Periodic Orbits and Transport: From the Three-Body Problem to Atomic Physics

Shane Ross

Martin Lo (JPL), Wang Sang Koon and Jerrold Marsden (Caltech)

CDS 280, November 13, 2000

shane@cds.caltech.edu
http://www.cds.caltech.edu/~shane/

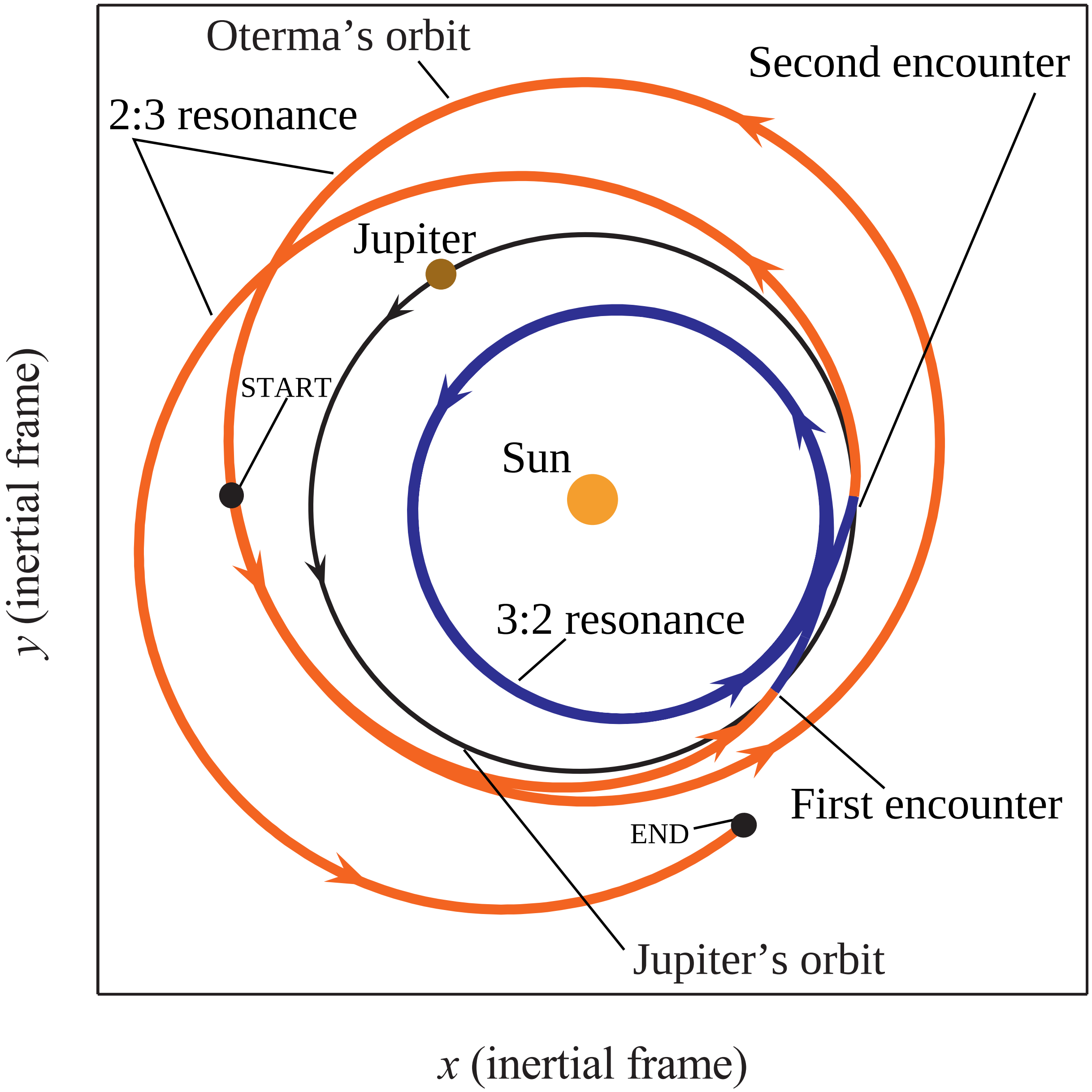
Outline

- ***Context***: Three-body problem (Hamiltonian)
- ***Equilibria***: Collinear libration points have saddle × center structure
- ***Periodic orbits***: Stable and unstable invariant manifolds divide the energy surface, channeling the flow in phase space
- ***Classification***: Interesting orbits can be classified and constructed using Poincaré sections and symbolic dynamics
- ***Atomic physics***: Similar behavior noticed in ionization of hydrogen atom (Jaffé, Farrelly & Uzer, 1999)

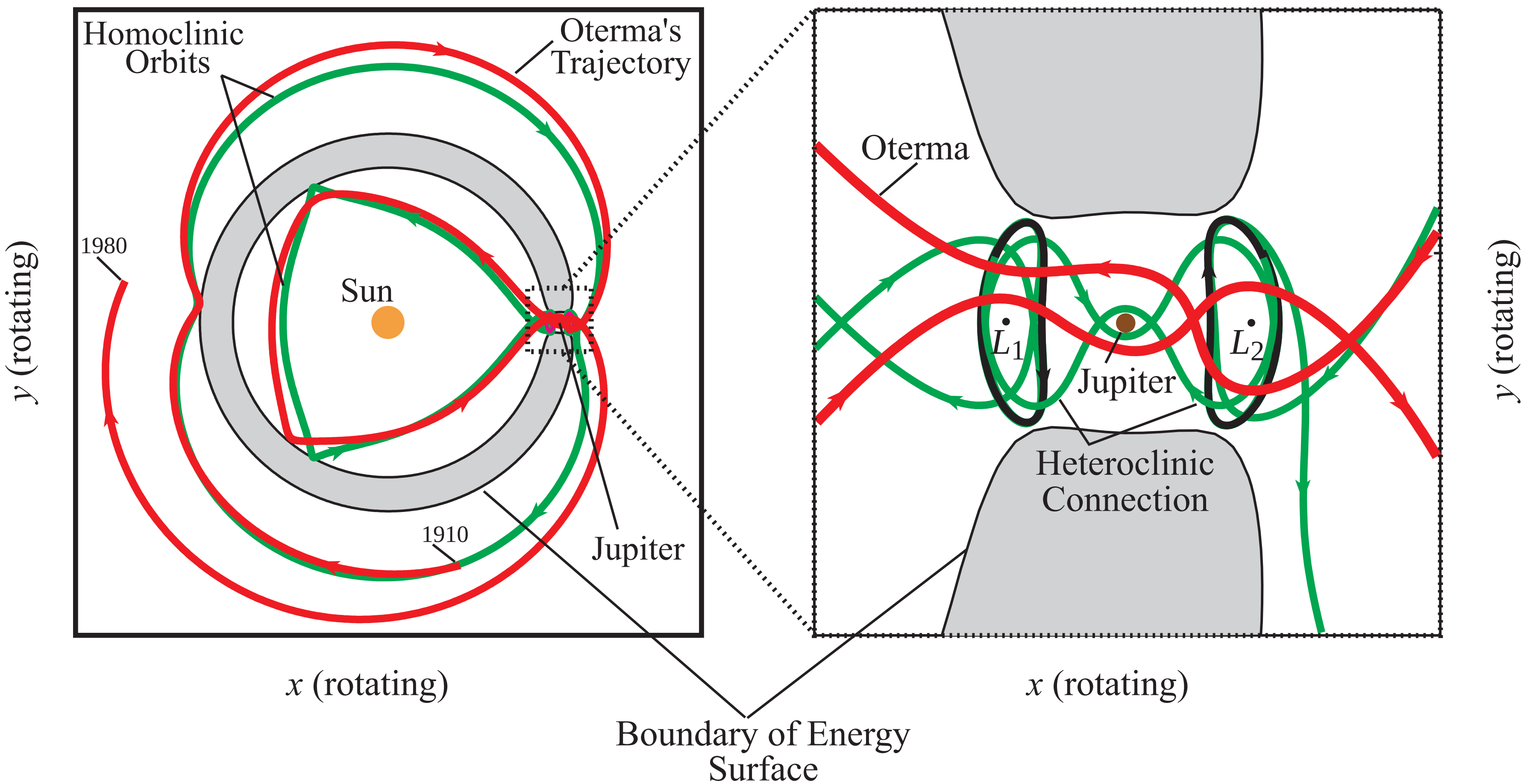
Motivation: Comet Transitions

Jupiter Comets–such as Oterma

- Comets moving in the vicinity of Jupiter do so mainly under the influence of Jupiter and the Sun–*i.e.*, in a three body problem.
- These comets sometimes make a ***rapid transition*** from **outside** to **inside** Jupiter's orbit.
- ***Captured temporarily*** by Jupiter during transition.
- **Exterior** (2:3 resonance) $\longrightarrow$ **Interior** (3:2 resonance).
- The next figure shows the orbit of Oterma (AD 1915–1980) in an inertial frame


Oterma's orbit
Second encounter
2:3 resonance
Jupiter
START
Sun
3:2 resonance
First encounter
END
Jupiter's orbit
y (inertial frame)
x (inertial frame)

- Next figure shows Oterma's orbit in a ***rotating frame*** (so Jupiter looks like it is standing still) and with some invariant manifolds in the ***three body problem*** superimposed.



Homoclinic Orbits
Oterma's Trajectory
1980
Sun
1910
Jupiter
y (rotating)
x (rotating)
Oterma
L_1
L_2
Jupiter
Heteroclinic Connection
y (rotating)
x (rotating)
Boundary of Energy Surface

- Now lets look at two ***movies of the trajectory of comet Oterma***, first in an ***inertial frame*** and then in a frame ***rotating with the Sun and Jupiter***.

Movie: Oterma in inertial frame

Movie: Oterma in a rotating frame

Planar Circular Restricted 3-Body Problem–PCR3BP

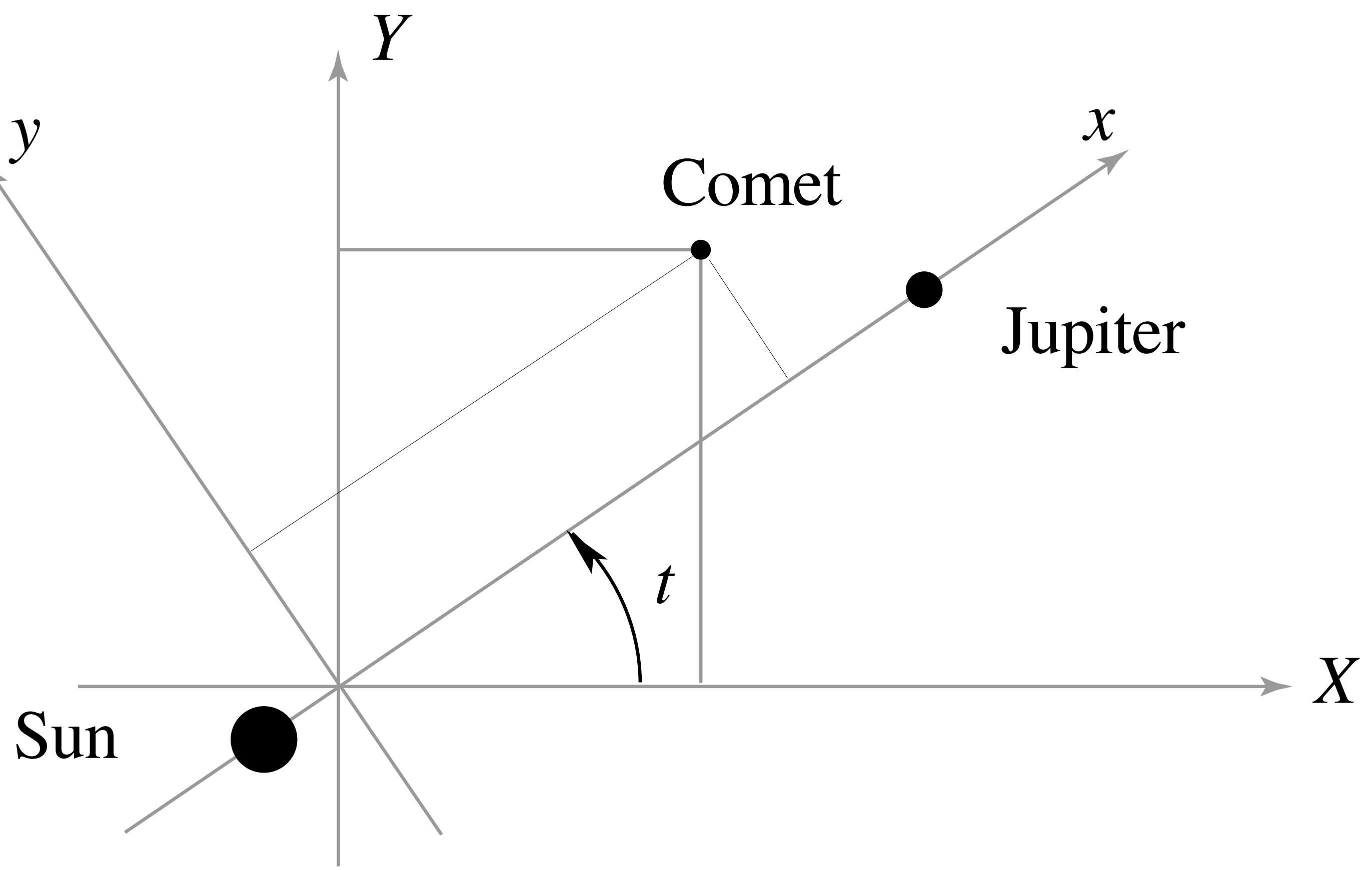
General Comments

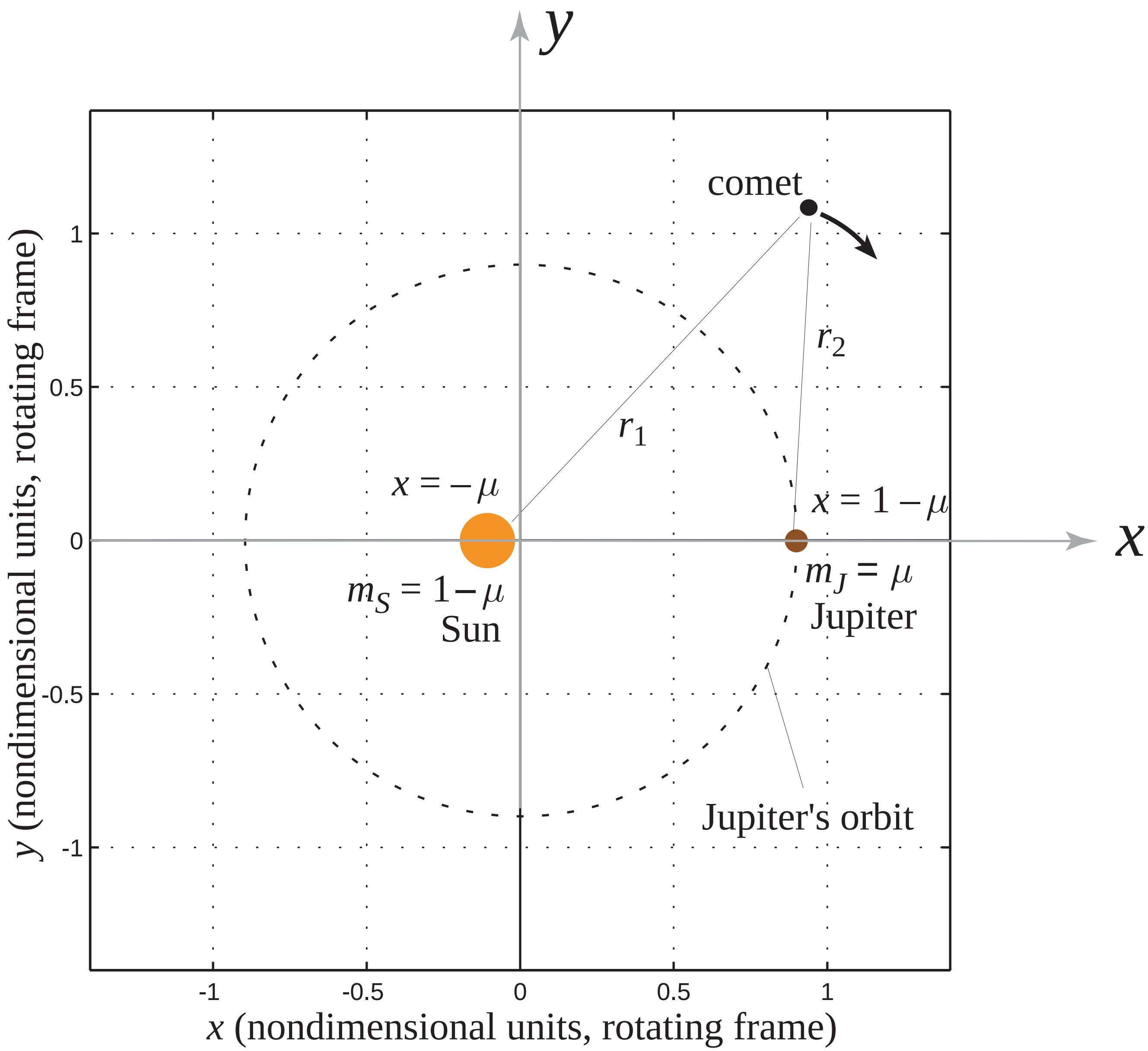
- The two main bodies could be the ***Sun and Jupiter***, or the ***Sun and Earth***, etc. The total mass is normalized to 1; they are denoted $m_S = 1 - \mu$ and $m_J = \mu$, so $0 < \mu \leq \frac{1}{2}$.
  - The two main bodies rotate in the plane in circles counterclockwise about their common center of mass and with angular velocity normalized to 1.
  - The third body, the ***comet or the spacecraft***, has mass zero and is free to move in the plane.
- The ***planar*** restricted three-body problem is used for simplicity. Generalization to the ***three-dimensional problem*** is of course important, but many of the effects can be described well with the planar model.

