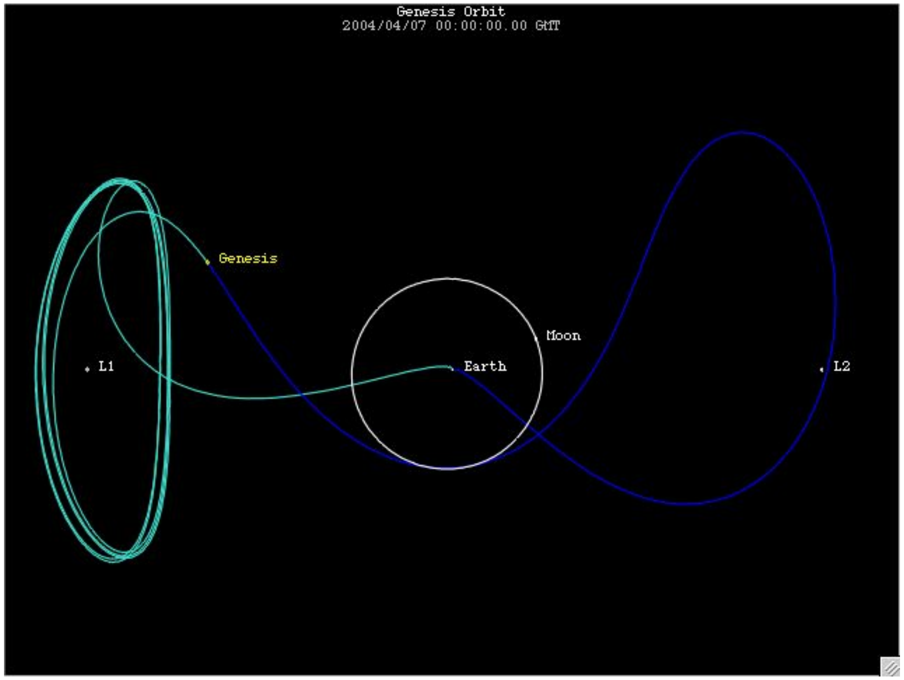
- Committee: M. Lo, J. Marsden, R. Murray, D. Scheeres
- W. Koon
- G. Gómez, J. Masdemont
- CDS staff & students
- JPL's Navigation & Mission Design section

Motivation

Low energy spacecraft trajectories

- **Genesis** has collected solar wind samples at the Sun-Earth L1 and will return them to Earth this September.
- First mission designed using dynamical systems theory.

Genesis Spacecraft



Where Genesis Is Today

Motivation

- Low energy transfer to the Moon

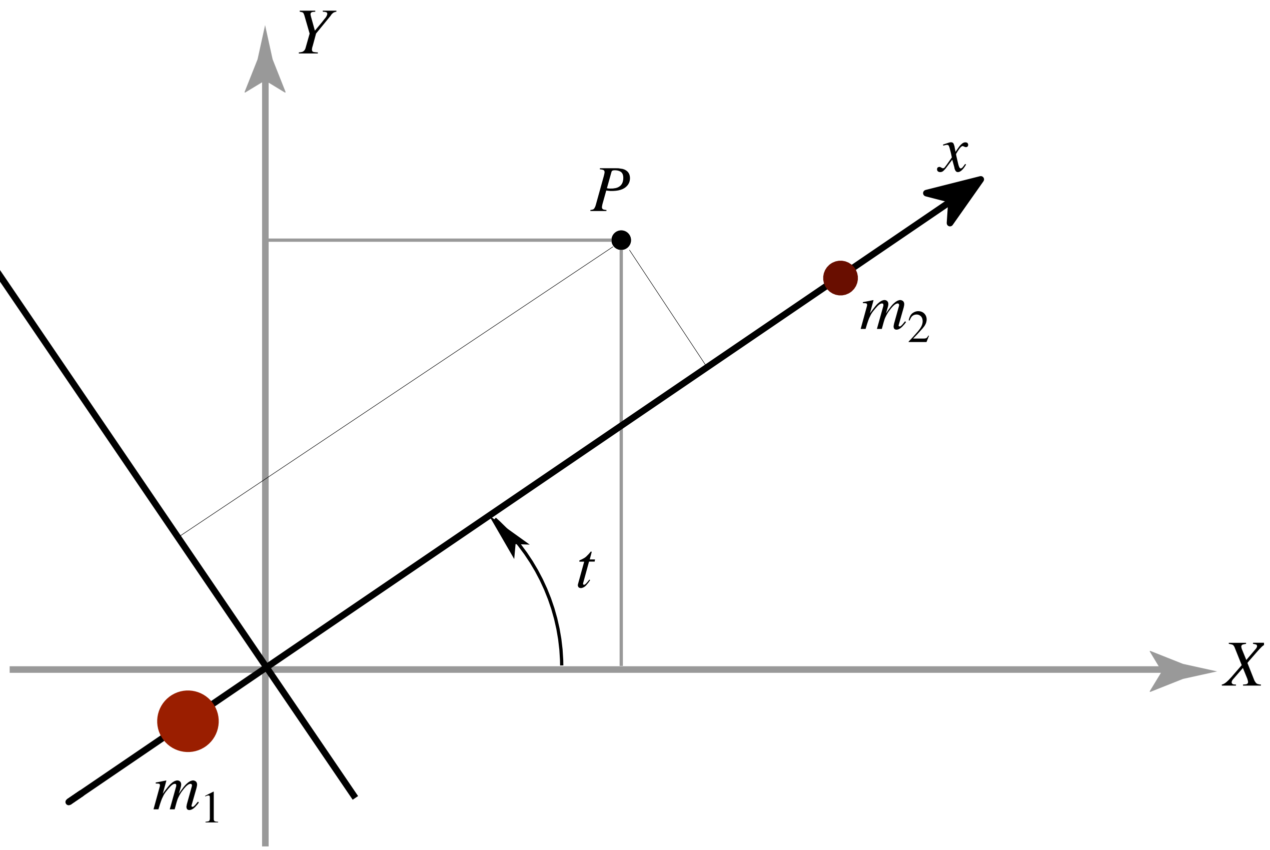
Outline of Talk

- ***Introduction and Background***
  - Planar circular restricted three-body problem
  - Motion near the collinear equilibria
- ***My Contribution***
  - Construction of trajectories with prescribed itineraries
  - Trajectories in the four-body problem
    – patching two three-body trajectories
    – e.g., low energy transfer to the Moon
  - Current and Ongoing Work
  - Summary and Conclusions

Three-Body Problem

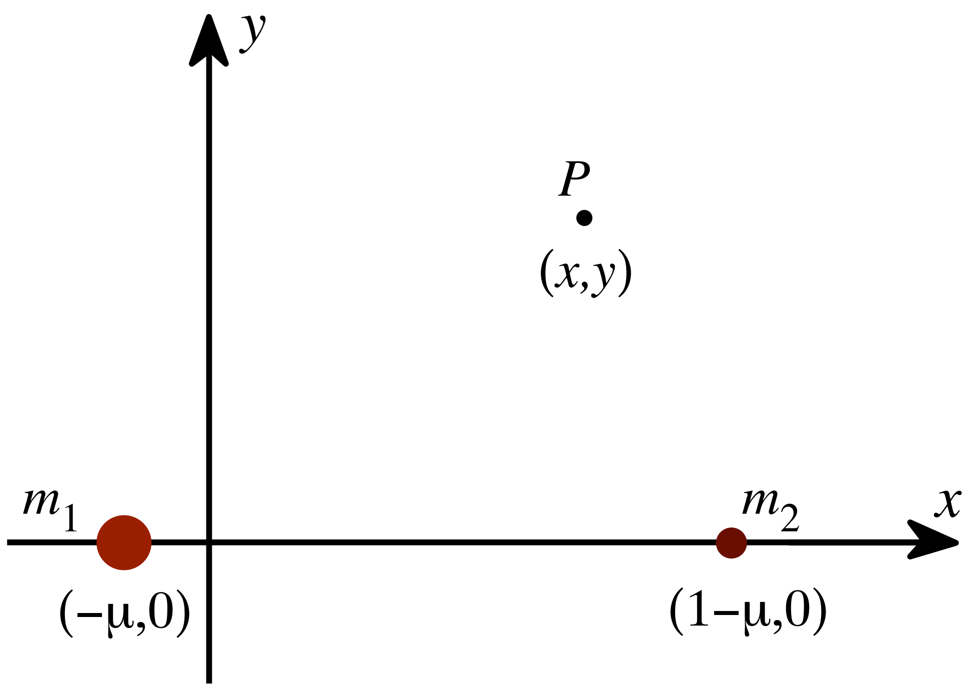
☐ Planar circular restricted three-body problem

- P in field of two bodies, m_1 and m_2
- x-y frame rotates w.r.t. X-Y inertial frame

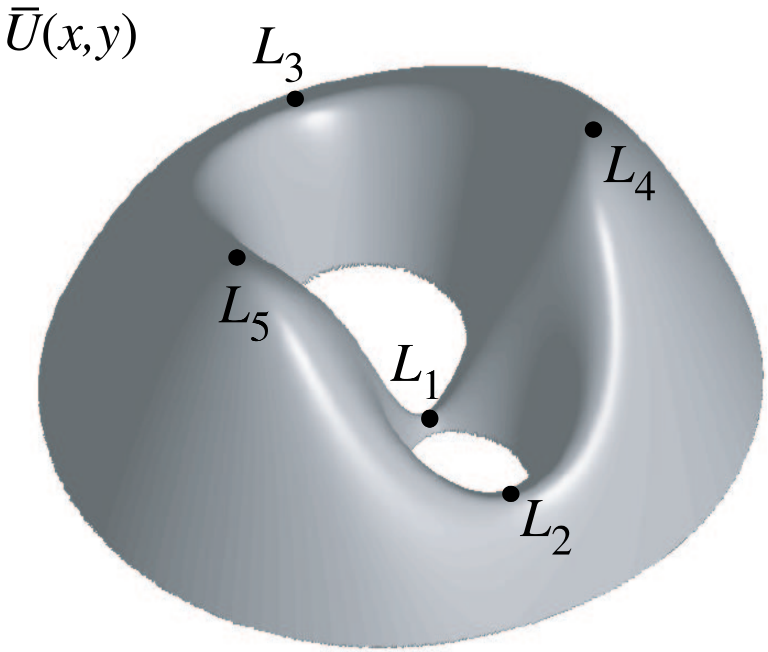


Three-Body Problem

- Equations of motion describe P moving in an effective potential plus a coriolis force



Position Space



Effective Potential

Hamiltonian System

- Hamiltonian function

$$H(x, y, p_x, p_y) = \frac{1}{2}((p_x + y)^2 + (p_y - x)^2) + \bar{U}(x, y),$$

where p_x and p_y are the conjugate momenta,

$$p_x = \dot{x} - y = v_x - y,$$
$$p_y = \dot{y} + x = v_y + x,$$

and

$$\bar{U}(x, y) = -\frac{1}{2}(x^2 + y^2) - \frac{1-\mu}{r_1} - \frac{\mu}{r_2}$$

where r_1 and r_2 are the distances of P from m_1 and m_2 and

$$\mu = \frac{m_2}{m_1 + m_2} \in (0, 0.5].$$

Equations of Motion

- Point in phase space: $q = (x, y, v_x, v_y) \in \mathbb{R}^4$
- Equations of motion, $\dot{q} = f(q)$, can be written as

$$\begin{aligned} \dot{x} &= v_x, \\ \dot{y} &= v_y, \\ \dot{v_x} &= 2v_y - \frac{\partial \bar{U}}{\partial x}, \\ \dot{v_y} &= -2v_x - \frac{\partial \bar{U}}{\partial y}, \end{aligned}$$

conserving an energy integral,

$$E(x, y, \dot{x}, \dot{y}) = \frac{1}{2}(\dot{x}^2 + \dot{y}^2) + \bar{U}(x, y).$$

Motion in Energy Surface

- Fix parameter μ
- Energy surface for energy e is

$$\mathcal{M}(\mu, e) = \{(x, y, \dot{x}, \dot{y}) \mid E(x, y, \dot{x}, \dot{y}) = e\}.$$

  For a fixed μ and energy e, one can consider the surface $\mathcal{M}(\mu, e)$ as a three-dimensional surface embedded in the four-dimensional phase space.