Equations of Motion

- **Notation:** Choose a ***rotating coordinate system*** so that
  - the origin is at the center of mass
  - the Sun and Jupiter are on the x-axis at the points $(-\mu, 0)$ and $(1-\mu, 0)$ respectively–i.e., the distance from the Sun to Jupiter is normalized to be 1.
  - Let (x, y) be the position of the comet in the plane relative to the positions of the Sun and Jupiter.
  - distances to the Sun and Jupiter:

$$r_1 = \sqrt{(x+\mu)^2 + y^2} \quad \text{and} \quad r_2 = \sqrt{(x-1+\mu)^2 + y^2}.$$


Y
y
x
Comet
Jupiter
t
X
Sun


y
x
comet
r_1
r_2
$x = -\mu$
$x = 1 - \mu$
$m_S = 1 - \mu$
Sun
$m_J = \mu$
Jupiter
Jupiter's orbit
1
0.5
0
-0.5
-1
-1
-0.5
0
0.5
1
x (nondimensional units, rotating frame)
y (nondimensional units, rotating frame)

- **Lagrangian approach–rotating frame**: In the rotating frame, the Lagrangian L is given by

$$L(x, y, \dot{x}, \dot{y}) = \frac{1}{2}((\dot{x} - y)^2 + (x + \dot{y})^2) - U(x, y)$$

where the ***gravitational potential*** in rotating coordinates is

$$U = -\frac{1-\mu}{r_1} - \frac{\mu}{r_2}.$$

Reason:

$$\begin{aligned} \dot{X} &= (\dot{x} - y)\cos t - (x + \dot{y})\sin t, \\ \dot{Y} &= (x + \dot{y})\cos t - (\dot{x} - y)\sin t \end{aligned}$$

which yields kinetic energy (wrt inertial frame)
$\dot{X}^2 + \dot{Y}^2 = (\dot{x} - y)^2 + (x + \dot{y})^2$.

Also, since both the distances r_1 and r_2 are invariant under rotation, we have

$$r_1^2 = (x+\mu)^2 + y^2,$$
$$r_2^2 = (x-(1-\mu))^2 + y^2.$$

- The theory of ***moving systems*** says that one can simply write down the Euler-Lagrange equations in the rotating frame and one will get the correct equations. It is a very efficient general method for computing equations for either moving systems or for systems seen from rotating frames (see Marsden & Ratiu, 1999).

- In the present case, the Euler-Lagrange equations are given by

$$\begin{aligned}\frac{d}{dt}(\dot{x} - y) &= x + \dot{y} - U_x,\\ \frac{d}{dt}(x + \dot{y}) &= -\dot{x} + y - U_y.\end{aligned}$$

After simplification, we have

$$\begin{aligned}\ddot{x} - 2\dot{y} &= -U_x^{\text{eff}},\\ \ddot{y} + 2\dot{x} &= -U_y^{\text{eff}}\end{aligned}$$

where

$$U^{\text{eff}} = -\frac{1}{2}(x^2 + y^2) + U(x, y)$$

is the augmented or ***effective potential*** and the subscripts denote its partial derivatives.

- ***Legendre transform*** to get ***Hamiltonian form***.
- The ***Hamiltonian*** ($\neq K.E. + P.E.$) is

$$H = \frac{1}{2}((p_x + y)^2 + (p_y - x)^2) + U^{\text{eff}}(x, y),$$

- Relationship between ***momenta*** and ***velocities***:

$$\dot{x} = \frac{\partial H}{\partial p_x} = p_x + y; \qquad \dot{y} = \frac{\partial H}{\partial p_y} = p_y - x.$$

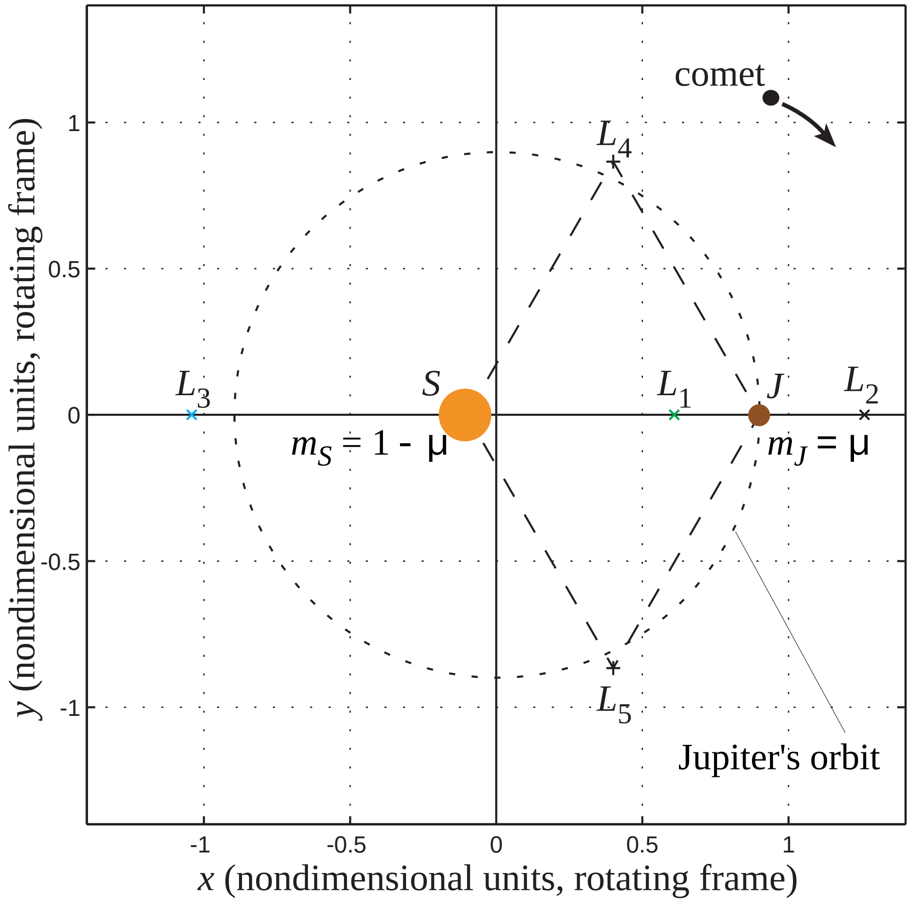
- ***Remaining dynamical equations***:

$$\dot{p}_x = -\frac{\partial H}{\partial x} = p_y - x - \frac{\partial U^{\text{eff}}}{\partial x},$$
$$\dot{p}_y = -\frac{\partial H}{\partial y} = -p_x - y - \frac{\partial U^{\text{eff}}}{\partial y}.$$

- Since equations of motion of PCR3BP are Hamiltonian and autonomous, they have an ***energy integral*** of motion, denoted E.

Five Equilibrium Points

- Three ***collinear*** (Euler, 1767) on the x-axis— L_1, L_2, L_3
- Two ***equilateral points*** (Lagrange, 1772)— L_4, L_5.



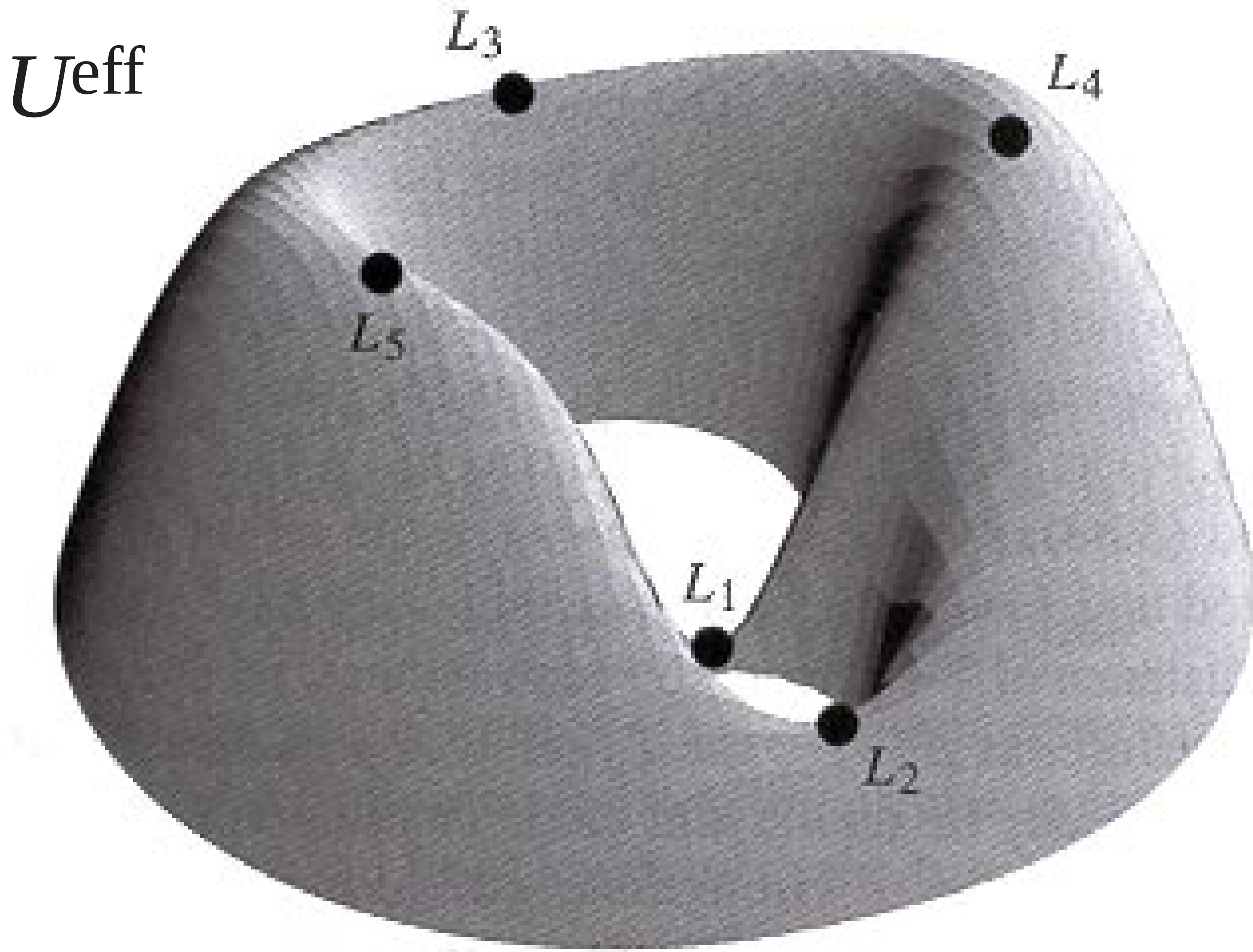
Energy Manifold

- The energy E is given by

$$\begin{aligned} E(x,y,\dot{x},\dot{y}) &= \frac{1}{2}(\dot{x}^2+\dot{y}^2)+U^{\text{eff}}(x,y) \\ &= \frac{1}{2}(\dot{x}^2+\dot{y}^2)-\frac{1}{2}(x^2+y^2)-\frac{1-\mu}{r_1}-\frac{\mu}{r_2}. \end{aligned}$$

  This energy integral will help us determine the region of ***possible motion***, i.e., the region in which the comet can possibly move along and the region which it is forbidden to move. The first step is to look at the surface of the effective potential U^{eff}.

- Note that the energy manifold is 3-dimensional.


U^{eff}
L_3
L_4
L_5
L_1
L_2

- Near either the Sun or Jupiter, we have a potential well.
- Far away from the Sun-Jupiter system, the term that corresponds to the centrifugal force dominates, we have another potential well.
- Moreover, by applying multivariable calculus, one finds that there are 3 saddle points at L_1, L_2, L_3 and 2 maxima at L_4 and L_5.
- Let E_i be the energy at L_i, then $E_5 = E_4 > E_3 > E_2 > E_1$.

- Let $\mathcal{M}$ be the ***energy surface*** given by setting the energy integral equal to a constant, i.e.,

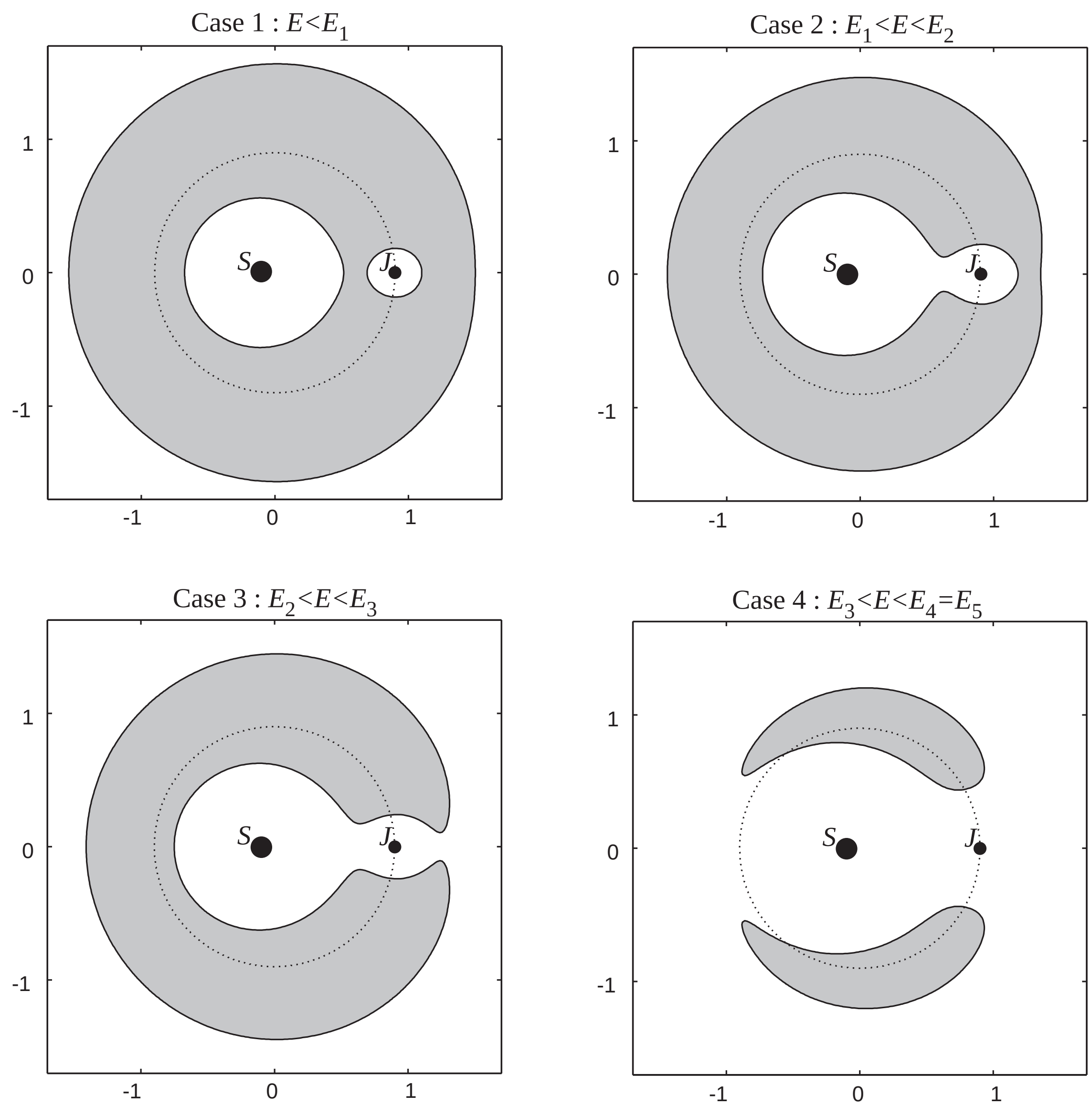
$$\mathcal{M}(\mu, e) = \{(x, y, \dot{x}, \dot{y}) \mid E(x, y, \dot{x}, \dot{y}) = e\} \tag{1}$$

where e is a constant.

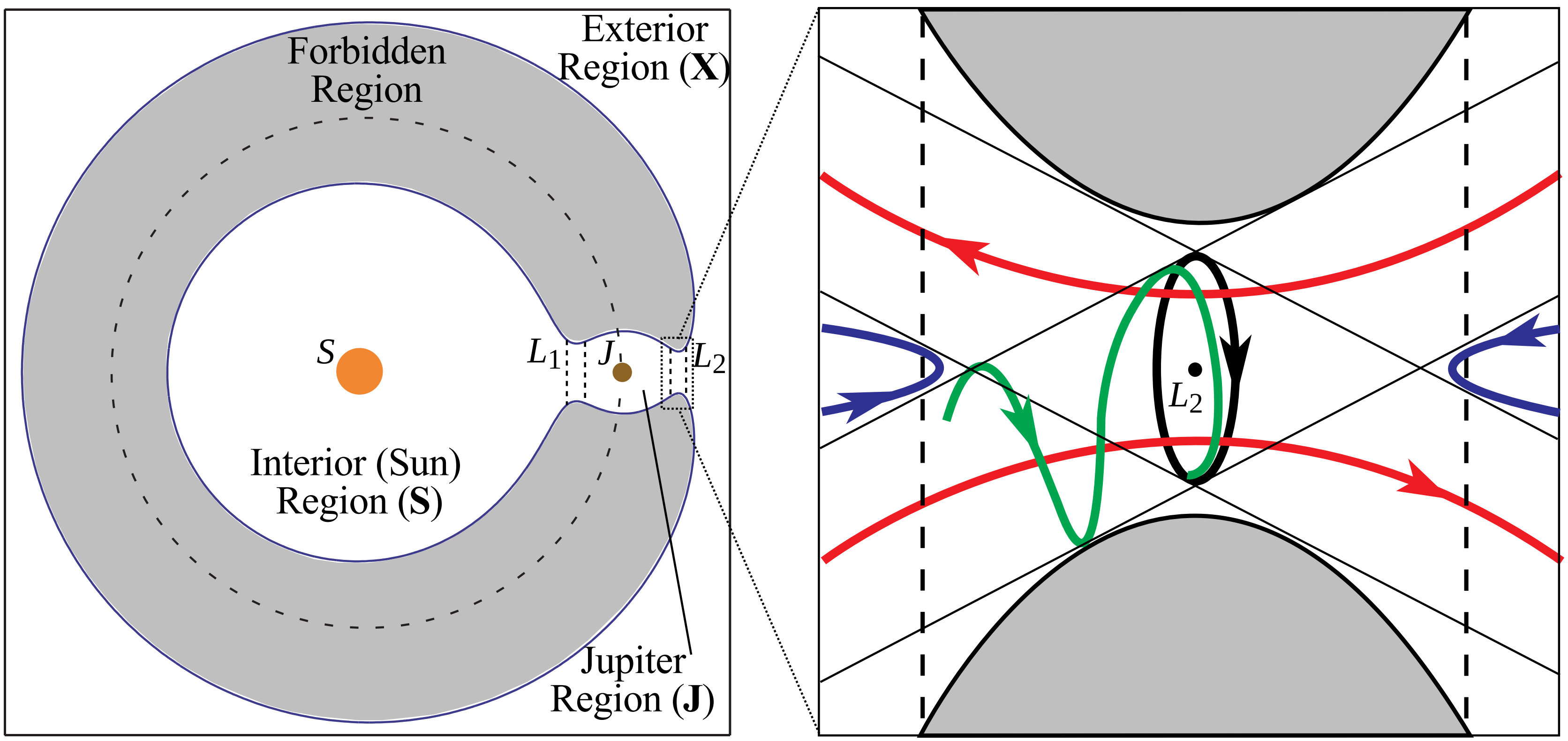
- The projection of this surface onto position space is called a ***Hill's region***

$$M(\mu, e) = \{(x, y) \mid U^{\text{eff}}(x, y) \leq e\}. \tag{2}$$

The boundary of $M(\mu, e)$ is the ***zero velocity curve***. The comet can move only within this region in the (x, y)-plane. For a given μ there are five basic configurations for the Hill's region, the first four of which are shown in the following figure.


Case 1 : $E<E_1$
Case 2 : $E_1<E<E_2$
Case 3 : $E_2<E<E_3$
Case 4 : $E_3<E<E_4=E_5$
S
J
1
0
-1

- [Conley] Orbits with energy just above that of L_2 can be **transit orbits**, passing through the neck region between the ***exterior region*** (outside Jupiter's orbit) and the ***temporary capture region*** (bubble surrounding Jupiter). They can also be **non-transit orbits** or **asymptotic orbits**.



The Flow near L_1 and L_2: Linearization

- [Moser] All the qualitative results of the linearized equations carry over to the full nonlinear equations.
- Recall equations of PCR3BP:

$$\begin{aligned} \dot{x} &= v_x, \qquad \dot{v}_x = 2v_y - U_x^{\text{eff}}, \\ \dot{y} &= v_y, \qquad \dot{v}_y = -2v_x - U_y^{\text{eff}}. \end{aligned}$$

- After linearization,

$$\begin{aligned} \dot{x} &= v_x, \qquad \dot{v}_x = 2v_y + ax, \\ \dot{y} &= v_y, \qquad \dot{v}_y = -2v_x - by. \end{aligned}$$

  - Eigenvalues have the form $\pm\lambda$ and $\pm i\nu$.

- Corresponding eigenvectors are

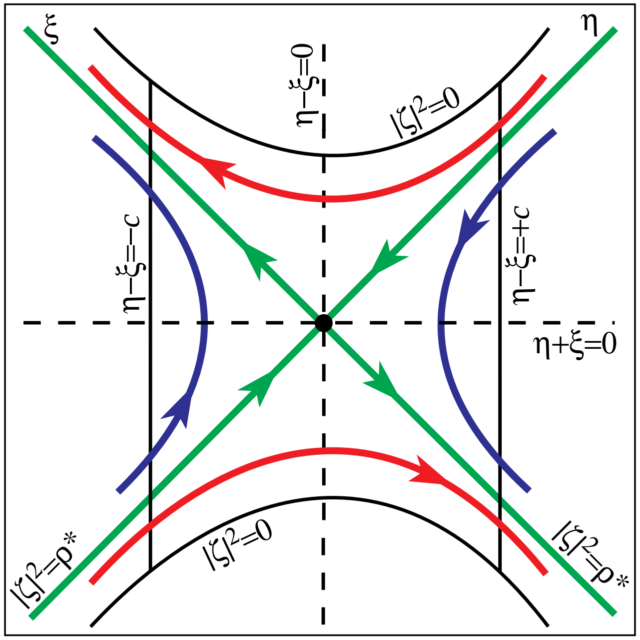
$$\begin{aligned} u_1 &= (1, -\sigma, \lambda, -\lambda\sigma), \\ u_2 &= (1, \sigma, -\lambda, -\lambda\sigma), \\ w_1 &= (1, -i\tau, i\nu, \nu\tau), \\ w_2 &= (1, i\tau, -i\nu, \nu\tau). \end{aligned}$$

- After ***linearization*** and making the ***eigenvectors*** the new coordinate axes, the equations of motion assume the simple form

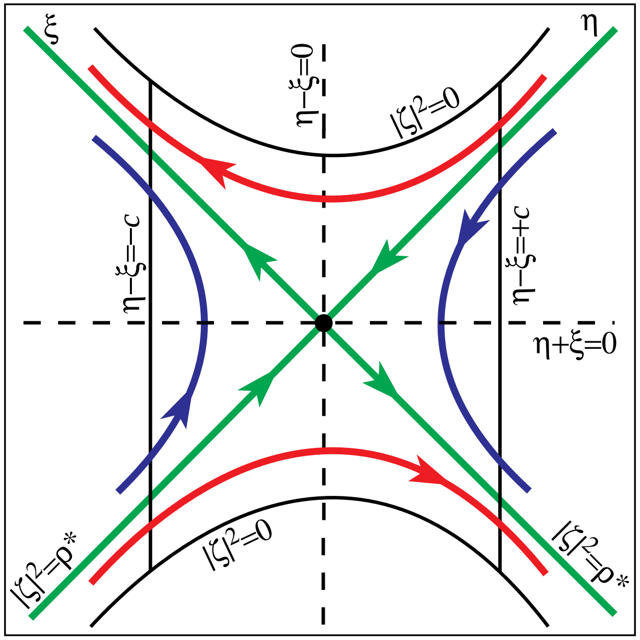
$$\dot{\xi} = \lambda\xi, \qquad \dot{\eta} = -\lambda\eta, \qquad \dot{\zeta}_1 = \nu\zeta_2, \qquad \dot{\zeta}_2 = -\nu\zeta_1,$$

with ***energy function*** $E_l = \lambda\eta\xi + \frac{\nu}{2}(\zeta_1^2 + \zeta_2^2)$.

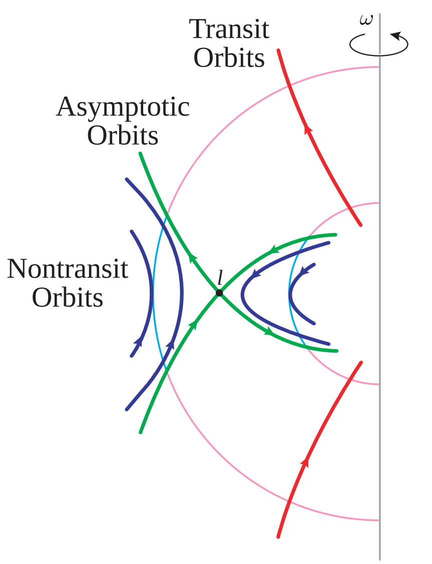
- The ***flow*** near L_1, L_2 has the form of a ***saddle***$\times$***center***.



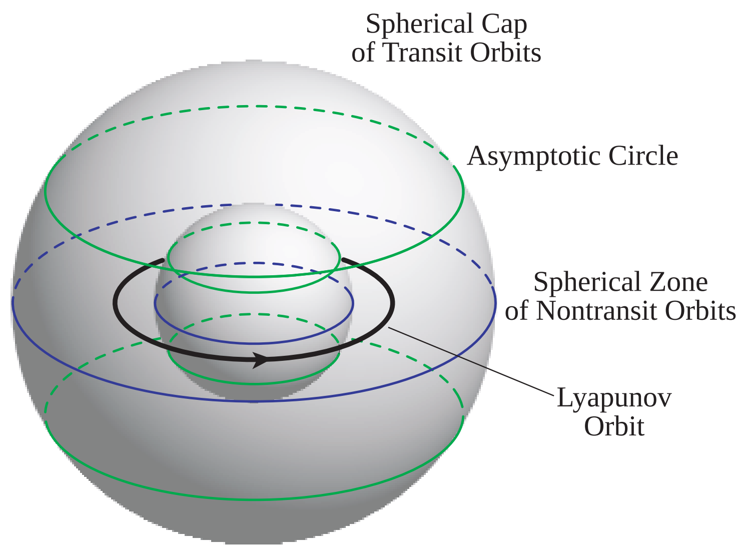
- For each fixed value of $\eta - \xi$ (vertical lines in figure below), $E_l = \mathcal{E}$ describes a ***2-sphere***.
- The ***equilibrium region*** $\mathcal{R}$ on the 3D energy manifold is homeomorphic to $S^2 \times I$.



- [McGehee] Can visualize **4 types** of orbits in $\mathcal{R} \simeq S^2 \times I$.
  - **Black** circle is the unstable **periodic** Lyapunov orbit.
  - 4 cylinders of **asymptotic** orbits form pieces of stable and unstable manifolds. They intersect the bounding spheres at asymptotic circles, separating ***spherical polar caps***, which contain **transit** orbits, from ***spherical equatorial zones***, which contain **nontransit** orbits.

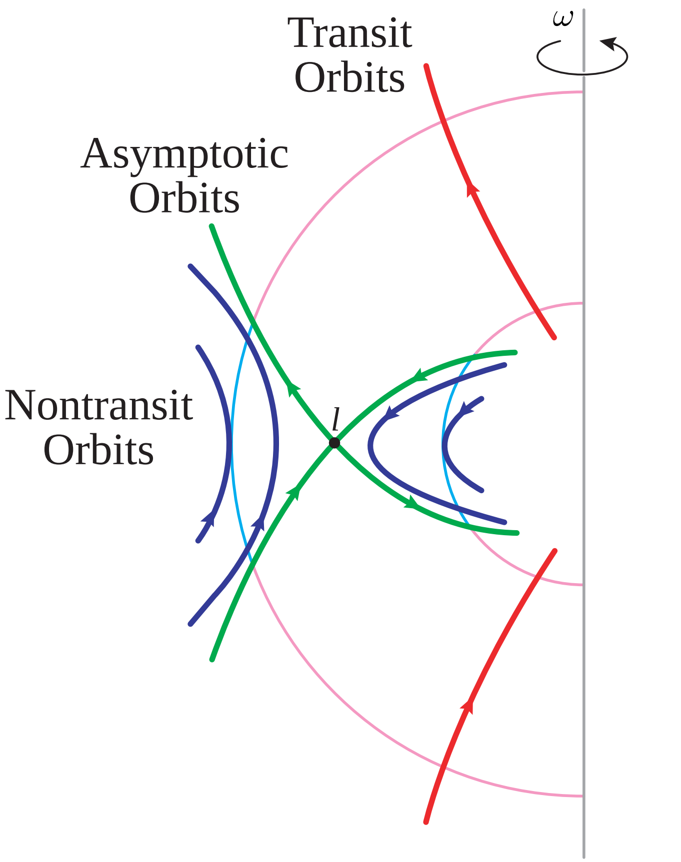


Cross-section of Equilibrium Region

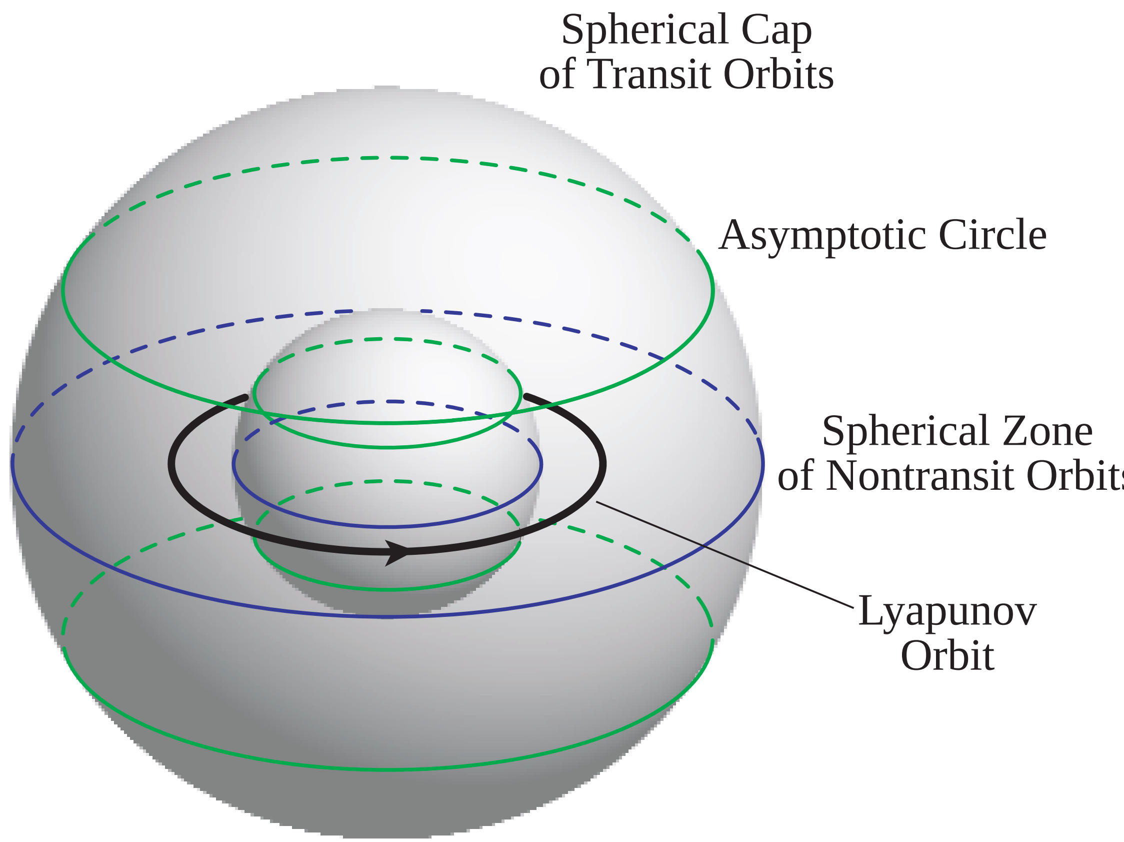


Equilibrium Region

- Roughly speaking, for fixed energy, the equilibrium region has the dynamics of a ***saddle*** $\times$ ***harmonic oscillator***.