- Projection of $\mathcal{M}(\mu, e)$ onto position space,

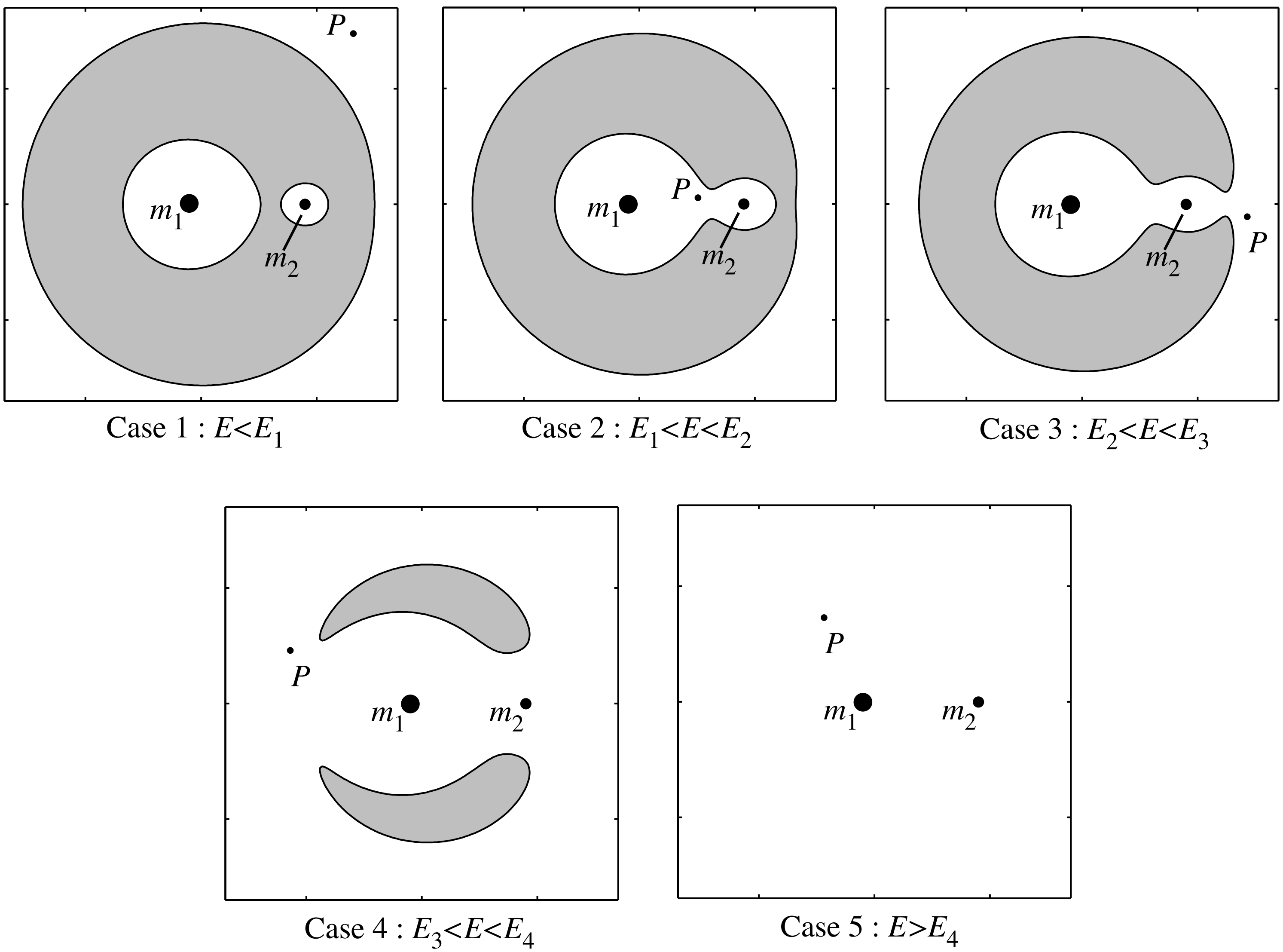
$$M(\mu, e) = \{(x, y) \mid \bar{U}(x, y; \mu) \leq e\},$$

  is the region of possible motion (Hill's region).

- Boundary of $M(\mu, e)$ places bounds on particle motion.

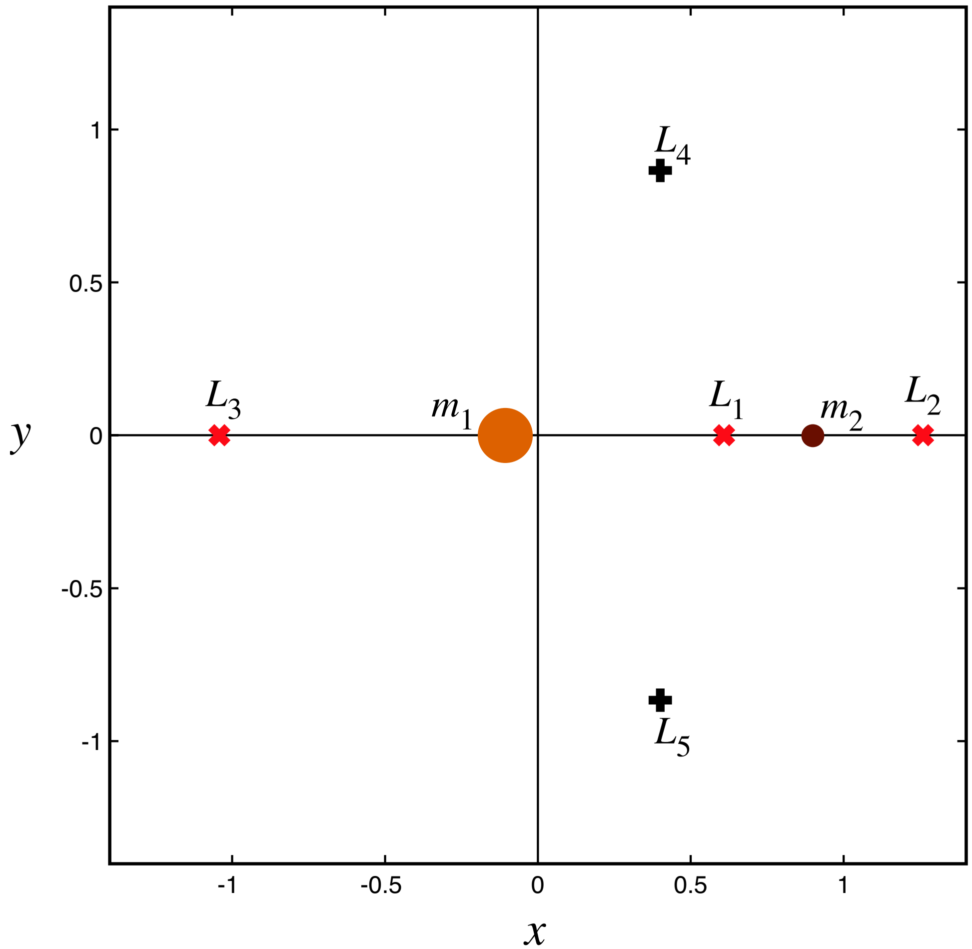
Realms of Possible Motion

- For fixed μ, e gives the connectivity of three **realms**



Realms of Possible Motion

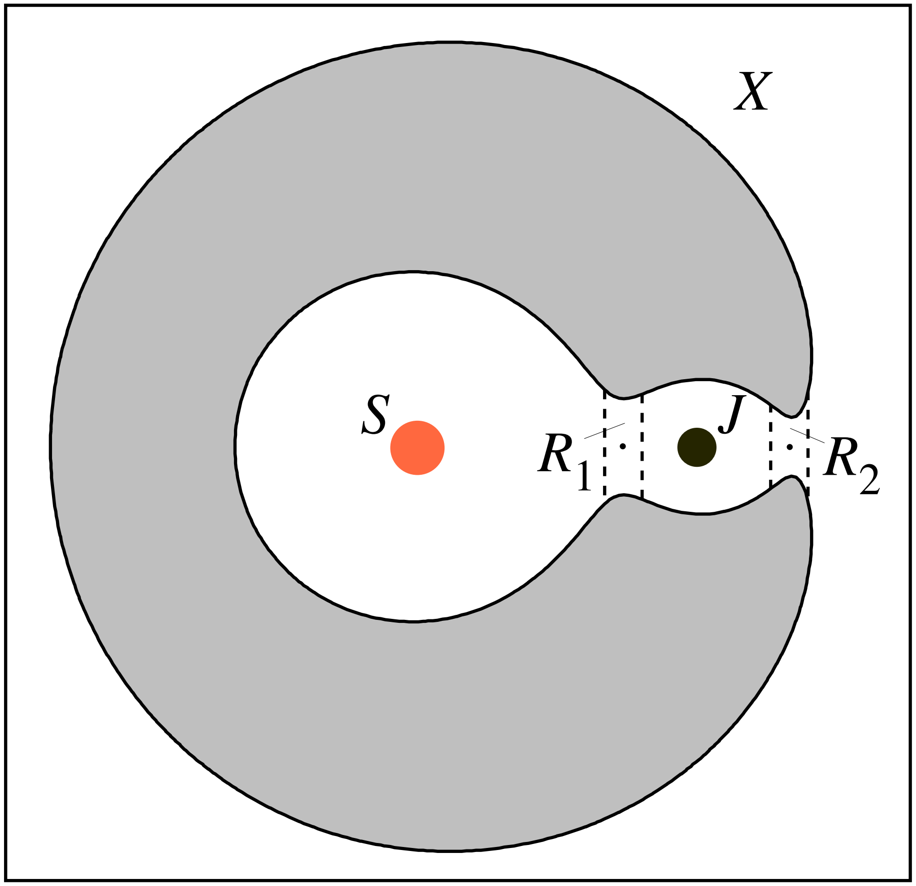
- Neck regions related to collinear unstable equilibria, **x**'s



The location of all the equilibria for $\mu = 0.3$

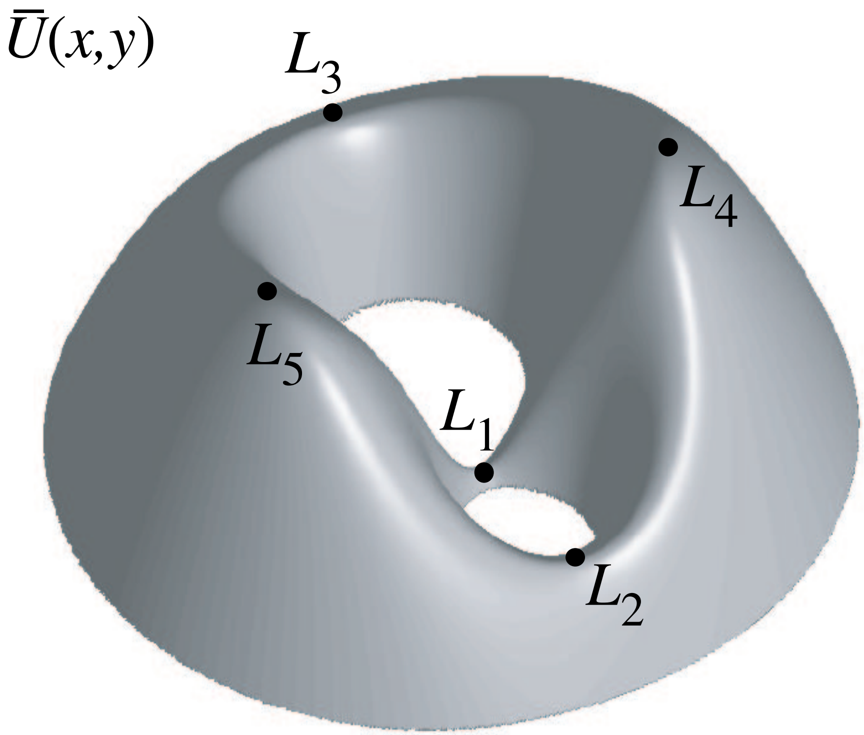
Realms of Possible Motion

- Energy Case 3: For $m_1 =$ Sun, $m_2 =$ Jupiter, we divide the Hill's region into five sets; three realms, S, J, X, and two equilibrium neck regions, R_1, R_2



Equilibrium Points

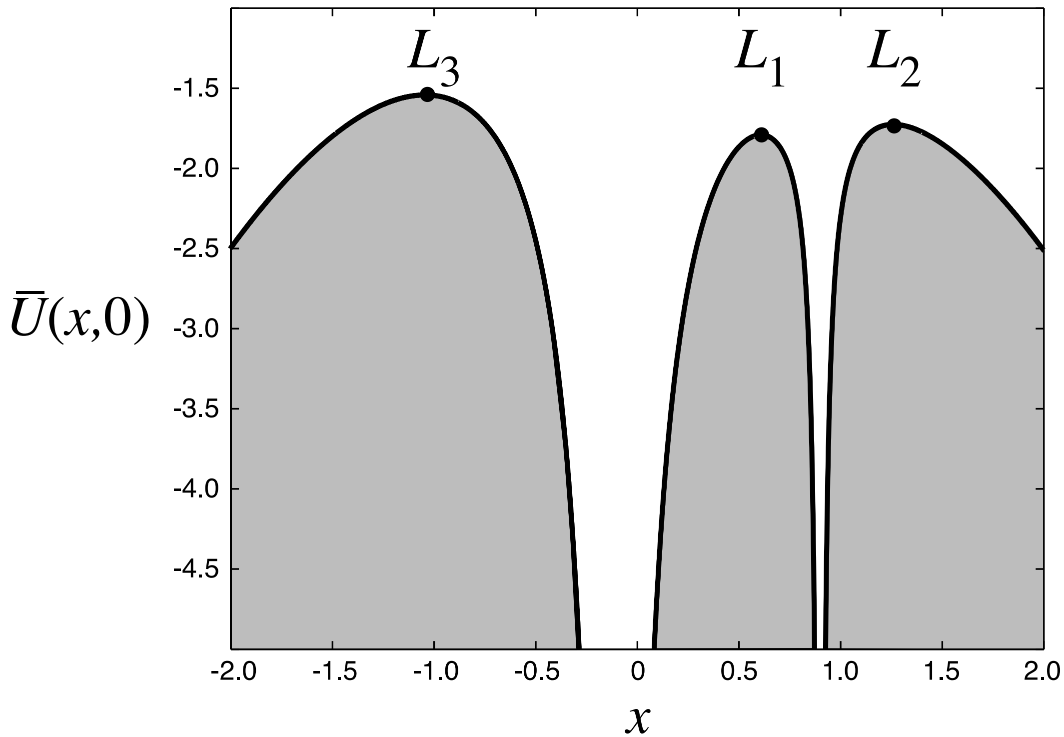
- Find $\bar{q} = (\bar{x}, \bar{y}, \bar{v}_x, \bar{v}_y)$ s.t. $\dot{\bar{q}} = f(\bar{q}) = 0$
- Have form $(\bar{x}, \bar{y}, 0, 0)$ where $(\bar{x}, \bar{y})$ are critical points of $\bar{U}(x, y)$, i.e., $\bar{U}_x = \bar{U}_y = 0$, where $\bar{U}_a \equiv \frac{\partial \bar{U}}{\partial a}$



Critical Points of $\bar{U}(x, y)$

Equilibrium Points

- Consider x-axis solutions; the collinear equilibria
- $\bar{U}_x = \bar{U}_y = 0 \Rightarrow$ polynomial in x
- depends on parameter μ



The graph of $\bar{U}(x, 0)$ for $\mu = 0.1$

Equilibrium Regions

Phase space near equilibrium points

- Consider the equilibrium $\bar{q} = L$ (either L_1 or L_2)
- Eigenvalues of linearized equations about L are $\pm\lambda$ and $\pm i\nu$ with corresponding eigenvectors u_1, u_2, w_1, w_2
- Equilibrium region has a saddle $\times$ center geometry

Equilibrium Regions

Eigenvectors Define Coordinate Frame

- Let the eigenvectors u_1, u_2, w_1, w_2 be the coordinate axes with corresponding new coordinates $(\xi, \eta, \zeta_1, \zeta_2)$. The differential equations assume the simple form

$$\dot{\xi} = \lambda \xi, \qquad \dot{\eta} = -\lambda \eta,$$
$$\dot{\zeta}_1 = \nu \zeta_2, \qquad \dot{\zeta}_2 = -\nu \zeta_1,$$

  and the energy function becomes

$$E_l = \lambda \xi \eta + \frac{\nu}{2} \left(\zeta_1^2 + \zeta_2^2 \right) .$$

- Two additional integrals: $\xi\eta$ and $\rho \equiv |\zeta|^2 = \zeta_1^2 + \zeta_2^2$, where $\zeta = \zeta_1 + i\zeta_2$

Equilibrium Regions

- For positive ε and c, the region $\mathcal{R}$ (either $\mathcal{R}_1$ or $\mathcal{R}_2$), is determined by

$$E_l = \varepsilon, \quad \text{and} \quad |\eta - \xi| \leq c,$$

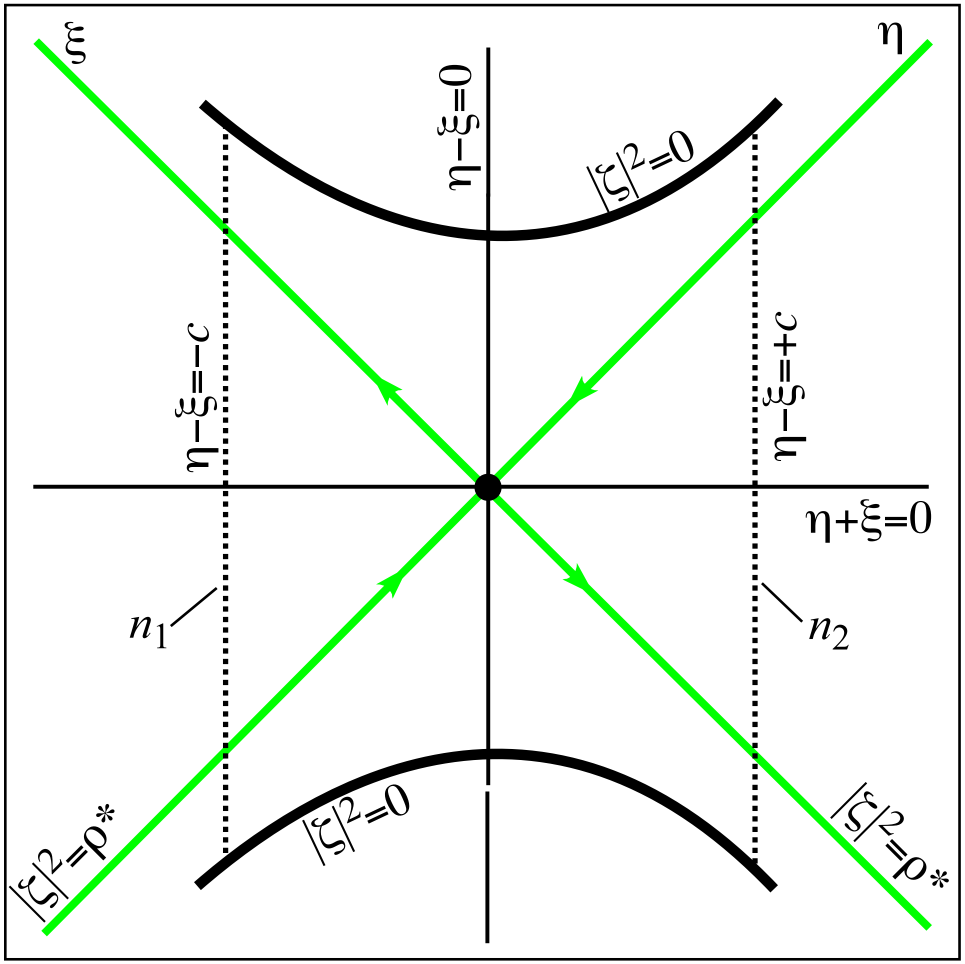
is homeomorphic to $S^2 \times I$; namely, for each fixed value of $(\eta - \xi)$ on the interval $I = [-c, c]$, the equation $E_l = \varepsilon$ determines the two-sphere

$$\frac{\lambda}{4}(\eta + \xi)^2 + \frac{\nu}{2}\left(\zeta_1^2 + \zeta_2^2\right) = \varepsilon + \frac{\lambda}{4}(\eta - \xi)^2,$$

in the variables $((\eta + \xi), \zeta_1, \zeta_2)$.

Bounding Spheres of $\mathcal{R}$

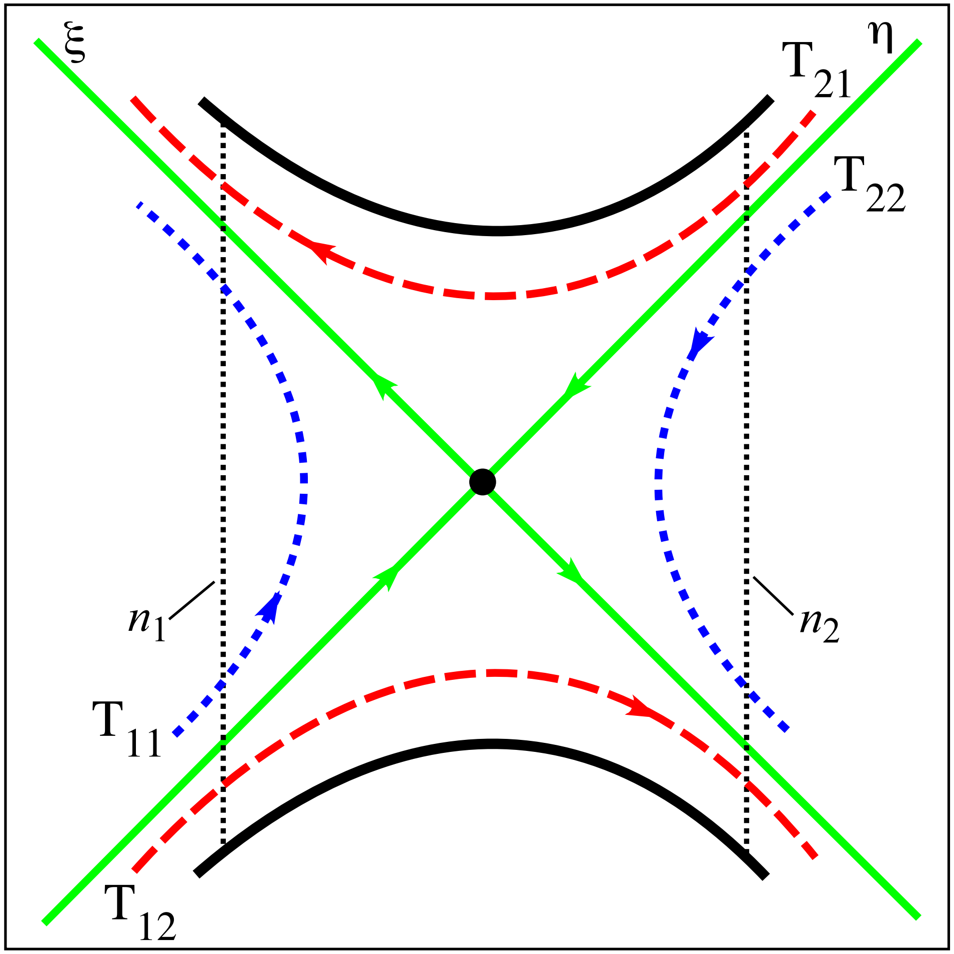
- n_1, the left side ($\eta - \xi = -c$)
  n_2, the right side ($\eta - \xi = c$)



The projection of the flow onto the η-ξ plane

Transit & Non-transit Orbits

- There are **transit** orbits, T_{12}, T_{21} and **non-transit** orbits, T_{11}, T_{22}, separated by **asymptotic** sets to a p.o.



Transit, non-transit, and asymptotic orbits projected onto the η-ξ plane

Twisting of Orbits

- We compute that

$$\frac{d}{dt}\arg\zeta = -\nu,$$

i.e., orbits "twist" while in $\mathcal{R}$ in proportion to the time T spent in $\mathcal{R}$, where

$$T = \frac{1}{\lambda}\left(\ln\frac{2\lambda(\eta^0)^2}{\nu} - \ln(\rho^* - \rho)\right),$$

where η^0 is the initial condition on the bounding sphere and $\rho = \rho^* = 2\varepsilon/\nu$ only for the asymptotic orbits.

- Amount of twisting depends sensitively on how close an orbit comes to the cylinders of asymptotic orbits, i.e., depends on $(\rho^* - \rho) > 0$.

Orbits in Position Space

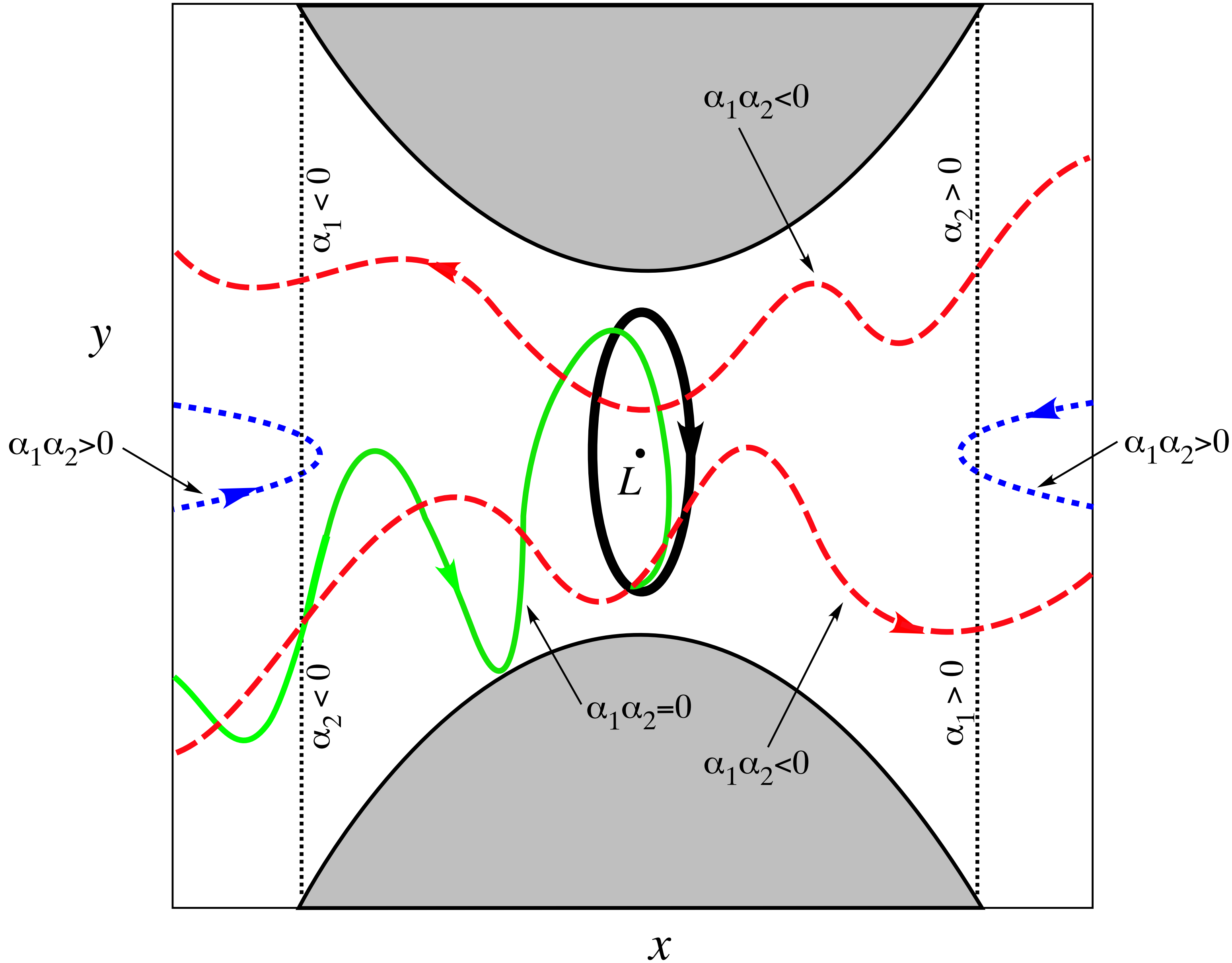
Appearance of orbits in position space

- The general (real) solution has the form

$$\begin{aligned} u(t) &= (x(t), y(t), v_x(t), v_y(t)), \\ &= \alpha_1 e^{\lambda t} u_1 + \alpha_2 e^{-\lambda t} u_2 + 2\mathrm{Re}(\beta e^{i\nu t} w_1), \end{aligned}$$

  where α_1, α_2 are real and $\beta = \beta_1 + i\beta_2$ is complex.
- Four categories of orbits, depending on the signs of α_1 and α_2.
- By a theorem of Moser [1958], all the qualitative results carry over to the nonlinear system.