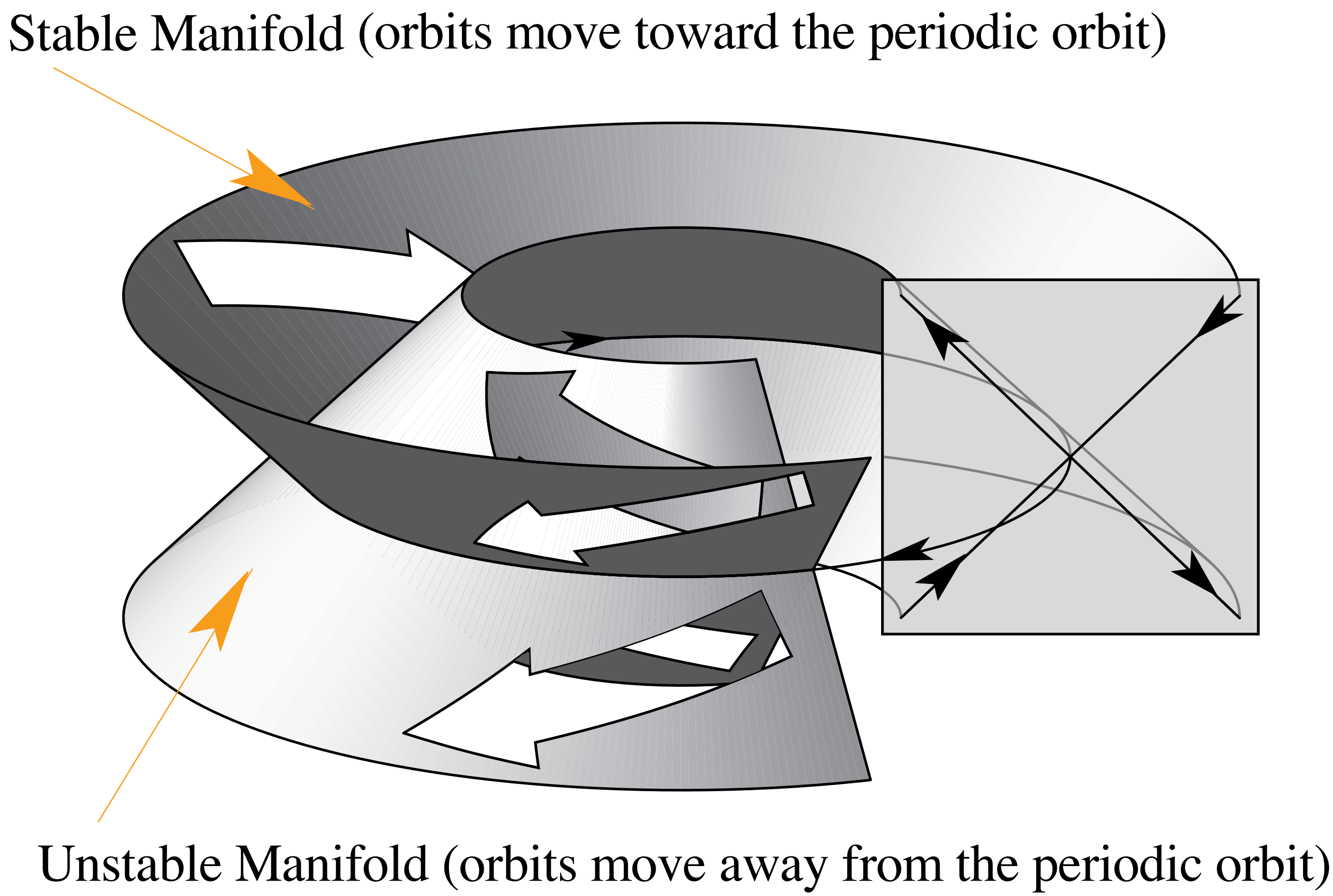


Cross-section of Equilibrium Region



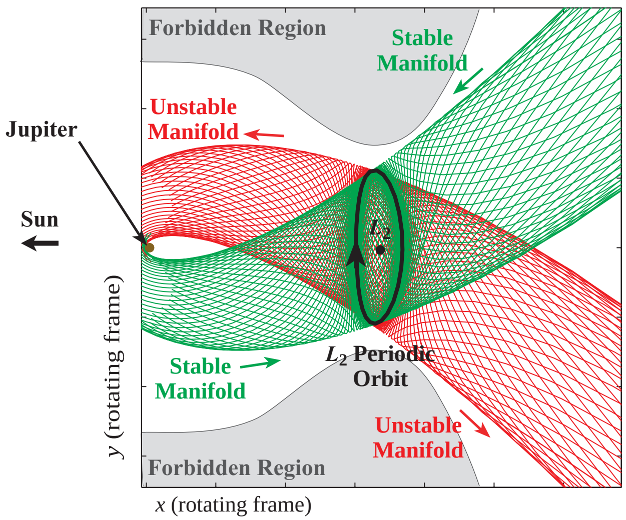
Equilibrium Region

- 4 cylinders of asymptotic orbits: ***stable and unstable manifolds***.

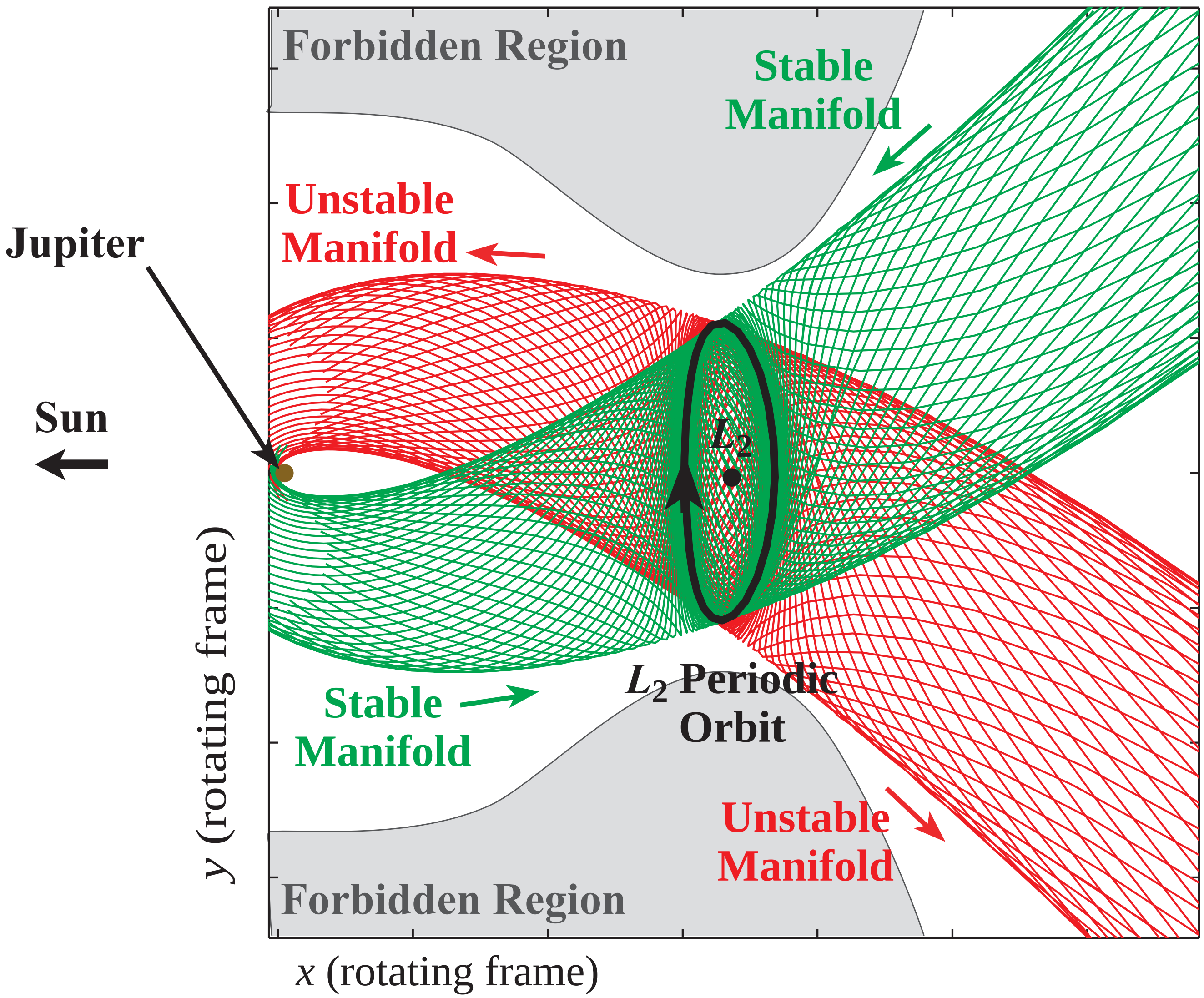


Invariant Manifold Tubes Partition the Energy Surface

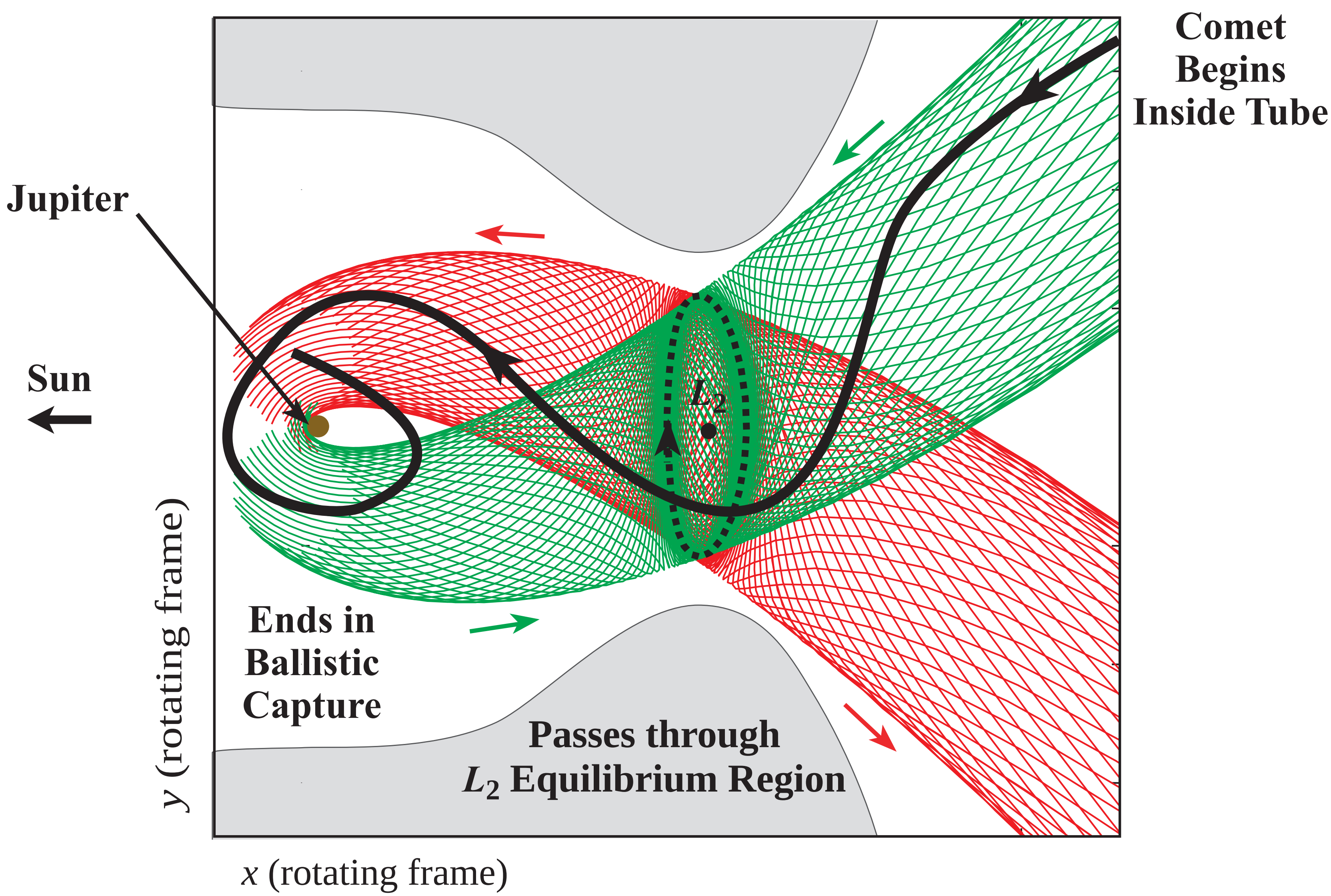
- **Stable** and **unstable** ***manifold tubes*** act as ***separatrices*** for the flow in the equilibrium region.
  - Those inside the tubes are ***transit*** orbits.
  - Those outside the tubes are ***nontransit*** orbits.
  - e.g., transit from outside Jupiter's orbit to Jupiter capture region possible ***only*** through L_2 periodic orbit stable tube.