Orbits in Position Space



Equilibrium Region: Summary

The Flow in the Equilibrium Region

- In summary, the phase space in the equilibrium region can be partitioned into four categories of distinctly different kinds of motion:

  (1) periodic orbits, a.k.a., Lyapunov orbits

  (2) asymptotic orbits, i.e., invariant stable and unstable cylindrical manifolds (henceforth called **tubes**)

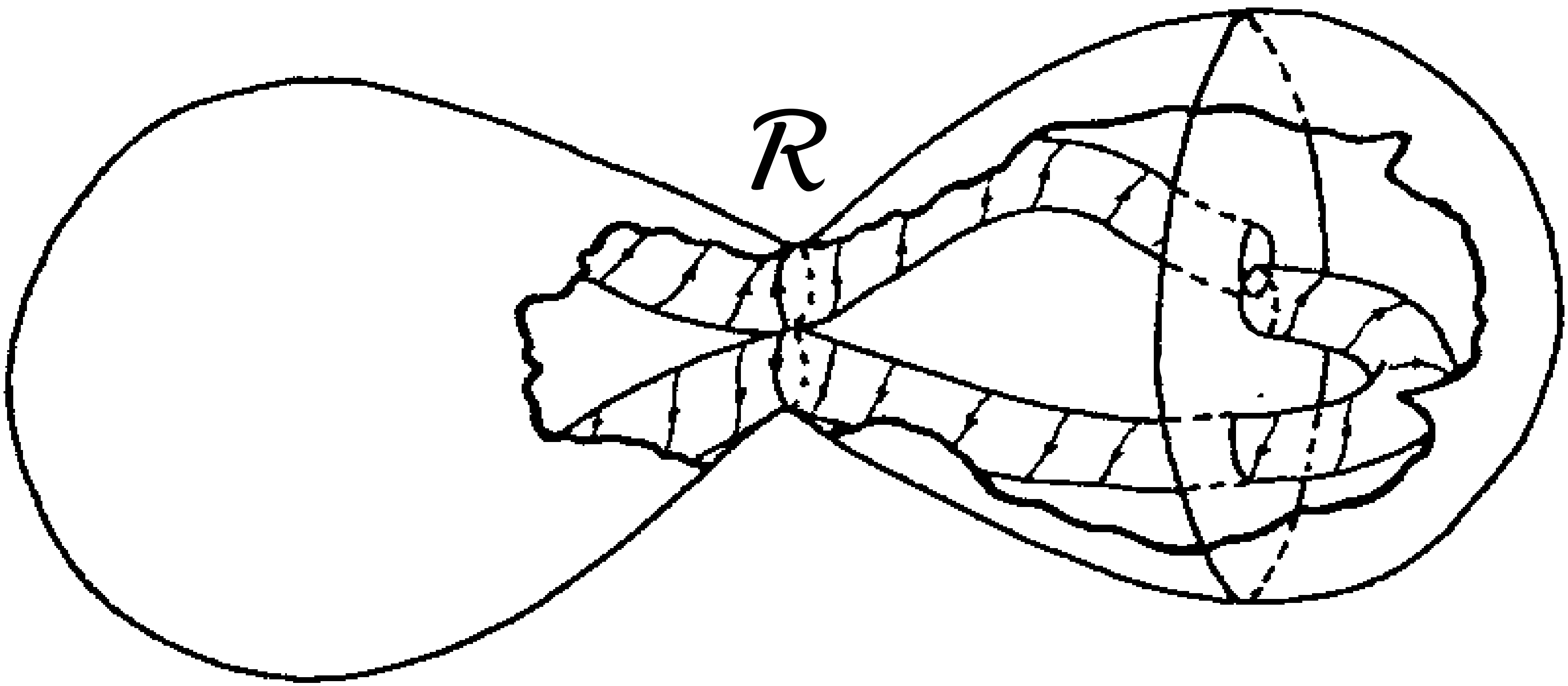
  (3) transit orbits, moving from one realm to another

  (4) non-transit orbits, returning to their original realm

- These categories help us understand the connectivity of the global phase space

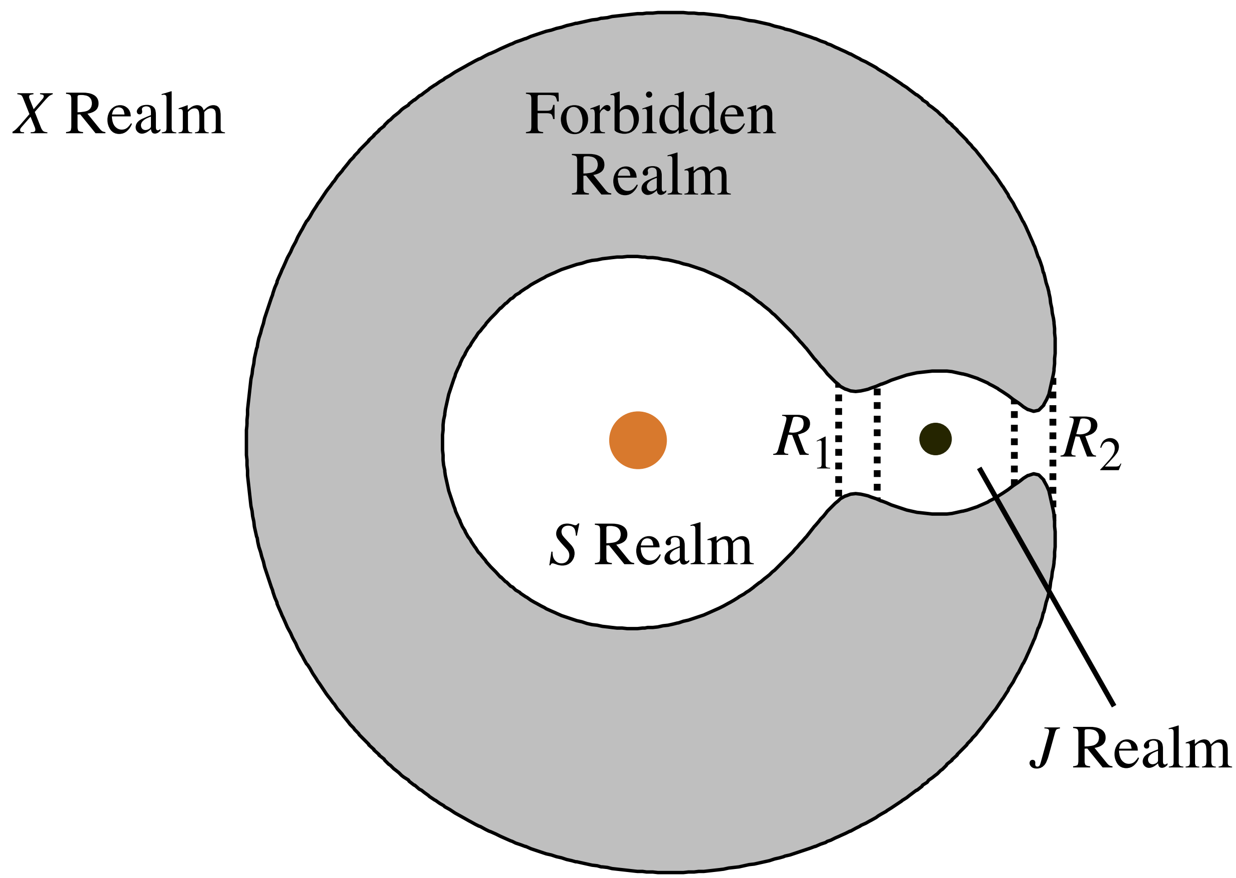
Tube Dynamics

- All motion between realms connected by equilibrium neck regions $\mathcal{R}$ must occur through the interior of the cylindrical stable and unstable manifold **tubes**



Tube Dynamics: Itineraries

- We can find/construct an orbit with any **itinerary**, e.g., $(\ldots, J, X, J, S, J, \ldots)$, where X, J and S denote the different realms (symbolic dynamics)



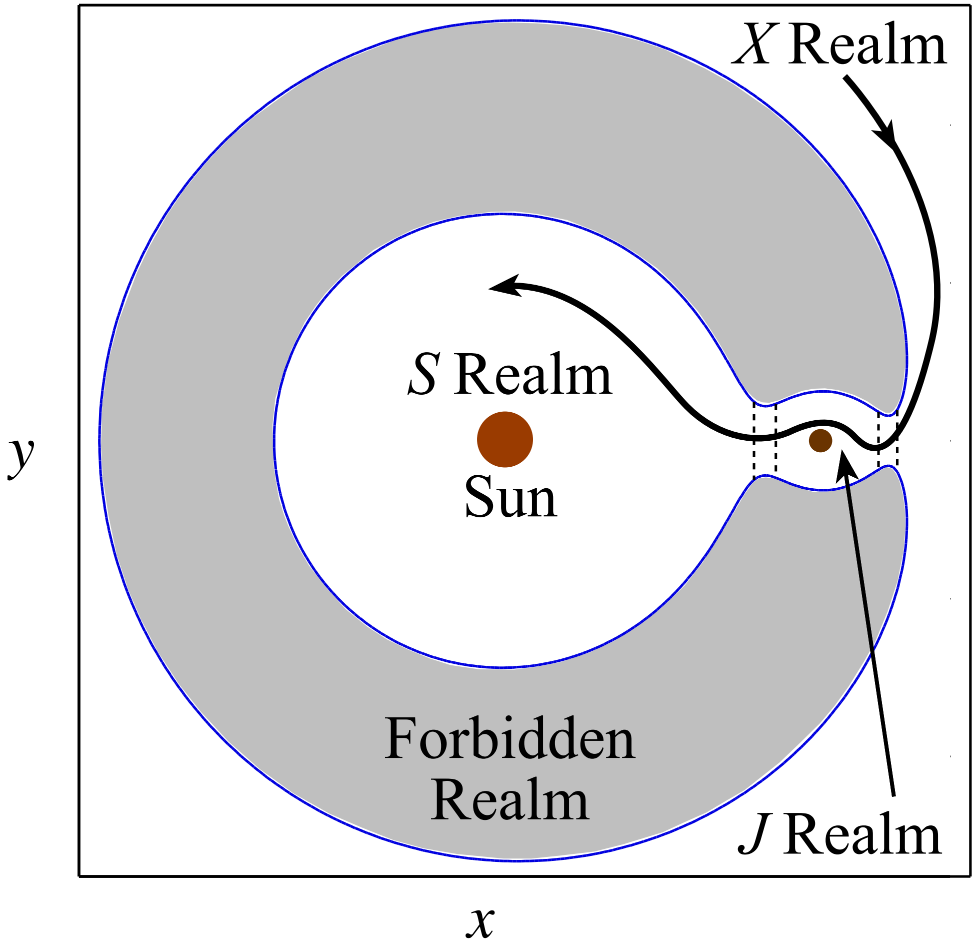
Construction of Trajectories

- Systematic construction of trajectories with desired itineraries – trajectories which use **no fuel.**

  – by linking tubes in the right order $\longrightarrow$ **tube hopping**

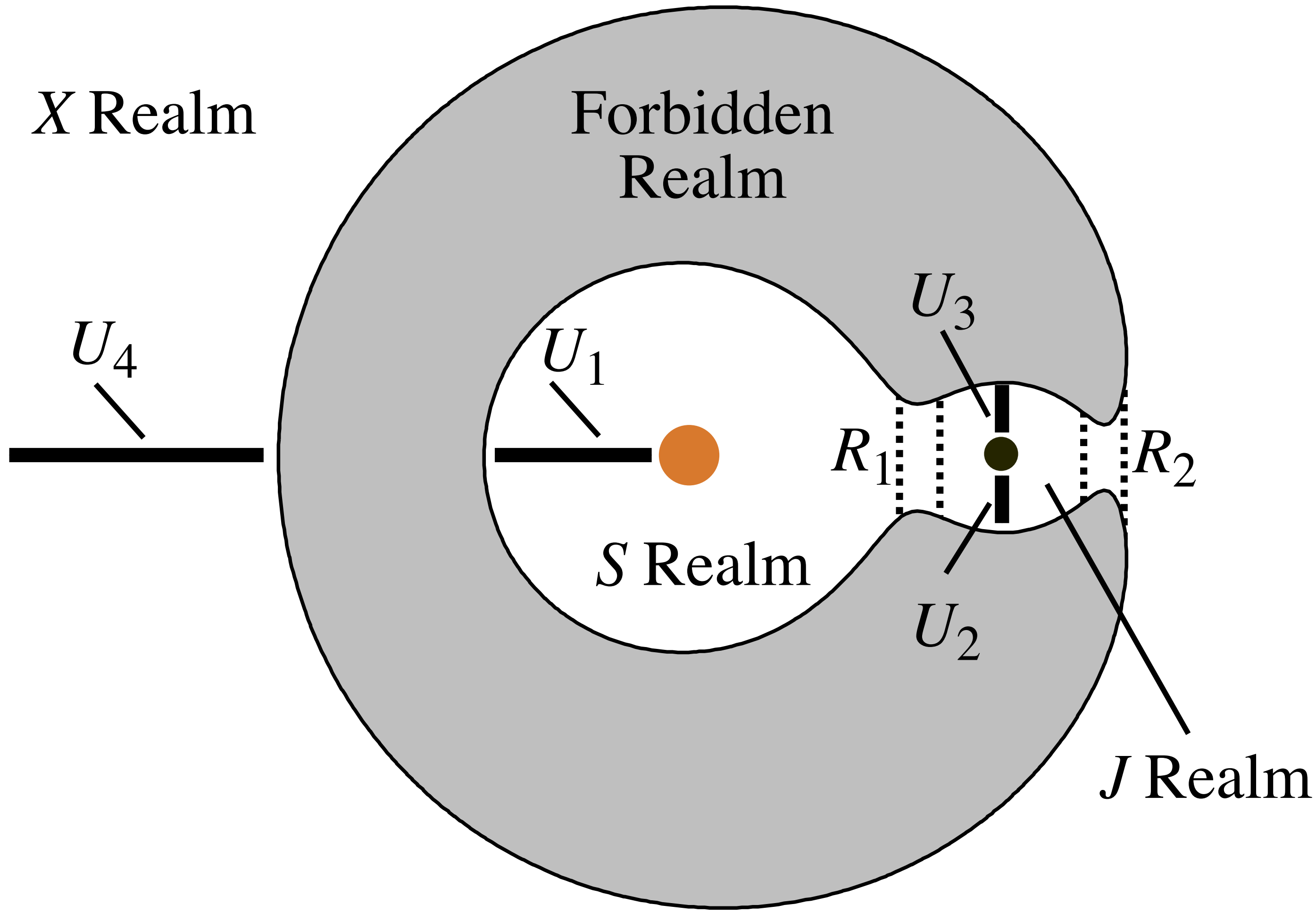
Construction of Trajectories

- ***Ex. Trajectory with Itinerary*** (X, J, S)
  - search for an initial condition with this itinerary



Construction of Trajectories

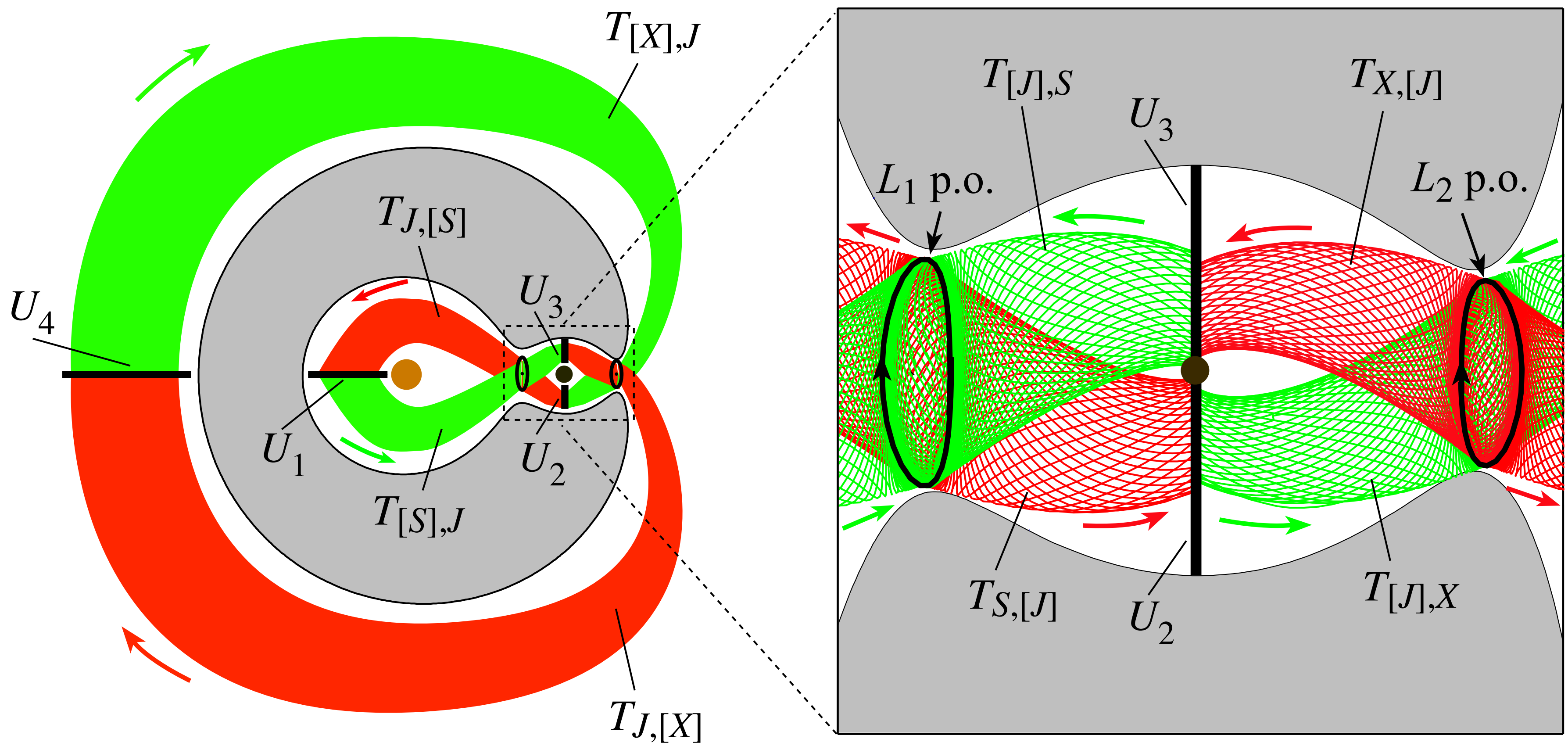
- seek area on 2D Poincaré section corresponding to (X, J, S) itinerary region; an "itinerarea"



The location of four Poincaré sections U_i

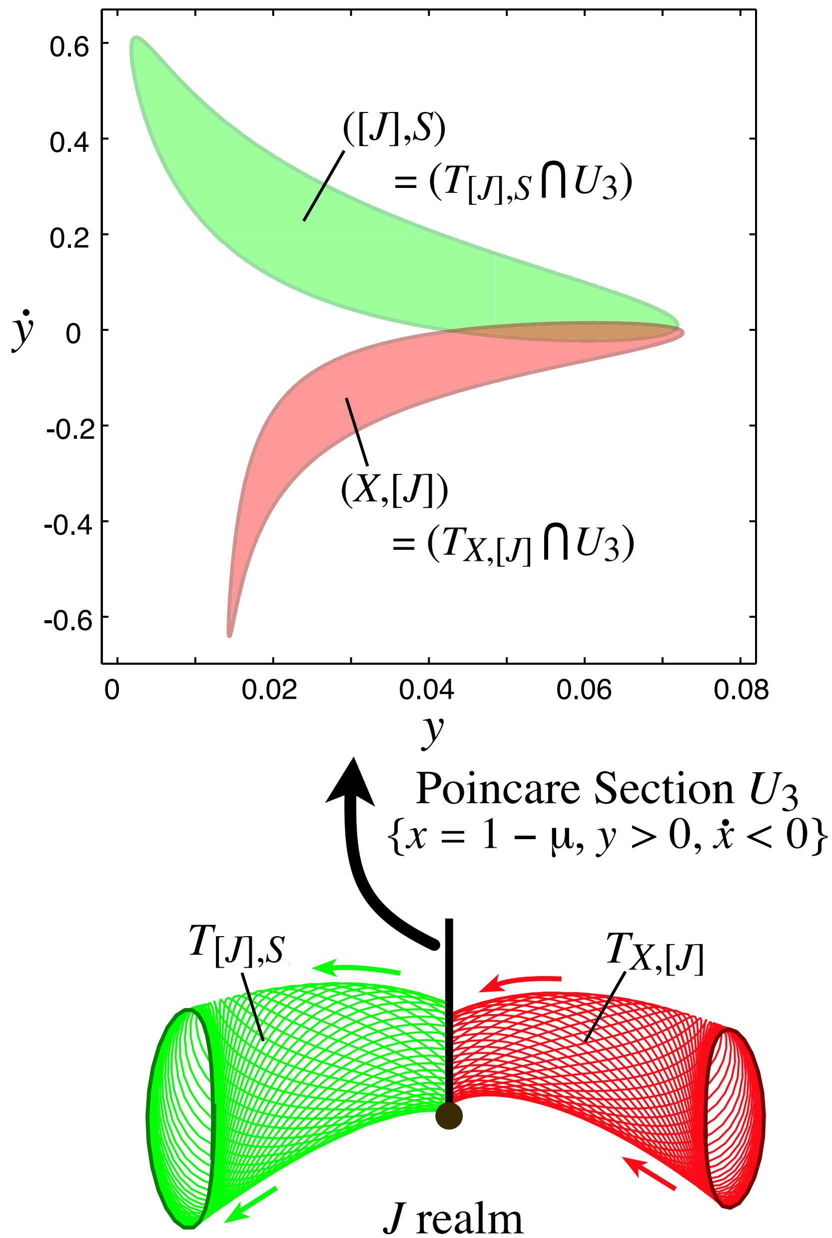
Construction of Trajectories

- $T_{[X],J}$ is the solid tube of trajectories currently in the X realm and heading toward the J realm
  - Let's seek itinerarea $(X, [J], S)$



How the tubes connect the U_i

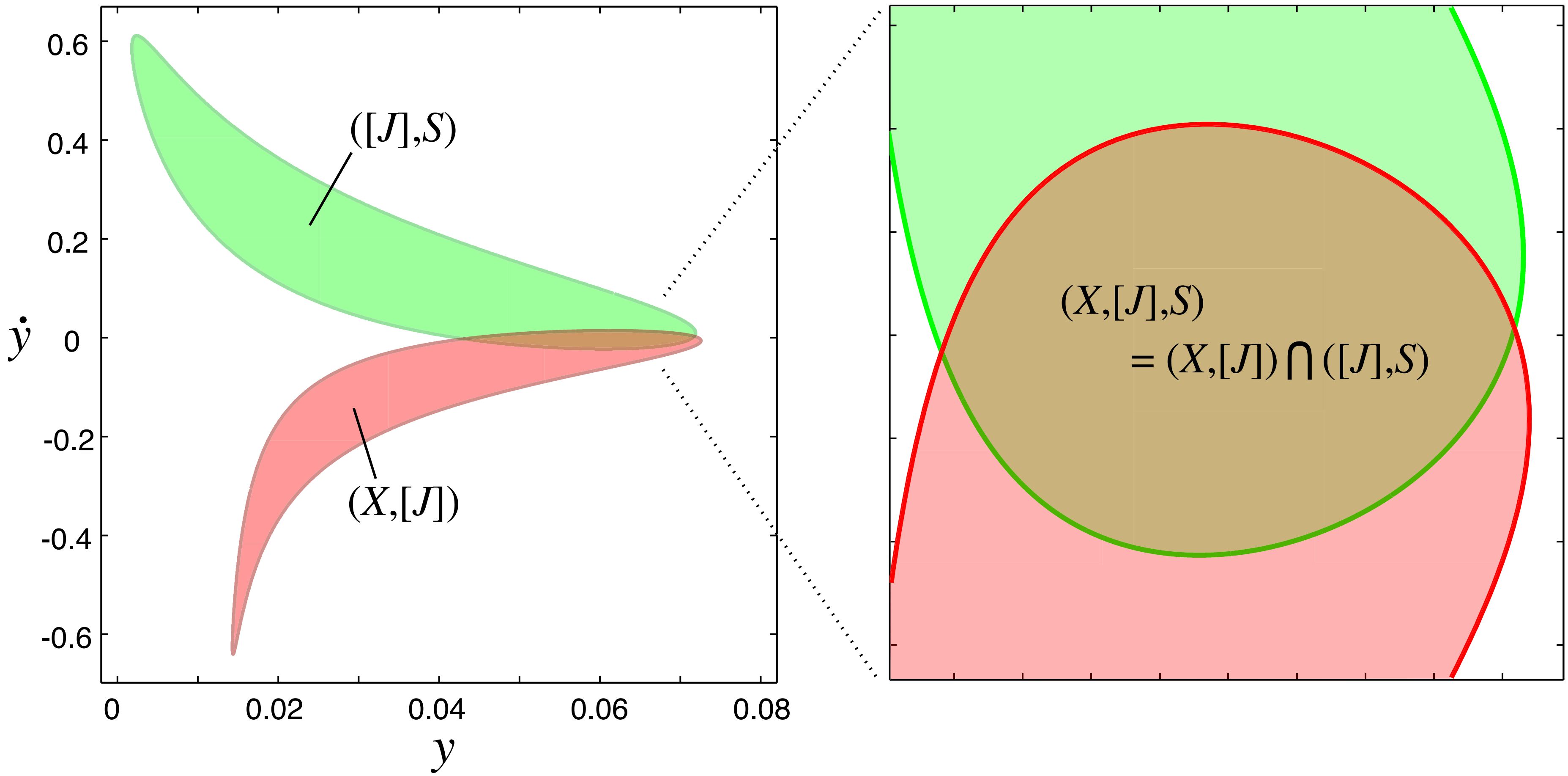
Construction of Trajectories



Construction of Trajectories

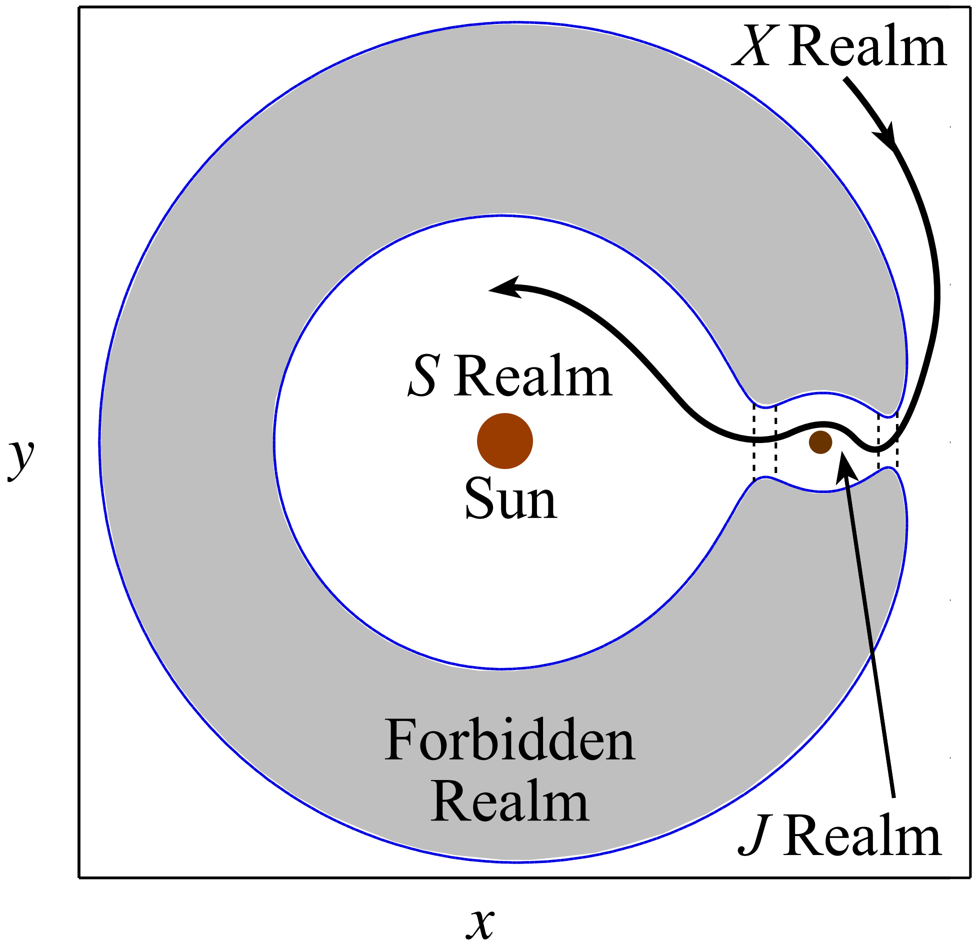
An itinerarea with label (X,[J],S)