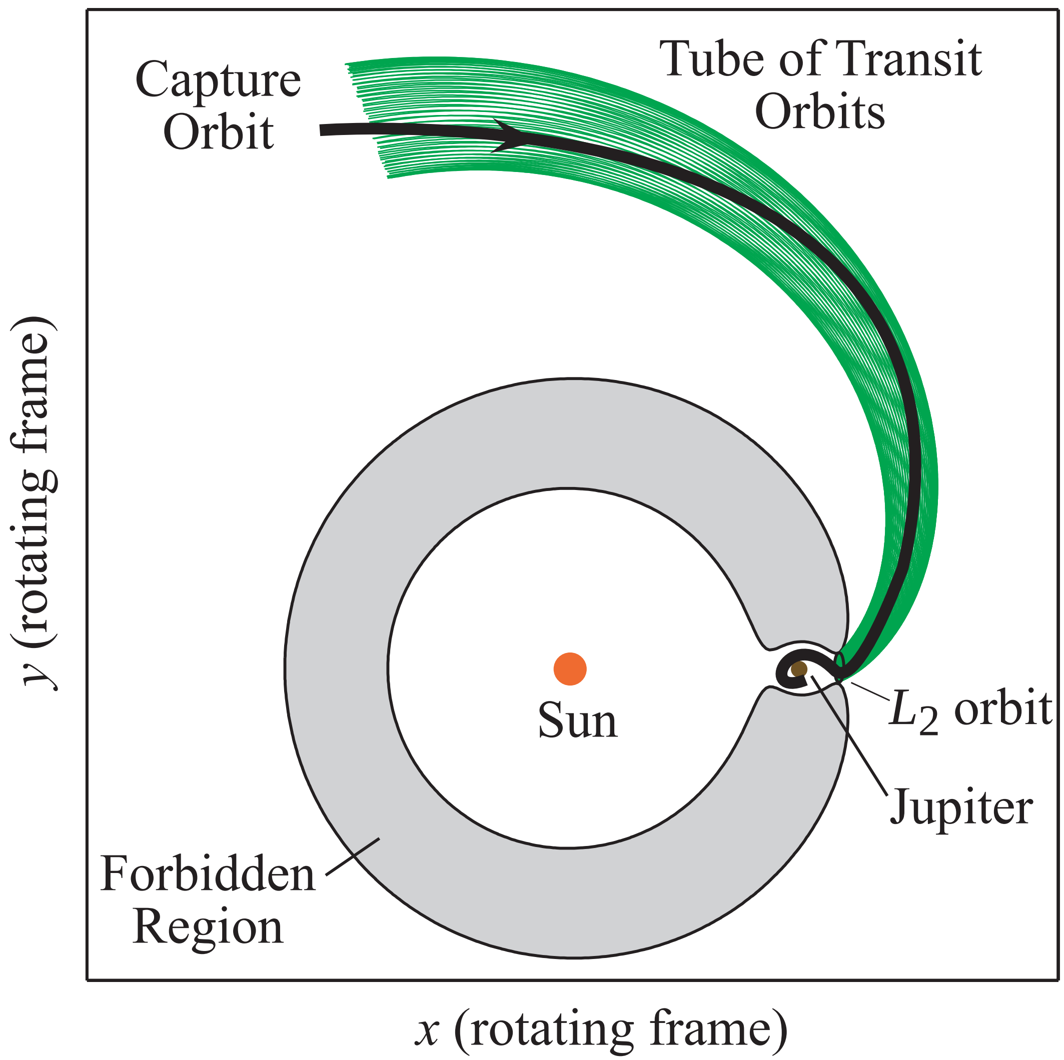
- **Stable** and **unstable** manifold tubes control the ***transport*** of material to and from the capture region.



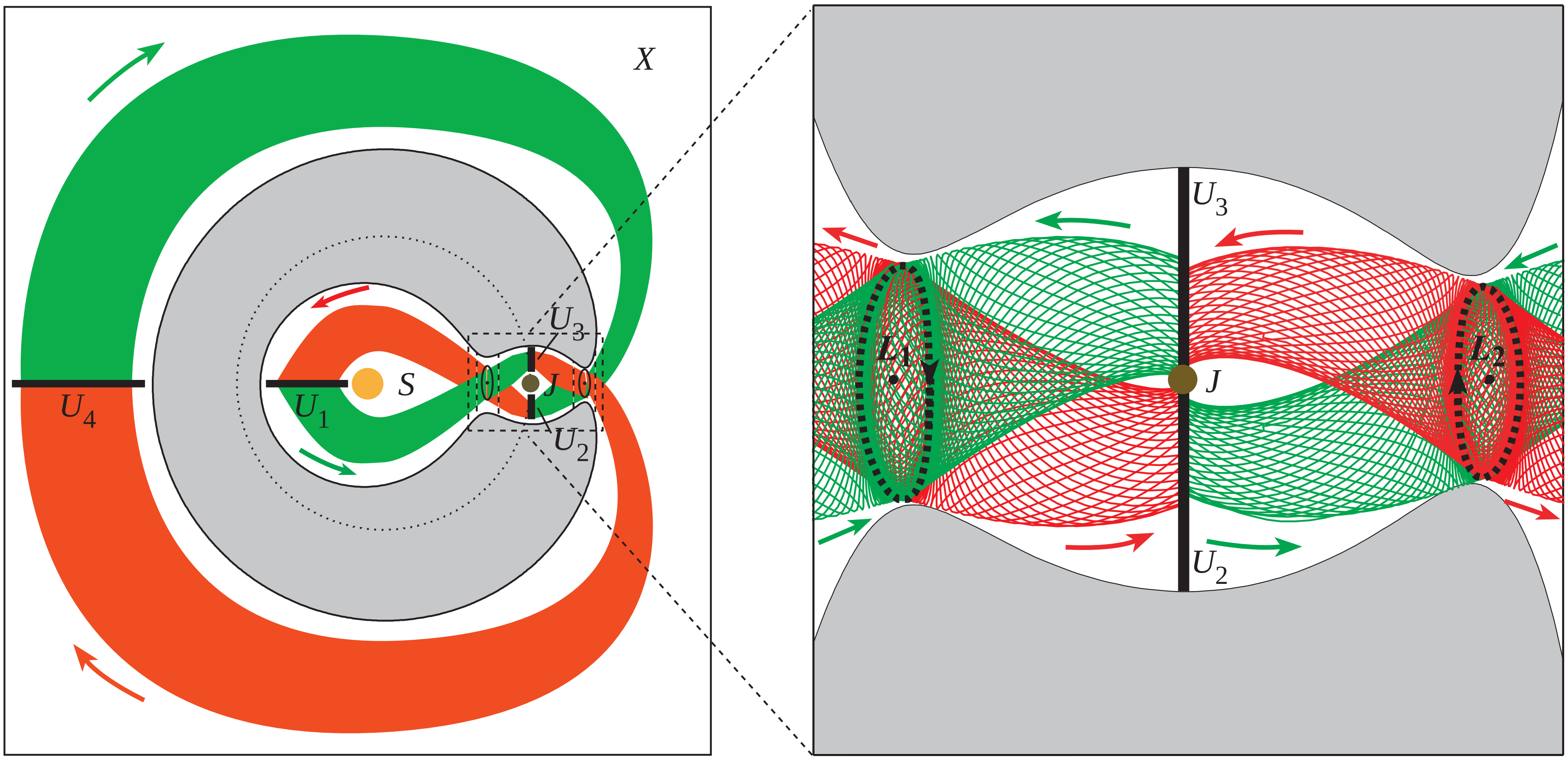
- Tubes of transit orbits contain ***ballistic capture*** orbits.



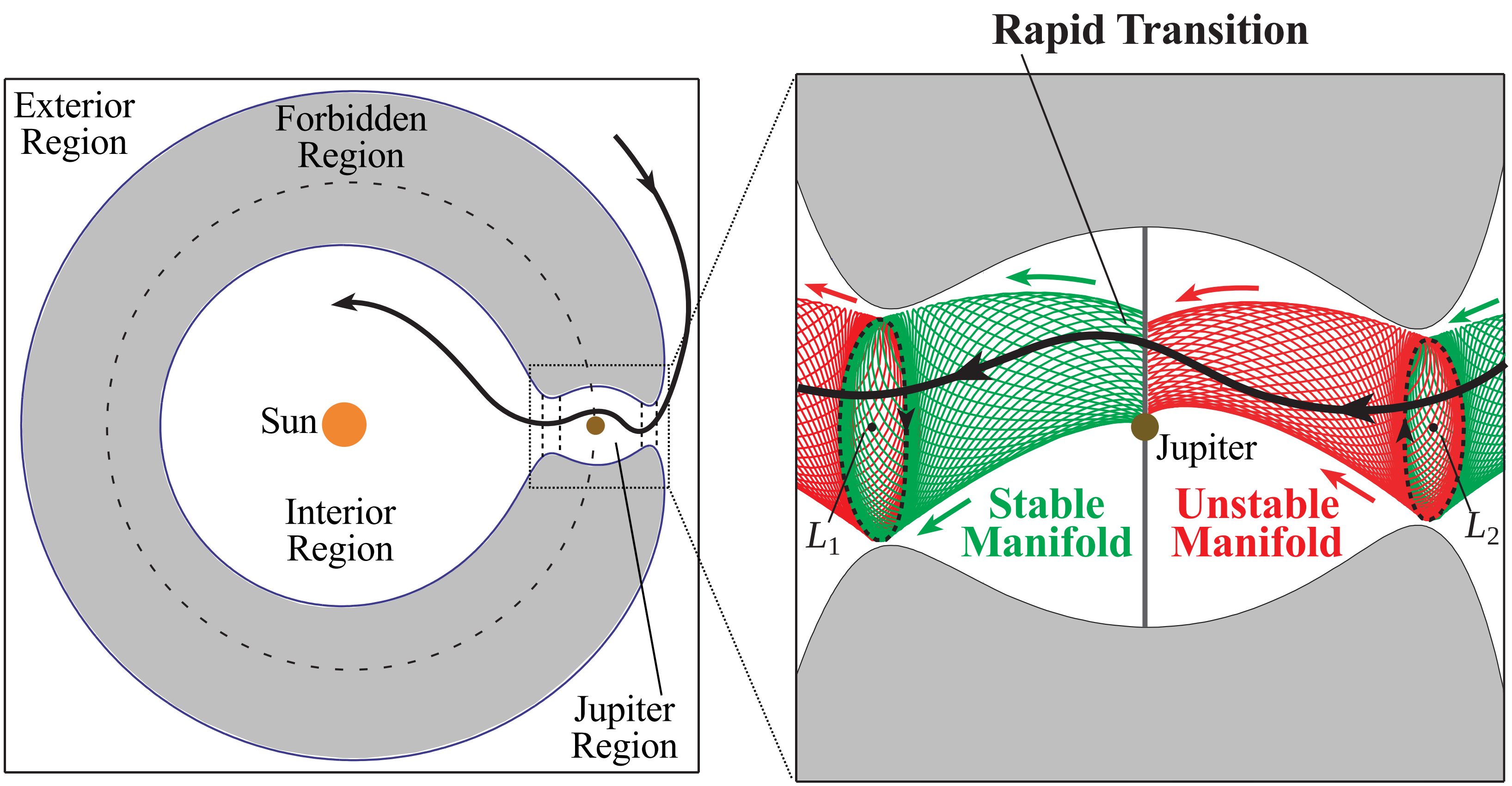
- Invariant manifold tubes are ***global objects*** — extend far beyond vicinity of libration points.



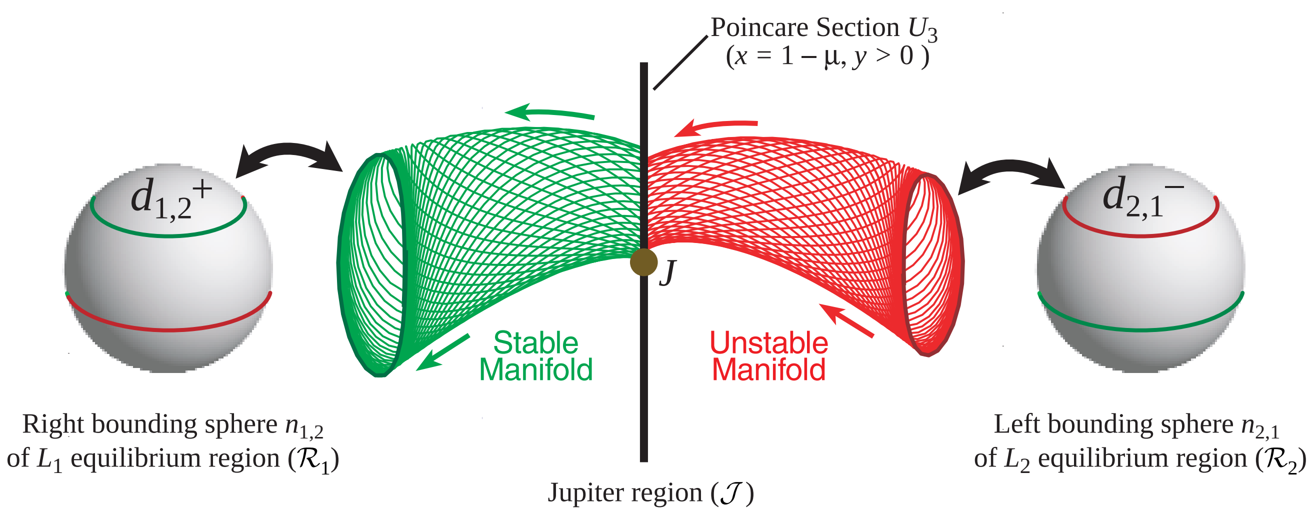
- Transport between all three regions (interior, Jupiter, exterior) is controlled by the intersection of stable and unstable manifold tubes.



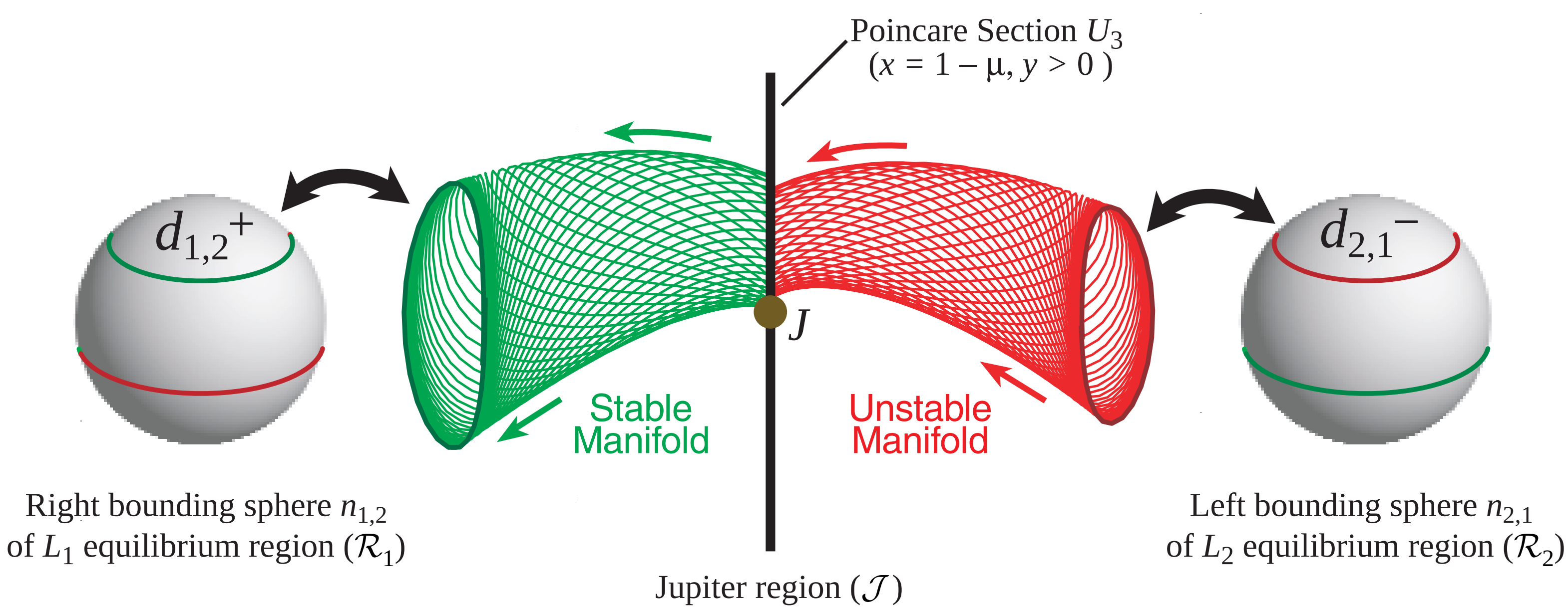
- In particular, rapid transport between outside and inside of Jupiter's orbit is possible.