- Denote the intersection $(X,[J])\bigcap([J],S)$ by $(X,[J],S)$



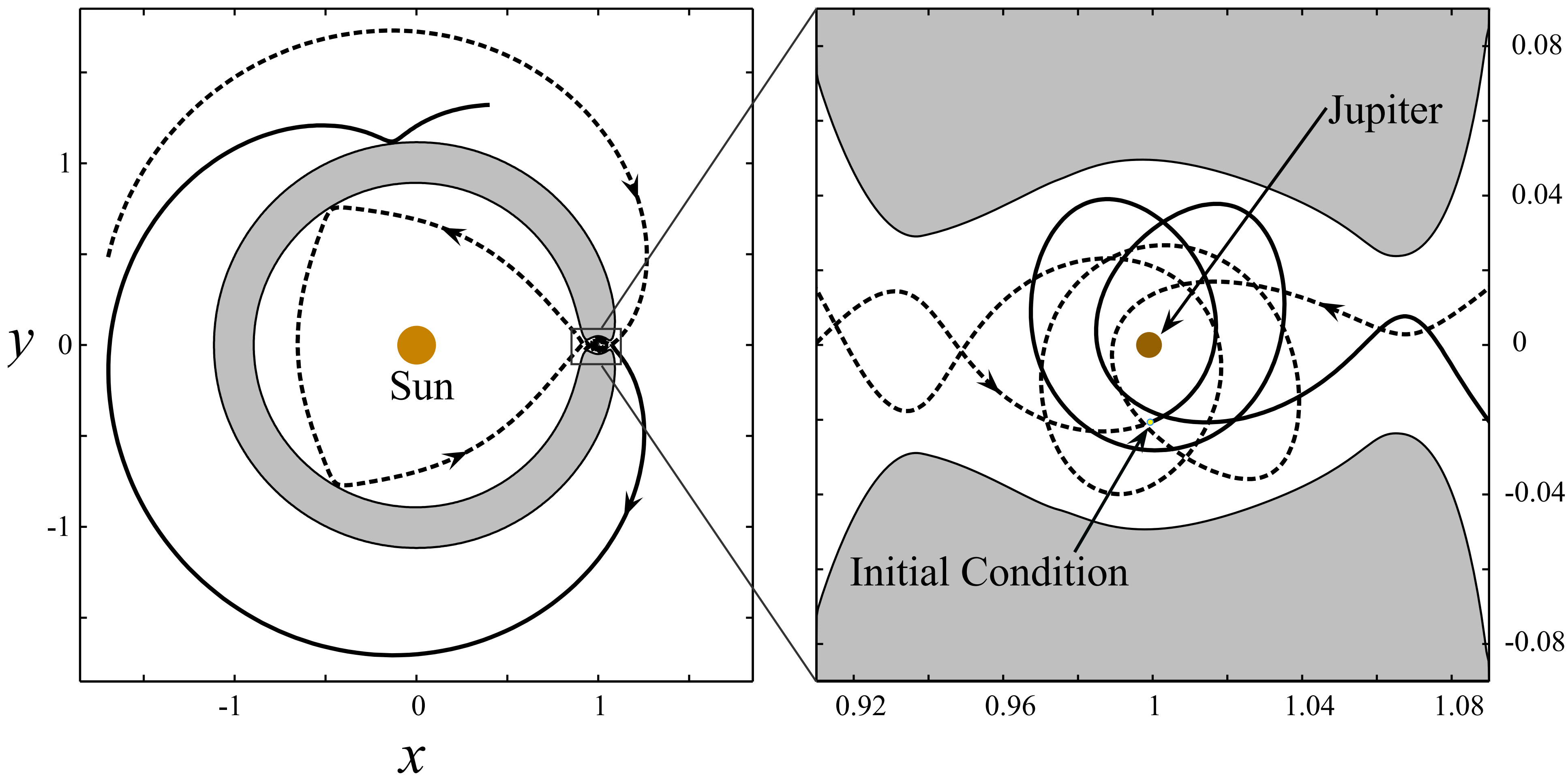
Construction of Trajectories

- Forward and backward numerical integration of any initial condition within the itinerarea yields a trajectory with the desired itinerary



Construction of Trajectories

- Trajectories with longer itineraries can be produced
  - e.g., (X, J, S, J, X)



Restricted 4-Body Problem

- Solutions to the restricted 4-body problem can be built up from solutions to the rest. 3-body problem
- One system of particular interest is a spacecraft in the Earth-Moon vicinity, with the Sun's perturbation
- Example mission: low energy transfer to the Moon

Low Energy to the Moon

- Motivation: systematic construction of trajectories like the 1991 Hiten trajectory. This trajectory uses significantly less on-board fuel than an Apollo-like transfer using third body effects.
- The key is ballistic, or unpropelled, capture by the Moon
- Originally found via a trial-and-error approach, before tube dynamics in the system was known (Belbruno and Miller [1993])

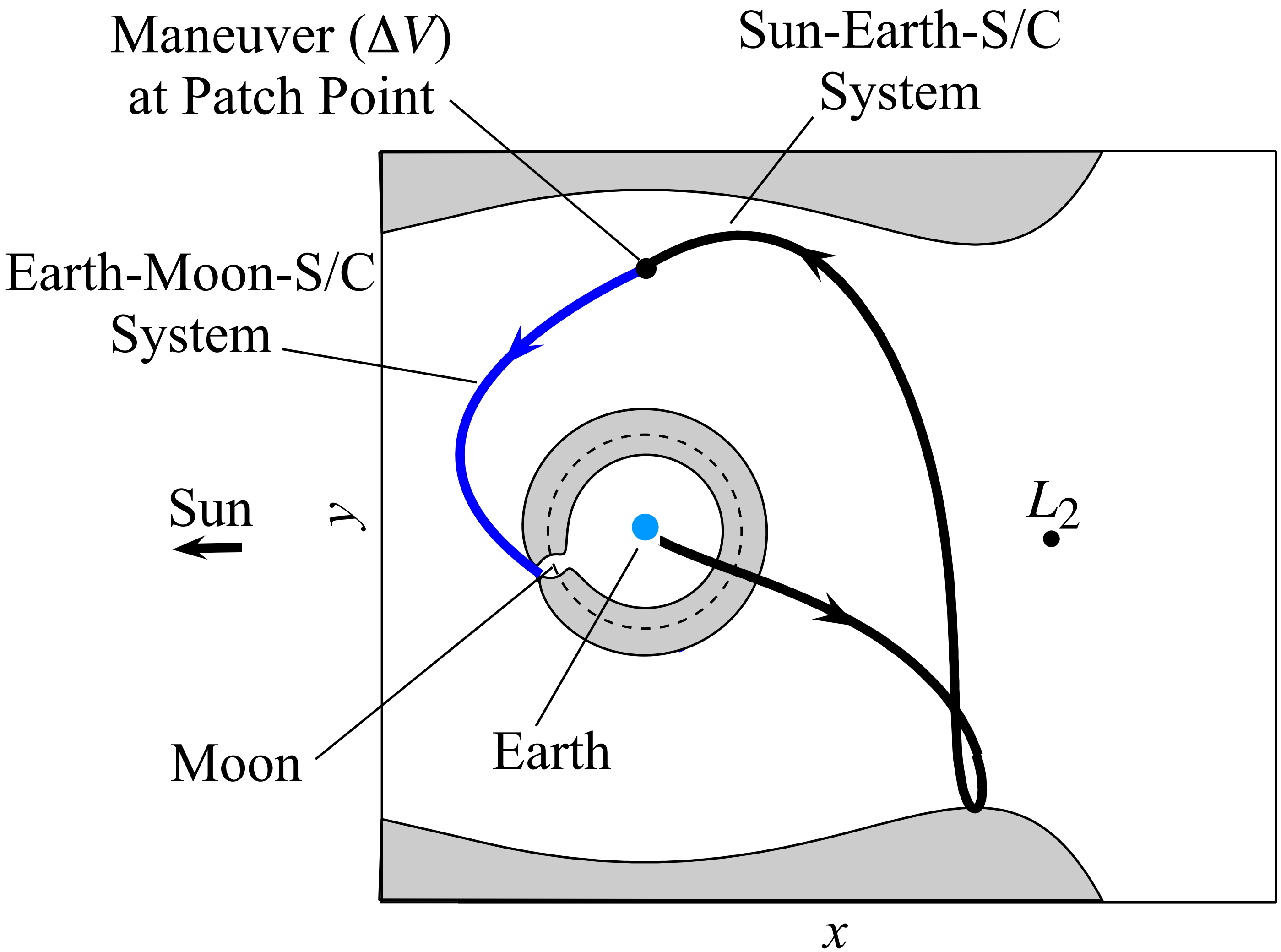
Low Energy to the Moon

- Patched three-body approximation: we assume the S/C's trajectory can be divided into two portions of rest. 3-body problem solutions