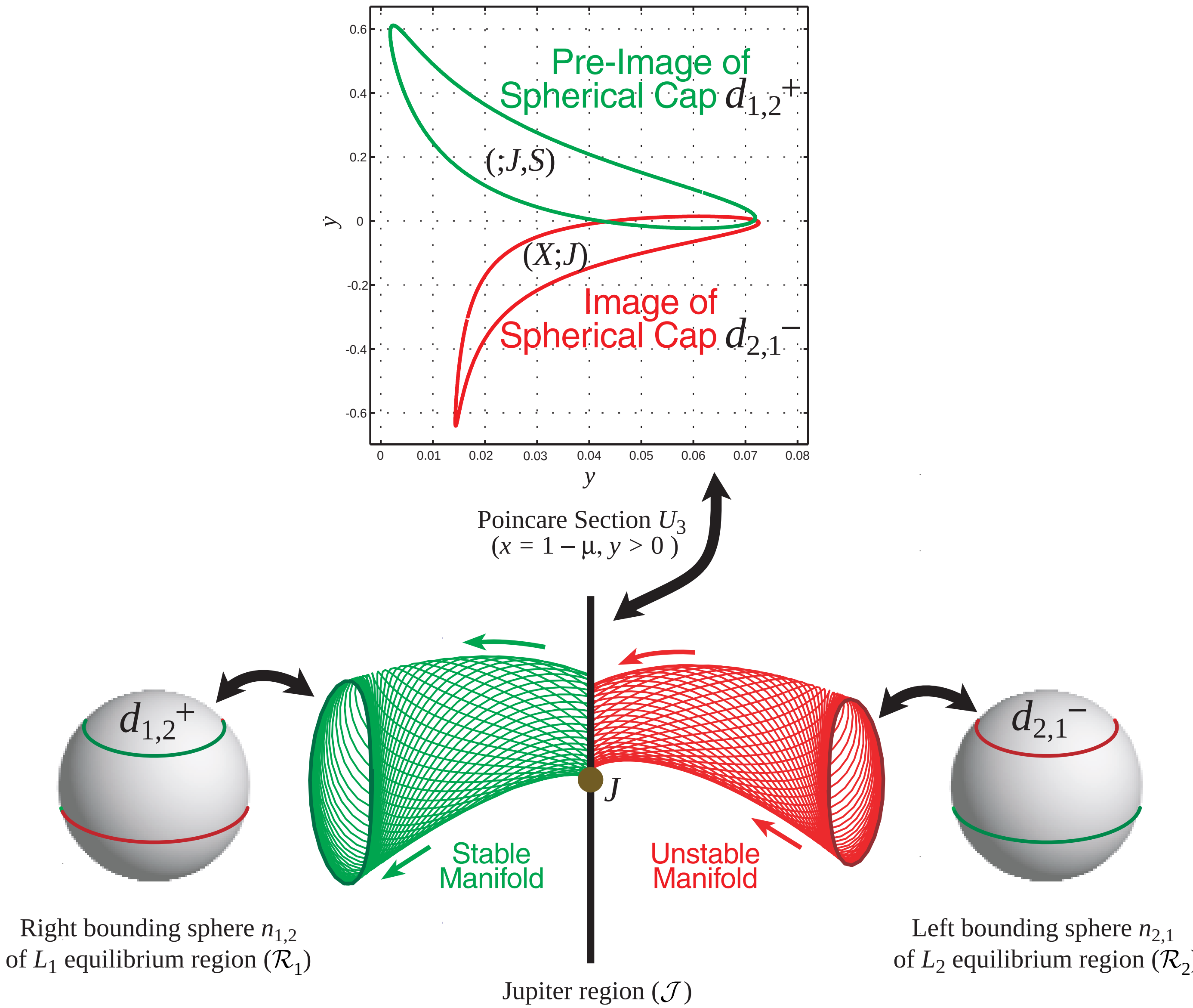


- This can be seen by recalling the bounding spheres for the equilibrium regions.
- We will look at the images and pre-images of the **spherical caps** of **transit orbits** on a suitable Poincaré section.
  - The images and pre-images of the spherical caps form the tubes that partition the energy surface.



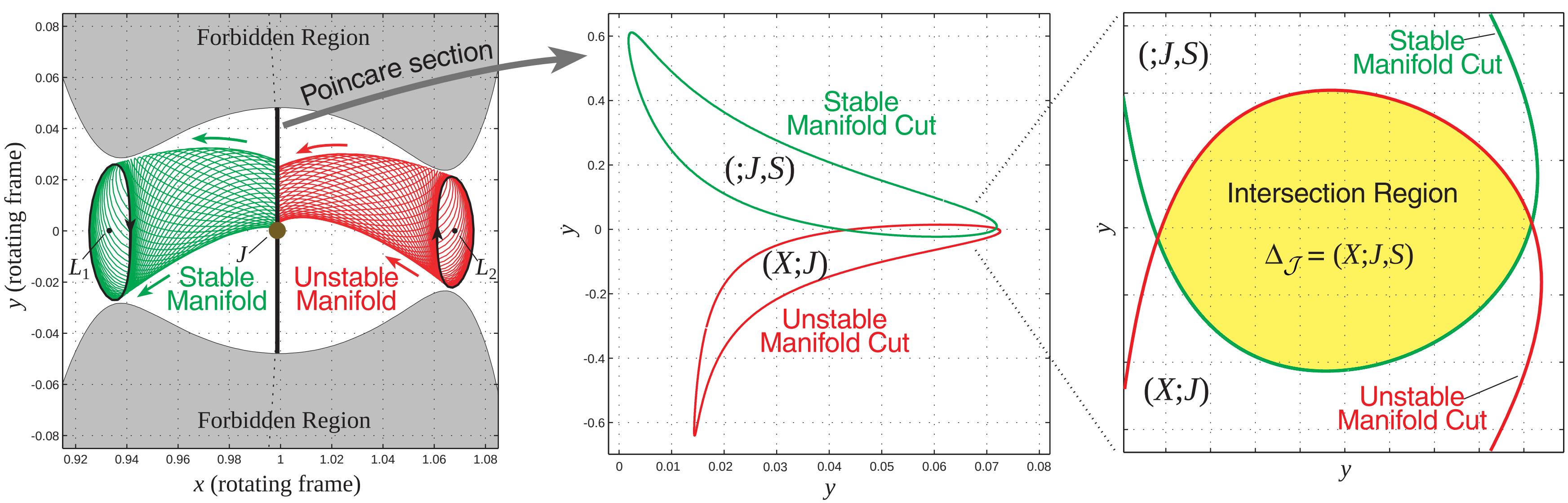
- For instance, on a Poincaré section between L_1 and L_2,
  - We look at the **image of the cap** on the left bounding sphere of the L_2 equilibrium region $\mathcal{R}_2$ containing **orbits leaving** $\mathcal{R}_2$.
  - We also look at the **pre-image of the cap** on the right bounding sphere of $\mathcal{R}_1$ containing **orbits entering** $\mathcal{R}_1$.



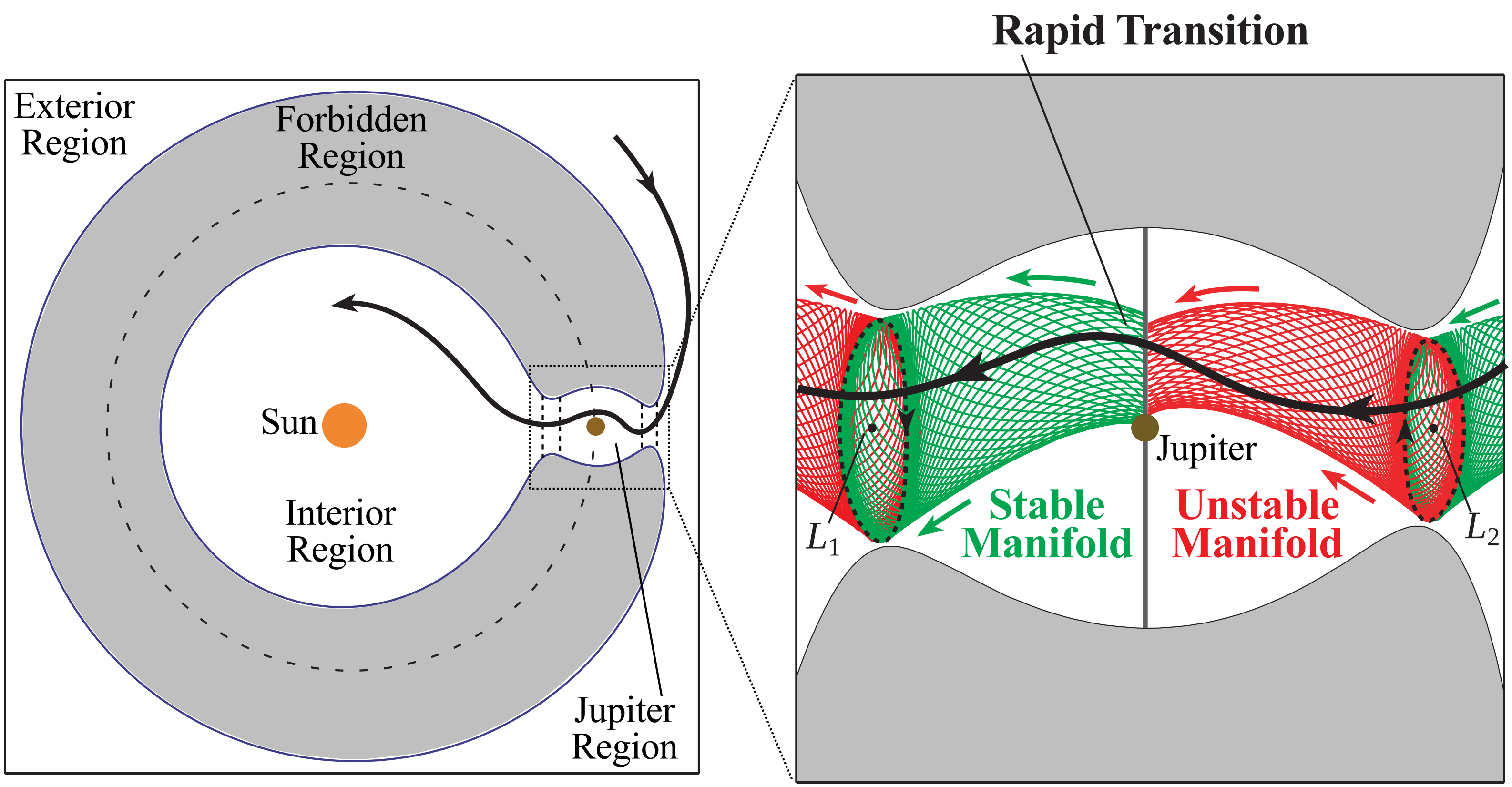


0.6
0.4
0.2
0
-0.2
-0.4
-0.6
y
0
0.01
0.02
0.03
0.04
0.05
0.06
0.07
0.08
y
Pre-Image of
Spherical Cap $d_{1,2}^+$
$(;J,S)$
$(X;J)$
Image of
Spherical Cap $d_{2,1}^-$
Poincare Section U_3
$(x = 1 - \mu, y > 0)$
$d_{1,2}^+$
$d_{2,1}^-$
J
Stable
Manifold
Unstable
Manifold
Right bounding sphere $n_{1,2}$
of L_1 equilibrium region ($\mathcal{R}_1$)
Jupiter region ($\mathcal{J}$)
Left bounding sphere $n_{2,1}$
of L_2 equilibrium region ($\mathcal{R}_2$)

- The Poincaré cut of the **unstable manifold** of the L_2 periodic orbit forms the boundary of the **image of the cap** containing transit orbits leaving $\mathcal{R}_2$.
  - All of these orbits came from the exterior region and are now in the Jupiter region, so we label this region $(\mathbf{X};\mathbf{J})$. Etc.
  - The dynamics of the invariant manifold tubes naturally suggest the ***itinerary*** representation.

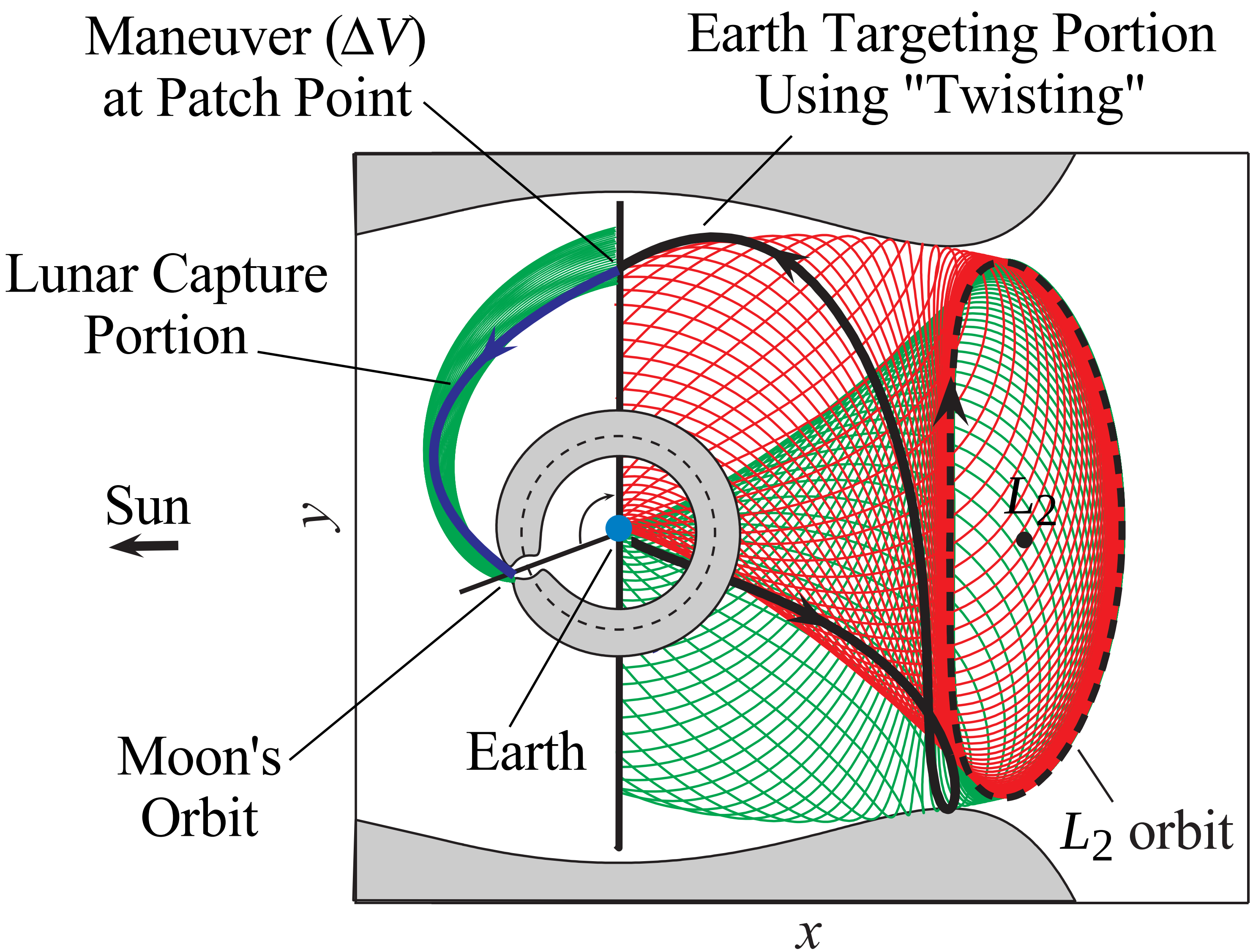


- Integrating an initial condition in the intersection region would give us an orbit with the desired itinerary $(\mathbf{X}, \mathbf{J}, \mathbf{S})$.



Lunar Capture: How to get to the Moon Cheaply

- Using the invariant manifold tubes as the building blocks, we can construct interesting, fuel saving space mission trajectories.
  - For instance, an Earth-to-Moon ballistic capture orbit.
  - Uses Sun's perturbation.
  - Jump from Sun-Earth-S/C system to Earth-Moon-S/C system.
  - Saves about 20% of onboard fuel compared to Apollo-like transfer.



Maneuver (ΔV) at Patch Point
Earth Targeting Portion Using "Twisting"
Lunar Capture Portion
Sun
y
L2
Moon's Orbit
Earth
L2 orbit
x

Movie: Shoot the Moon in rotating frame

Movie: Shoot the Moon in inertial frame

Future Work and Directions

- For a single 3-body system:
  - When is 3-body effect more important than 2-body?
  - Find "sweet spot" within tubes where transport is most efficient/fastest?
  - Consider continuous low-thrust control, optimal control.
- For coupling multiple 3-body systems:
  - Where to jump from one 3-body system to another?
  - Optimal control: trade off between travel time and fuel.
- Planetary science/astronomy applications:
  - Statistics: transport rates, capture probabilities, etc.
- How general is this method?
  - Is similar behavior seen in other systems?

Ionization of Rydberg Atoms in External Fields

- Similar behavior seen in the motion of loosely bound electrons in the presence of external fields (Jaffé, Farrelly & Uzer, 1999).
- ***Rydberg electrons*** are very weakly bound, residing an immense distance from the atomic core. They live in the poorly charted territory where quantum physics transforms into classical physics.
- Experiments reveal that the ***planar*** motion of the electron in the presence of crossed magnetic and electric fields is the most important.
- Crossed field situations exist in diverse areas of physics ranging from excitonic systems to plasmas and neutron stars.
- When the electric field is comparable to the atomic Coulomb field sensed by the Rydberg electron, interesting dynamical properties can be studied.

- The dynamics are similar to that of comets which are weakly bound to Jupiter, i.e., the dynamics of the planar 3-body problem where the invariant manifold tubes control the transport.
- An amazing ***20 orders of magnitude*** separate the length scales of these two similar phenomena! (atomic radius compared with solar system)

Chemical Physics Approach: Transition State Theory

- Once an electron is "activated" into an initial excitation state, energy flows into the ionization channel and the electron is detached.
- A central question concerns the ***rate*** at which energy migrates into the ionizing mode.
- The key to describing this or any chemical reaction is the recognition of the ***importance of phase space structures*** (bottlenecks, turnstiles, etc.) that govern the progress of the reaction.
- Chemists think in terms of the "transition state" – a minimal set of states that all reactive trajectories must pass through and which is never encountered by any nonreactive trajectories (Marcelin, 1915).

- The ***transition state*** has been viewed as a saddle between two valleys, one associated with the products, the other with the reactants (Eyring and Polanyi, 1931).
- It was shown that the transition state must be an unstable periodic orbit whose projection connects two branches of the equipotential – a periodic orbit dividing surface or PODS (Pechukas, 1976).
- The partitioning of phase space can be accomplished using the manifolds of the PODS (Davis et al., 1980s).
- Jaffé, Farrelly & Uzer, 1999 found the first actual example of a PODS occurring in phase space, which is not a dividing surface when projected into configuration space – Rydberg atoms in the presence of crossed fields.

- The Hamiltonian (in nondimensional form) for the planar hydrogen atom in crossed magnetic and electric fields in Cartesion coordinates is

$$H = \frac{1}{2}(p_x^2 + p_y^2) - \frac{1}{r} + \frac{1}{2}(xp_y - yp_x) + \frac{1}{8}(x^2 + y^2) - \epsilon x$$

where $r = \sqrt{x^2 + y^2}$ is the distance of the electron from the atomic core and ϵ is the electric field strength.

Note: compare the above with Hamiltonian for PCR3BP,

$$H = \frac{1}{2}(p_x^2 + p_y^2) - \frac{1-\mu}{r_1} - \frac{\mu}{r_2} - (xp_y - yp_x).$$

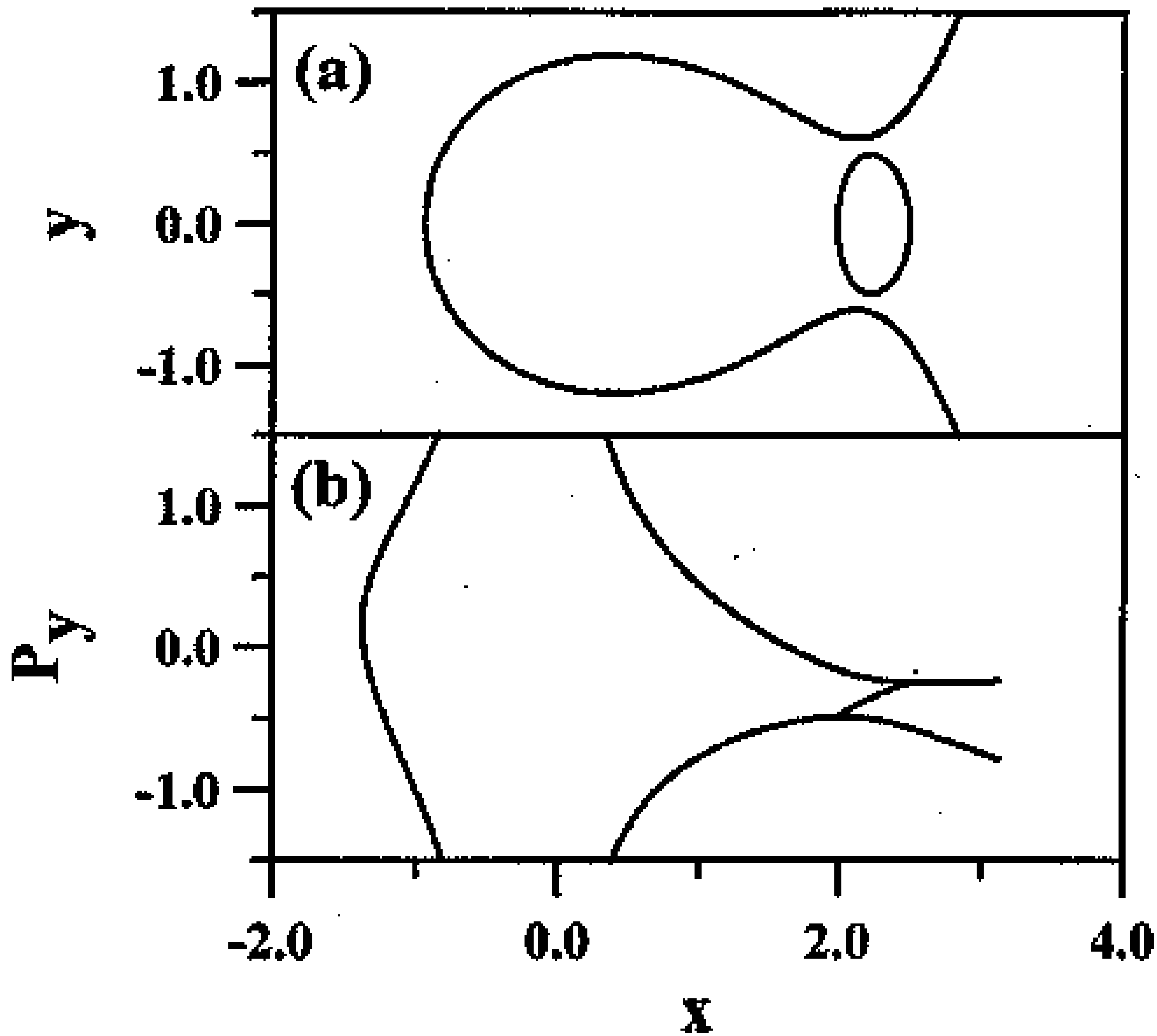
- The Hamiltonian has a single critical point, the ***Stark saddle point***, above which ionization becomes possible. Unstable periodic orbits exist around this point.

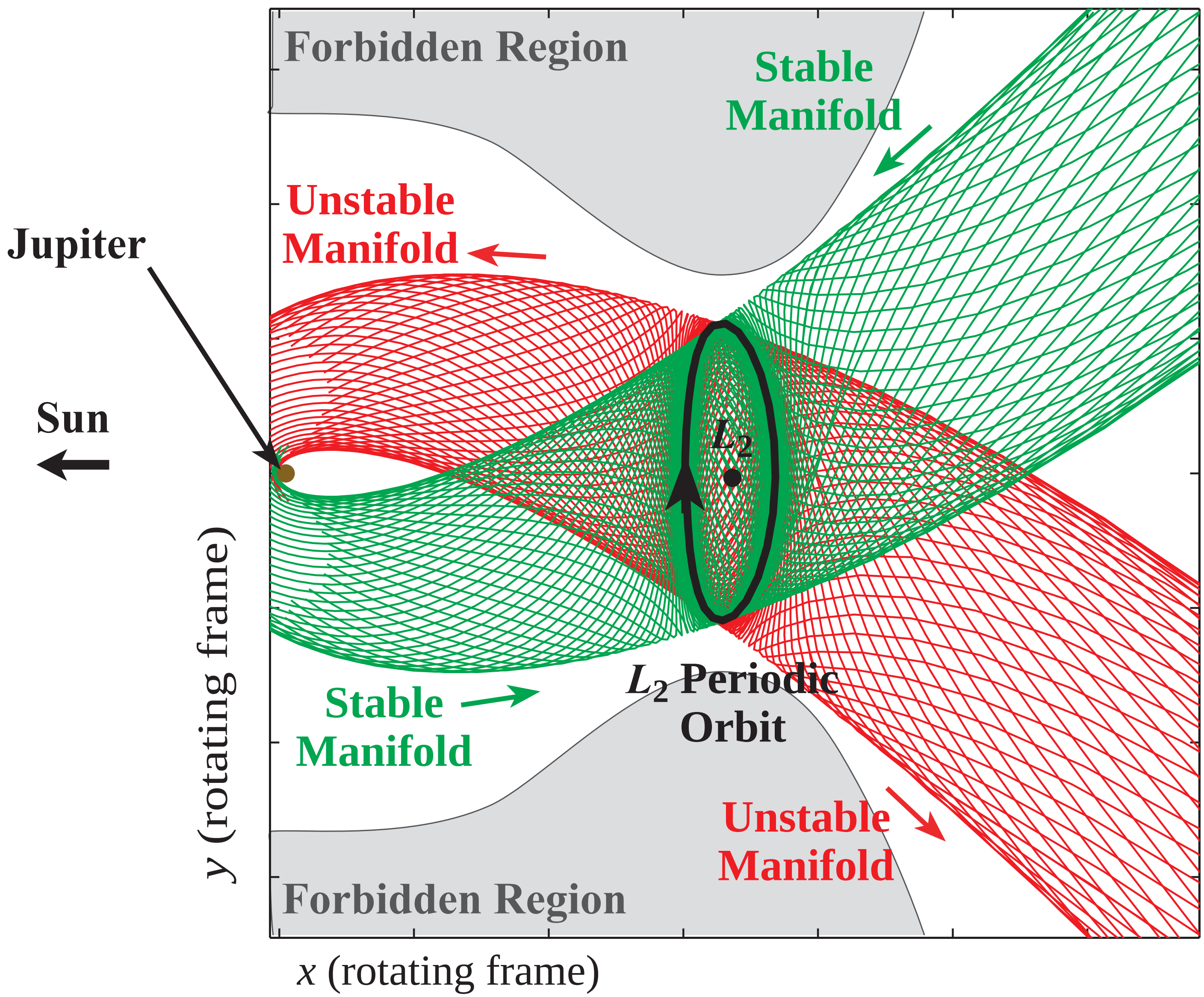
- The Hamiltonian gives rise to equations of motion which are not symmetric with respect to ***time-reversal***, but under the canonical transformation

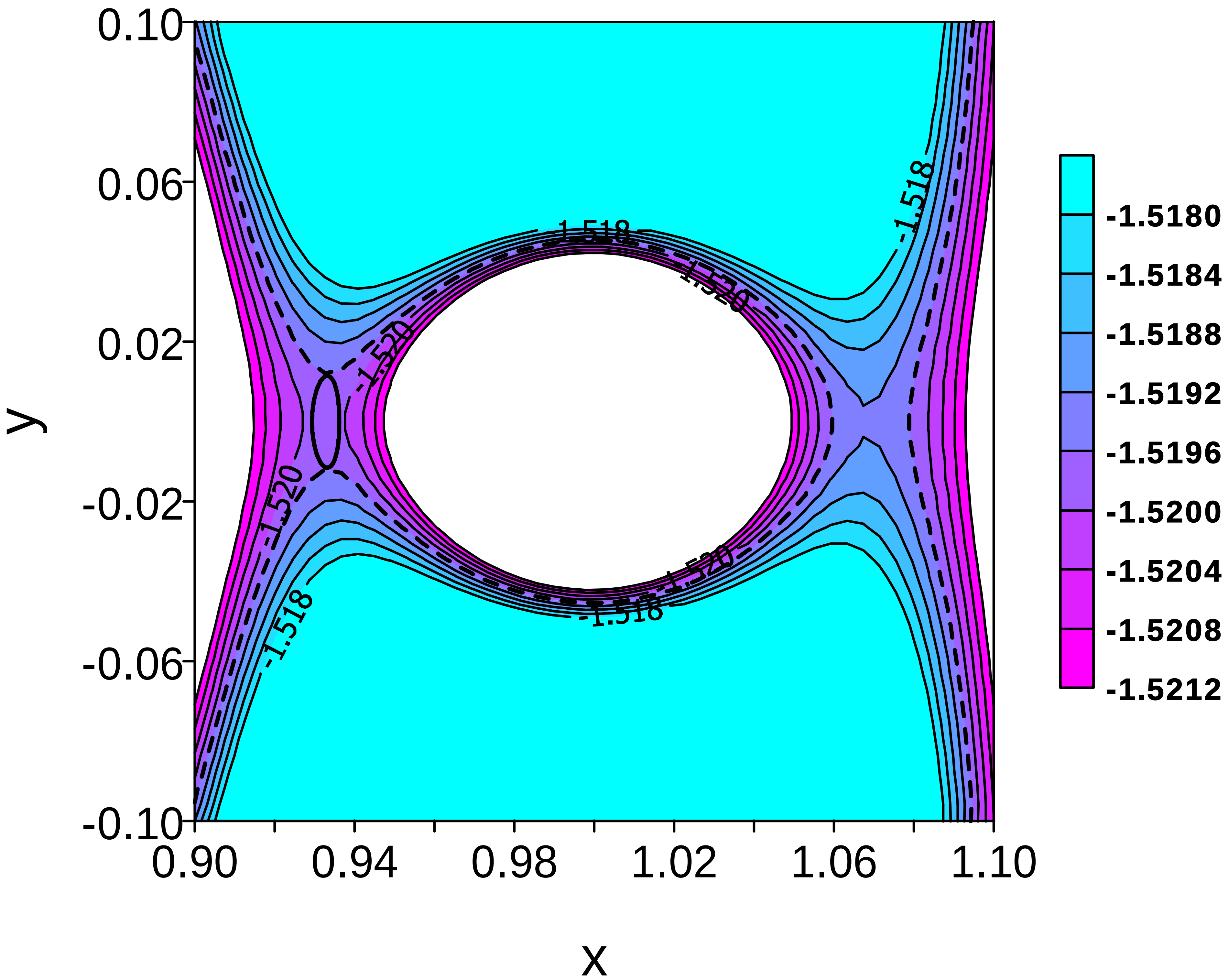
$$y \longrightarrow p_v$$
$$p_y \longrightarrow -v.$$

  the equations of motion do become time-reversal symmetric. Any trajectory which encounters the ***zero velocity surface*** in the new coordinates will retrace its path in configuration space.