Low Energy to the Moon

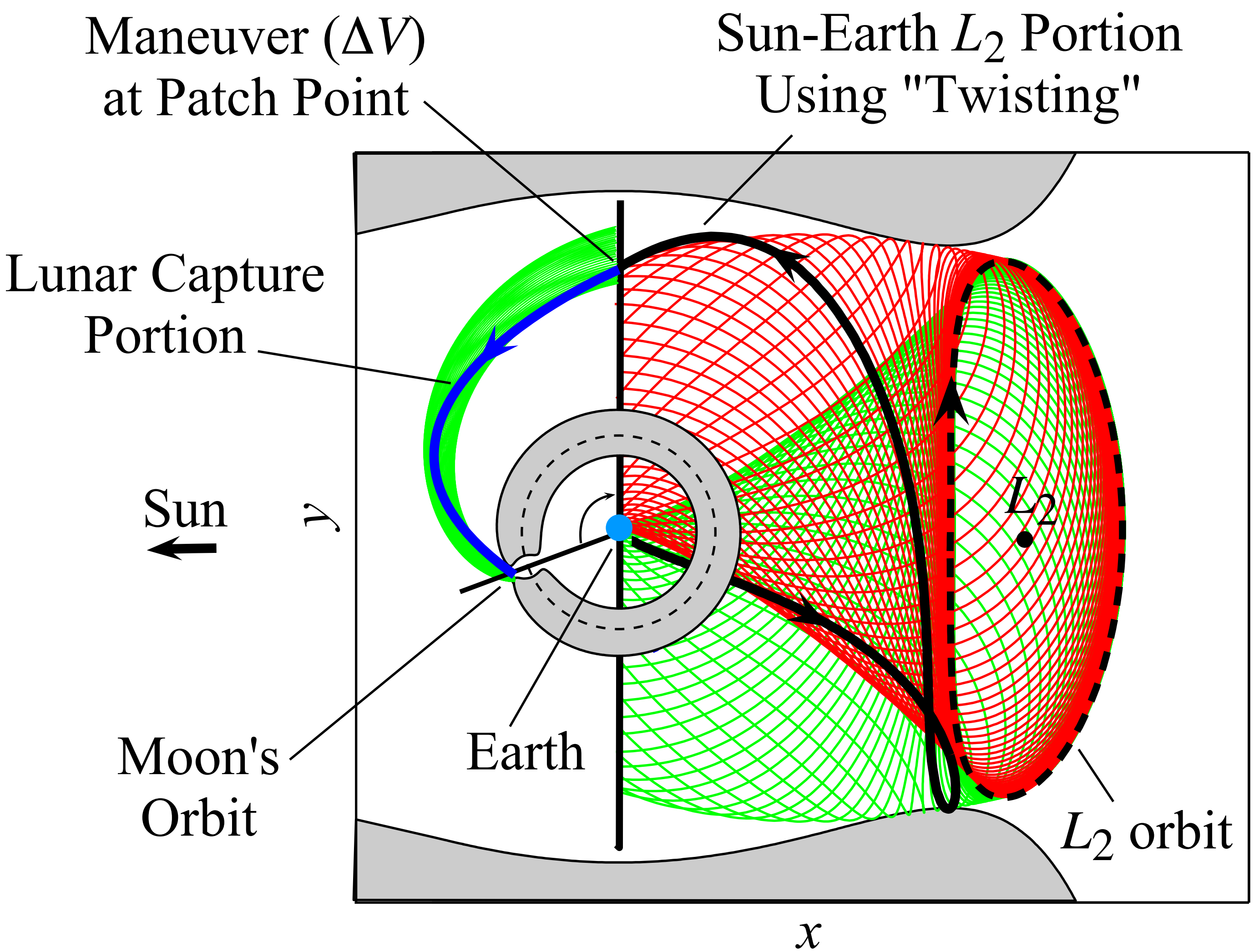
(1) Sun-Earth-S/C

(2) Earth-Moon-S/C



Low Energy to the Moon

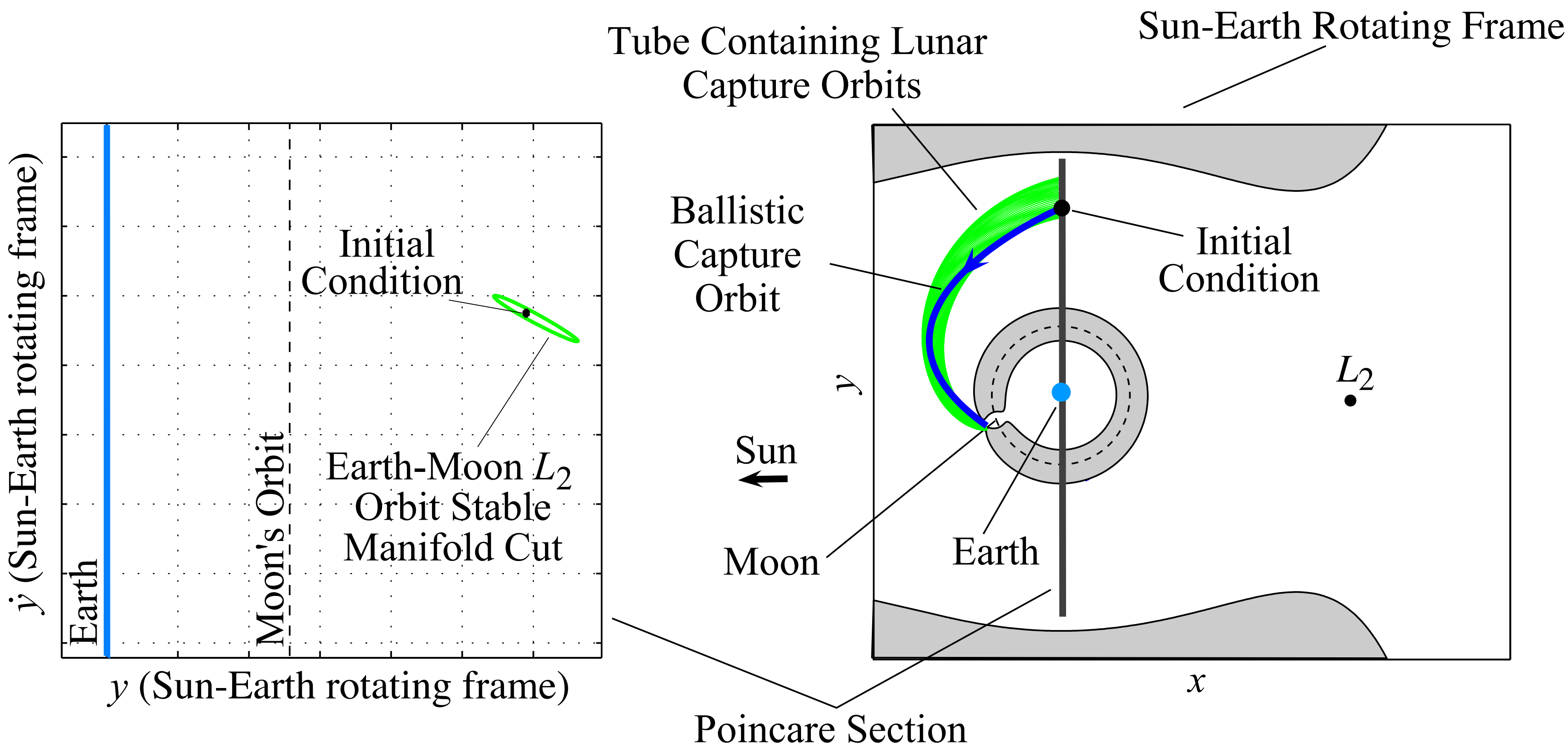
- Consider the intersection of tubes in these two systems (if any exists) on a Poincaré section



Low Energy to the Moon

Earth-Moon-S/C – Ballistic capture

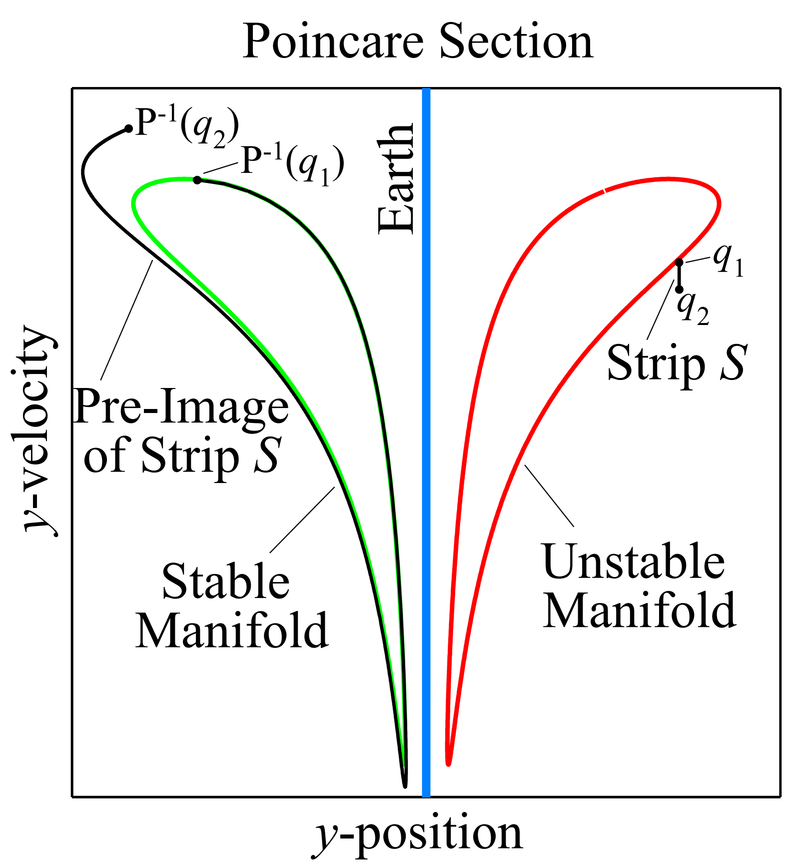
- Find boundary of tube of lunar capture orbits

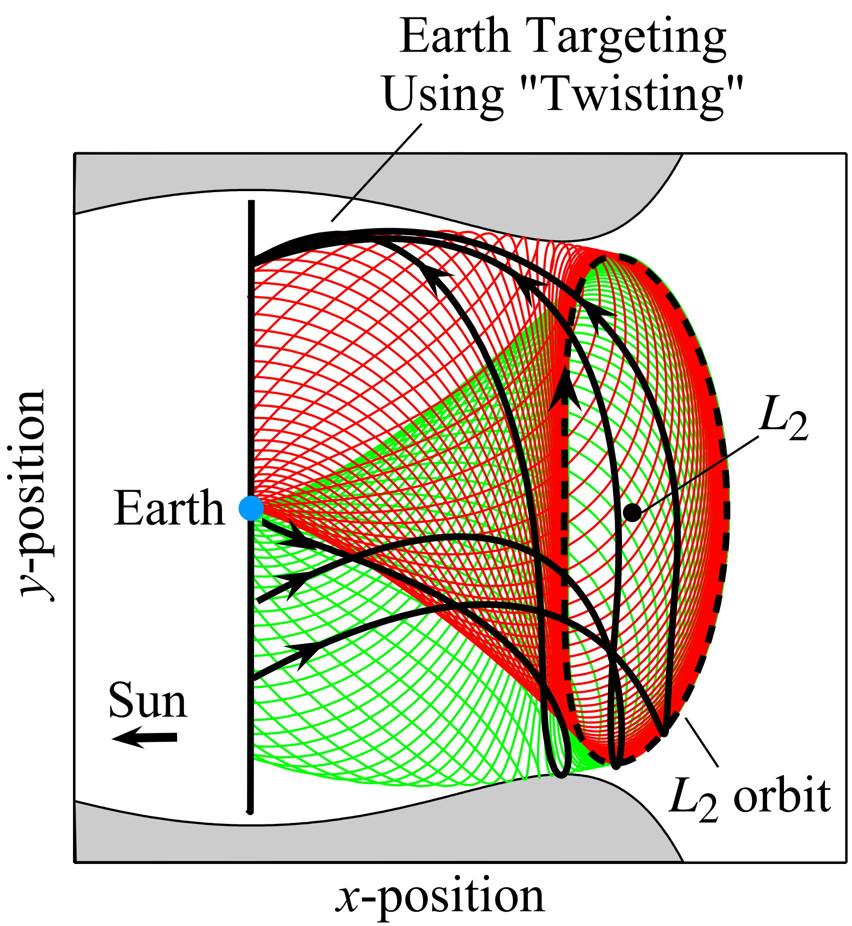


Low Energy to the Moon

■ *Sun-Earth-S/C – Twisting of orbits*

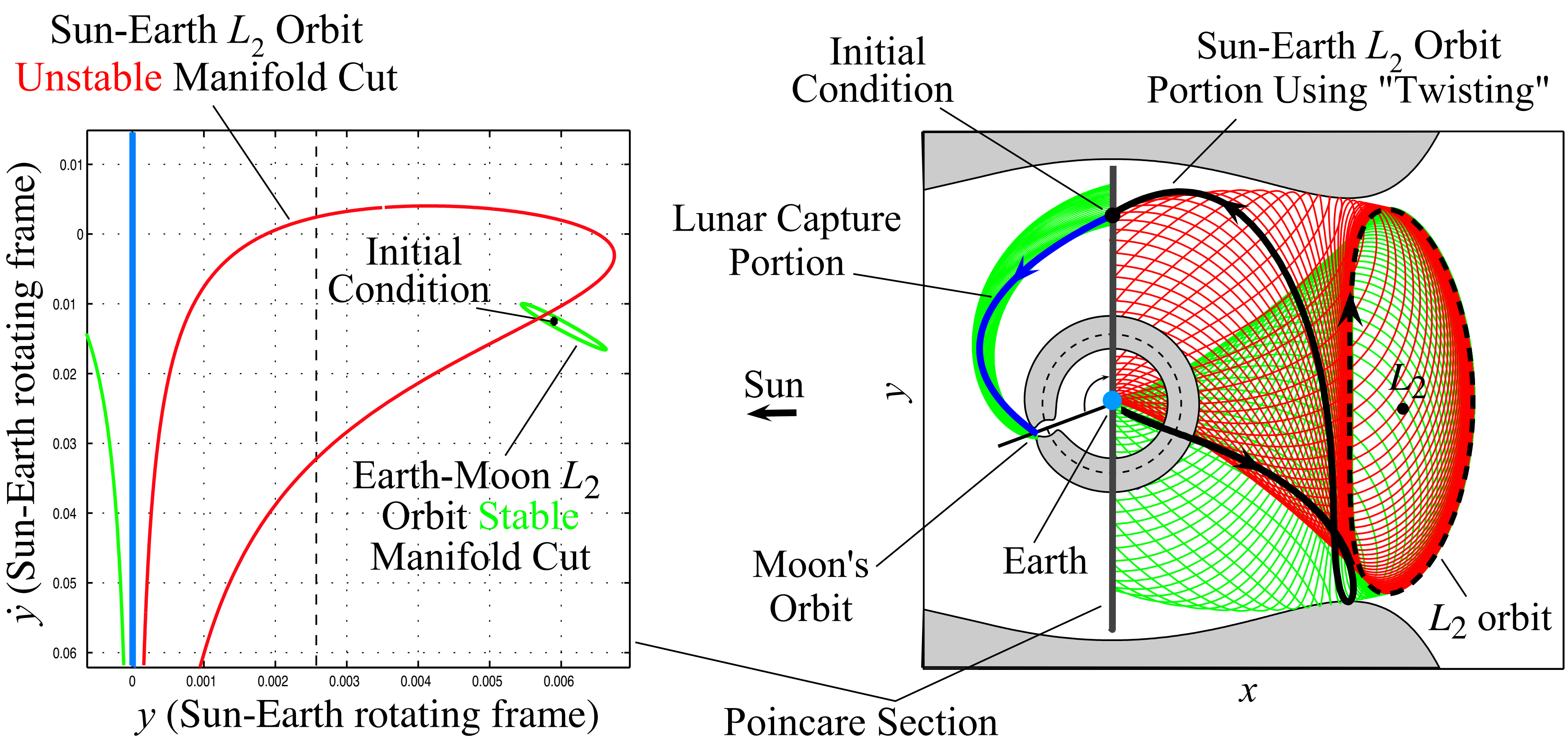
- Amount of twist depends sensitively on distance from tube boundary; use this to target Earth parking orbit





Low Energy to the Moon

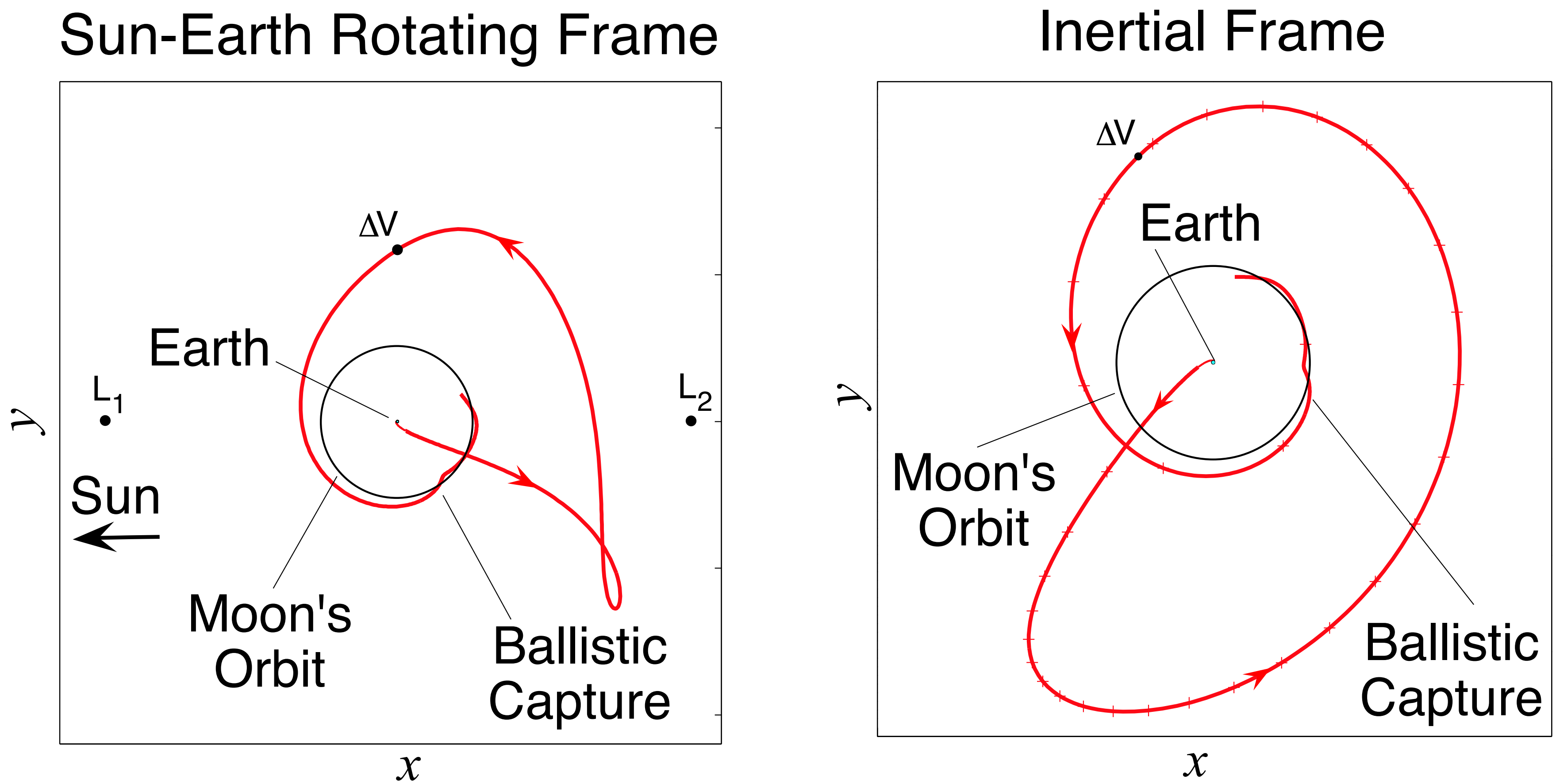
- Integrate initial conditions forward and backward to generate desired trajectory, allowing for velocity discontinuity (maneuver of size ΔV to "tube hop")



Low Energy to the Moon

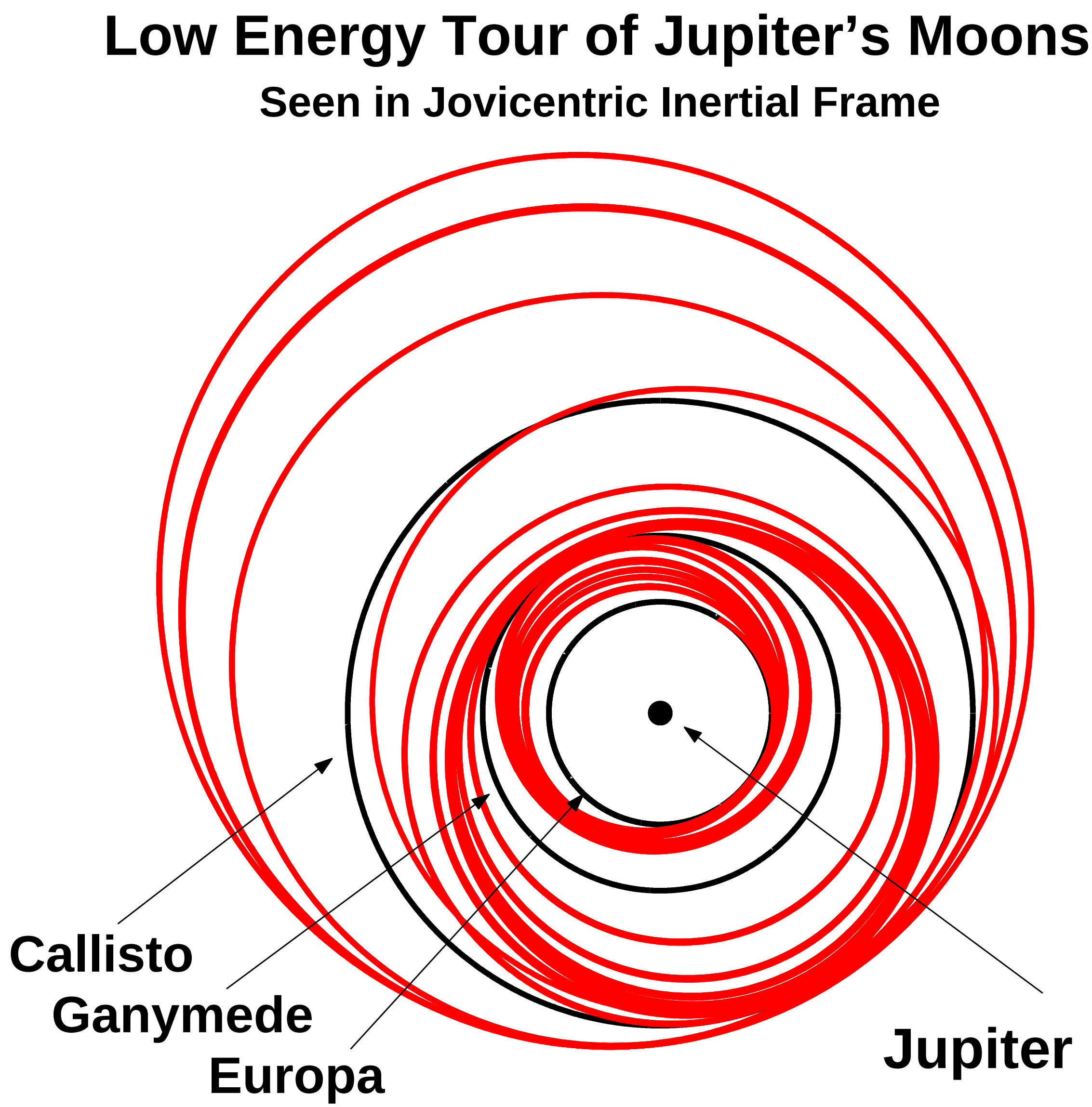
- Verification: use these initial conditions as an initial guess in restricted 4-body model, the bicircular model
- Small velocity discontinuity at patch point: $\Delta V = 34$ m/s
- Uses 20% less on-board fuel than an Apollo-like transfer – the trade-off is a longer flight time

Low Energy to the Moon



Current and Ongoing Work

- Multi-moon orbiter, $\Delta V = 22$ m/s (!!!) $\Rightarrow$ JIMO



Current and Ongoing Work

Ongoing challenges

- Make an automated algorithm for trajectory generation
- Consider model uncertainty, unmodeled dynamics, noise
- Trajectory correction when errors occur
  - – Re-targeting of original (nominal) trajectory vs. re-generation of nominal trajectory
  - – Trajectory correction work for *Genesis* is a first step

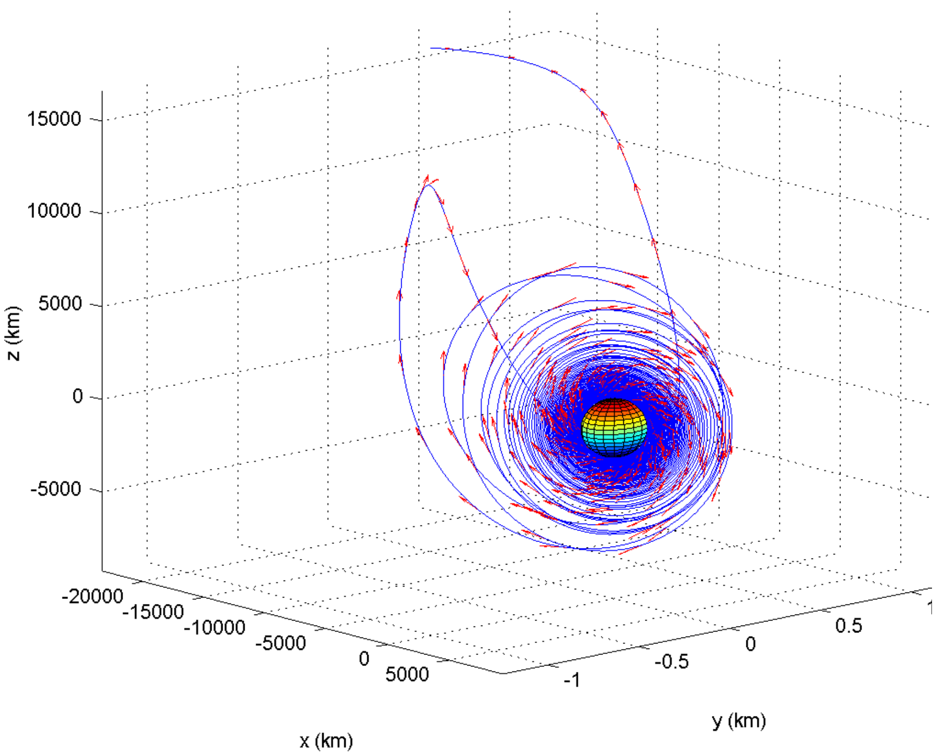
Current and Ongoing Work

- Getting *Genesis* onto the destination orbit at the right time, while minimizing fuel consumption

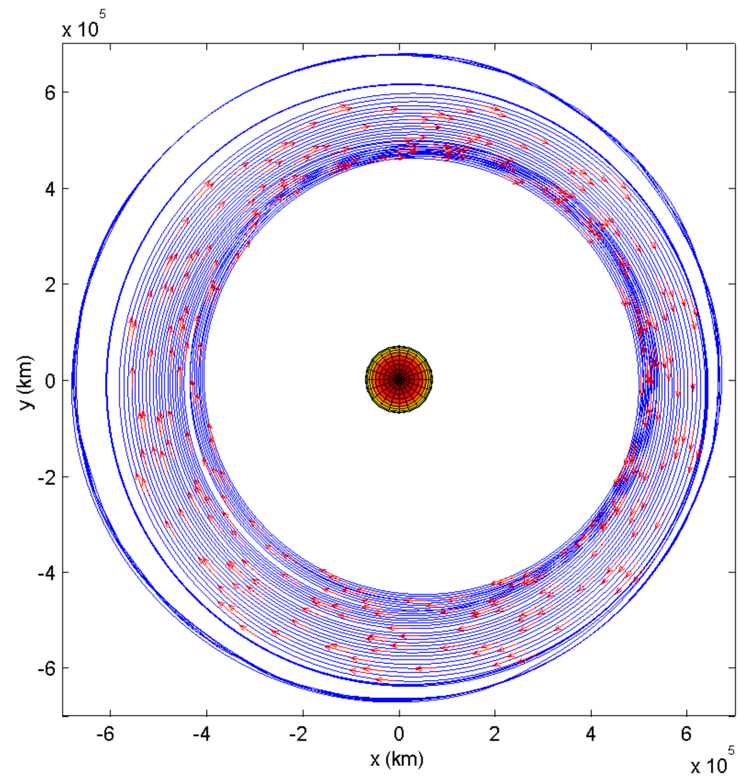
from Serban, Koon, Lo, Marsden, Petzold, Ross, and Wilson [2002]

Current and Ongoing Work

- Incorporation of low-thrust



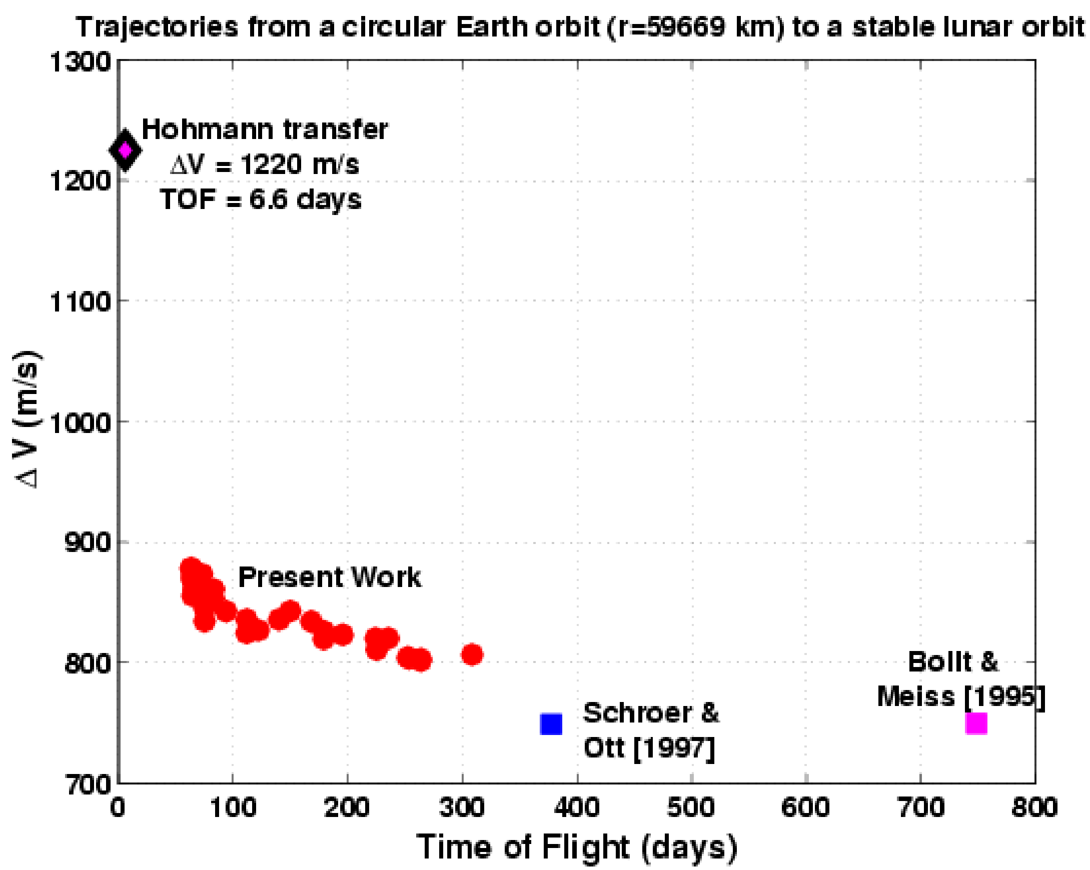
Spiral out from Europa



Europa to Io transfer

Current and Ongoing Work

- Coordination with goals/constraints of real missions
  e.g., time at each moon, radiation dose, max. flight time
- Decrease flight time: evidence suggests large decrease in time for small increase in ΔV



Current and Ongoing Work

- Spin-off: Results also apply to mathematically similar problems in chemistry and astrophysics
  - phase space transport
- Applications
  - chemical reaction rates
  - asteroid collision prediction

Summary and Conclusions

- For certain energies of the planar circular rest. 3-body problem, the phase space can be divided into sets; three large realms and equilibrium regions connecting them
- We consider stable and unstable manifolds of p.o.'s in the equil. regions
- The manifolds have a cylindrical geometry and the physical property that all motion from one realm to another must pass through their interior
- The study of the cylindrical manifolds, tube dynamics, can be used to design spacecraft trajectories
- Tube dynamics applicable in other physical problems too