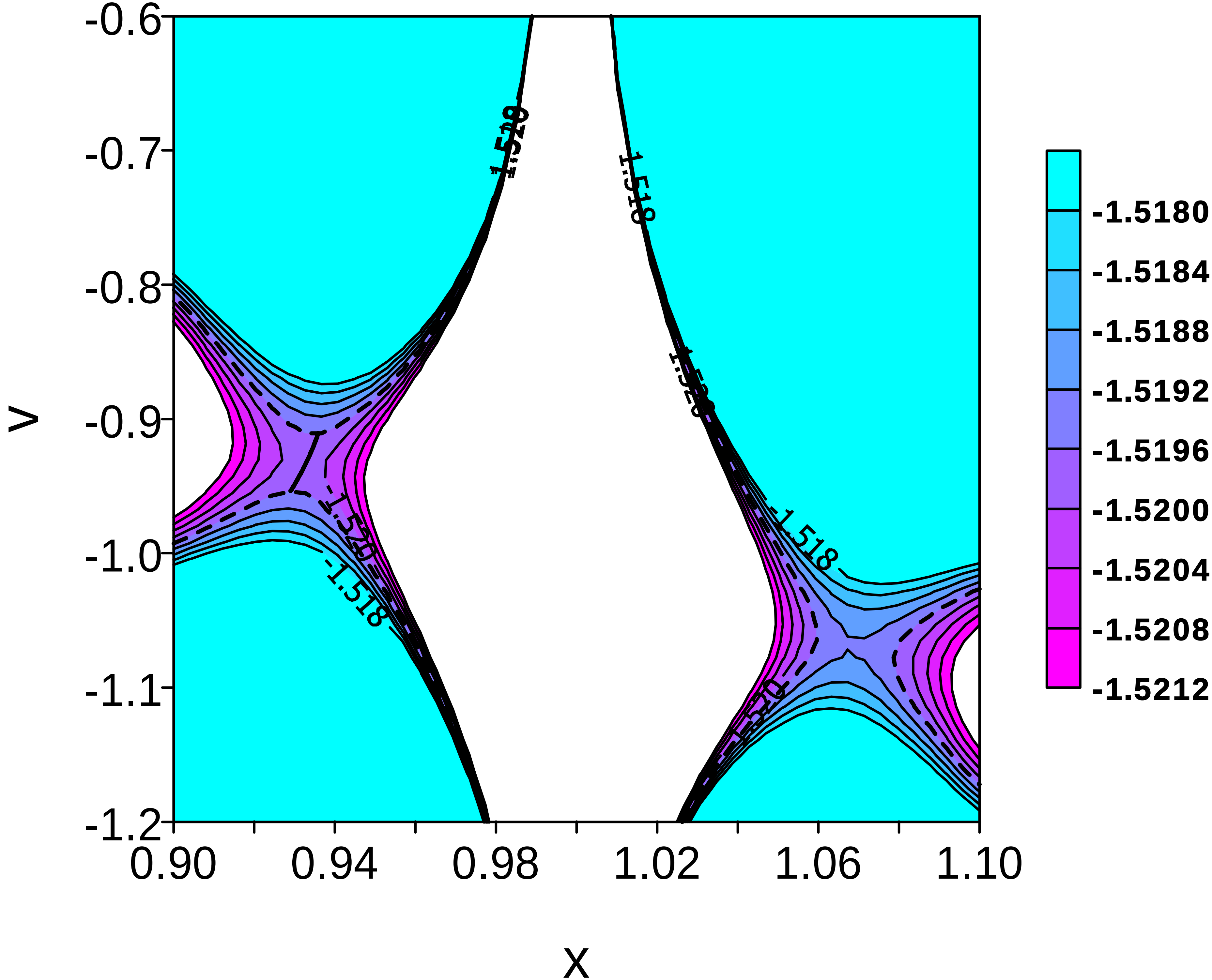
- One desires time-reversal symmetry in order to define the transition state. The basic idea is that any trajectory that goes from the bound region to the unbound region must cross this orbit – the PODS.


(a)
1.0
0.0
-1.0
y
(b)
1.0
0.0
-1.0
P_y
-2.0
0.0
2.0
4.0
x

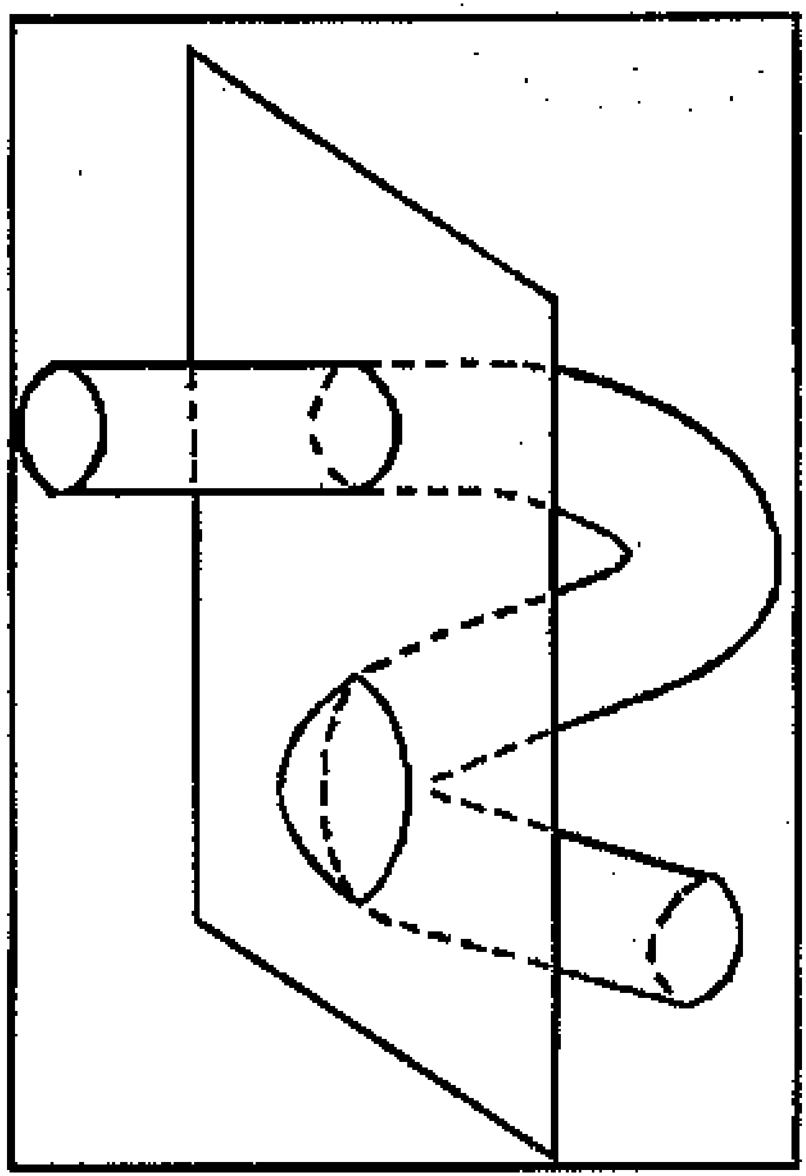

Forbidden Region
Stable
Manifold
Unstable
Manifold
Jupiter
Sun
L_2
Stable
Manifold
L_2 Periodic
Orbit
Unstable
Manifold
Forbidden Region
y (rotating frame)
x (rotating frame)

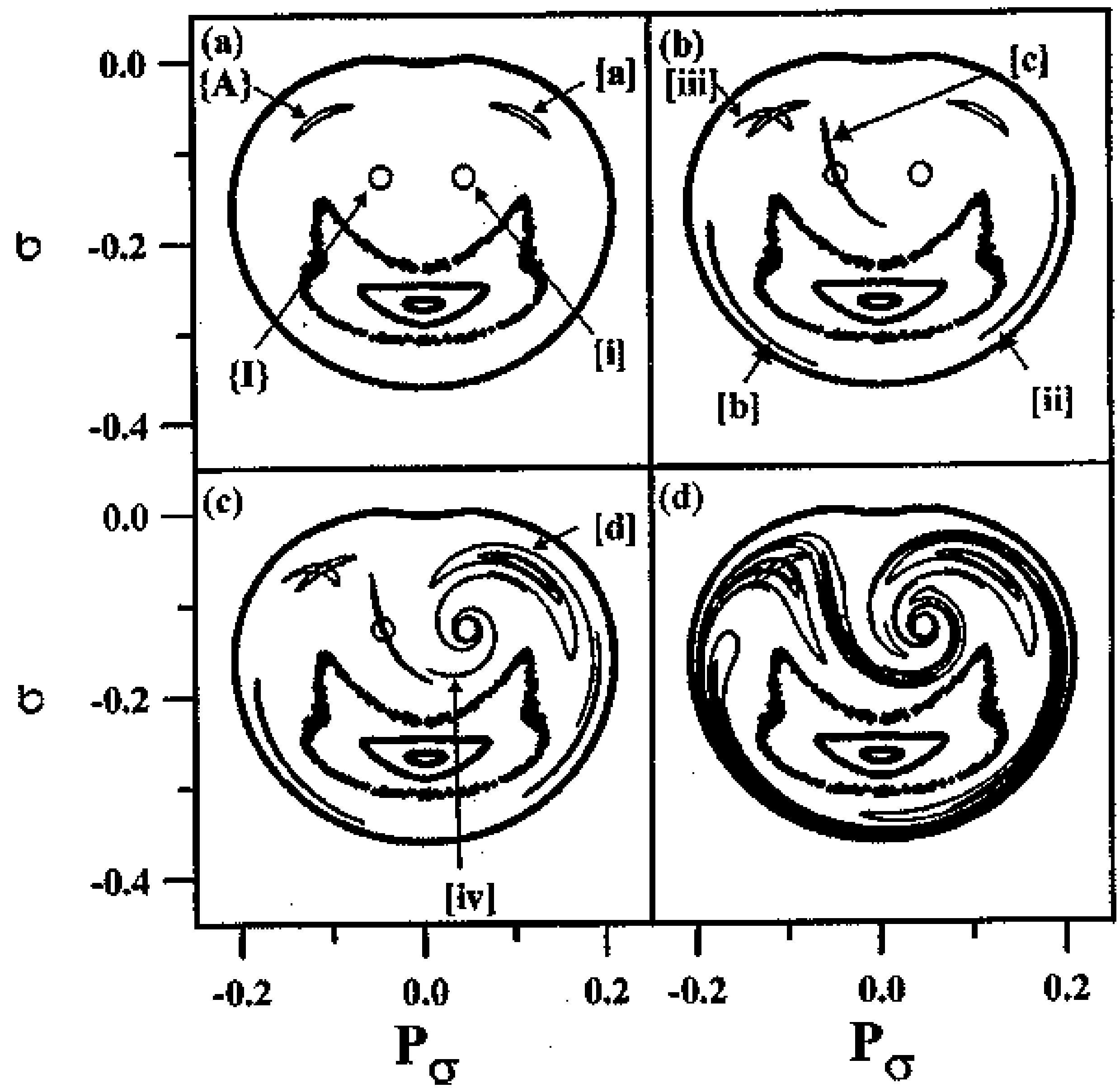


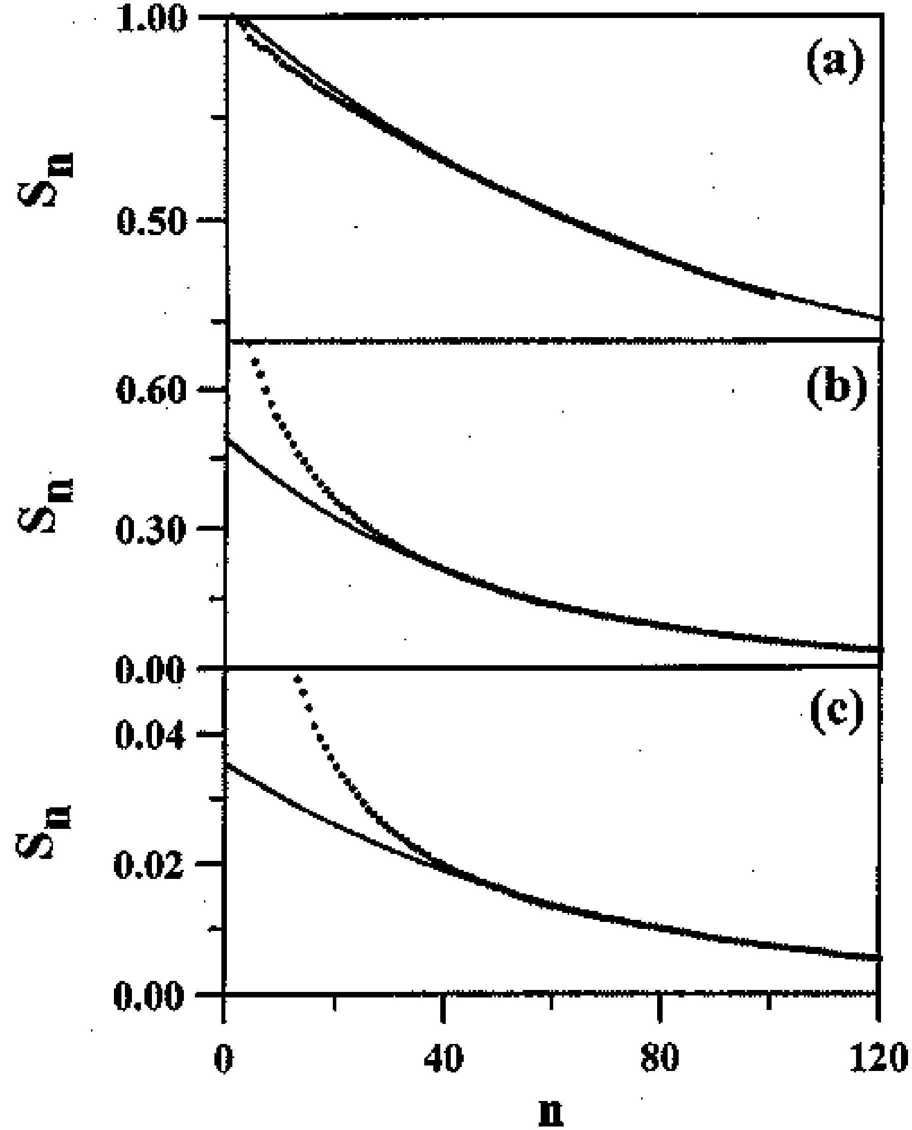
0.10
0.06
0.02
-0.02
-0.06
-0.10
y
0.90
0.94
0.98
1.02
1.06
1.10
x
-1.518
-1.520
-1.5180
-1.5184
-1.5188
-1.5192
-1.5196
-1.5200
-1.5204
-1.5208
-1.5212



-0.6
-0.7
-0.8
-0.9
-1.0
-1.1
-1.2
v
0.90
0.94
0.98
1.02
1.06
1.10
x
-1.518
-1.520
-1.5180
-1.5184
-1.5188
-1.5192
-1.5196
-1.5200
-1.5204
-1.5208
-1.5212




(a)
{A}
[a]
{I}
[i]
(b)
[iii]
[c]
[b]
[ii]
(c)
[d]
[iv]
(d)
σ
0.0
-0.2
-0.4
-0.2
0.0
0.2
P_σ



(a)
(b)
(c)
S_n
S_n
S_n
1.00
0.50
0.60
0.30
0.00
0.04
0.02
0.00
0
40
80
120
n

Celestial Mechanics vs. Chemical/Atomic Physics

- Astronomers and space mission designers have traditionally been interested in ***individual trajectories*** of particles.
  - For mission designers, parameters which describe the system (e.g., the masses of the major bodies) are fixed, control is performed on-board via changes in velocity, and control must be precise (e.g., capture instead of crash). Only short time dynamics is important.
- Chemists are more interested in ***ensembles of trajectories***, from which statistical information may be obtained (e.g., reaction rates, ionization probability).
  - Chemical engineers have control as to how system is prepaired (e.g., the kinds and numbers of reactants, temperature, applied electric field), but little or no control over individual trajectories. Long time asymptotic behavior is important.

- It's possible to merge the two approaches. Astronomers/planetary scientists are increasingly more interested in the statistical properties of the solar system (and extra-solar systems), such as mass accretion rates, comet capture probablity, likely planetary distributions around extra-solar stars, etc.
- **Main Point**: For a class of Hamiltonian systems which have phase space bottlenecks containing unstable periodic orbits, the unstable and stable manifolds of those periodic orbits partition the part of the energy surface where transport is possible. The manifolds not only provide a picture of the global behavior of the system, but are the starting point for obtaining the statistical properties of the system.

Further Information

- **Koon, W.S., M.W. Lo, J.E. Marsden and S.D. Ross [2000]**, Heteroclinic connections between periodic orbits and resonance transitions in celestial mechanics, **Chaos**, vol. **10(2)**, pp. 427-469.
- **Koon, W.S., M.W. Lo, J.E. Marsden and S.D. Ross [2000]**, Low energy transfer to the Moon.
- **Jaffé, C., D. Farrelly and T. Uzer [1999]**, Transition state in atomic physics, **Phys. Rev. A**, vol. **60(5)**, pp. 3833-3850.
- **http://www.cds.caltech.edu/~shane/**