CALTECH
Control & Dynamical Systems

Dynamics of Binary Asteroid Pairs

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Motivation

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Dactyl in orbit about Ida, discovered in 1994 during the Galileo mission.

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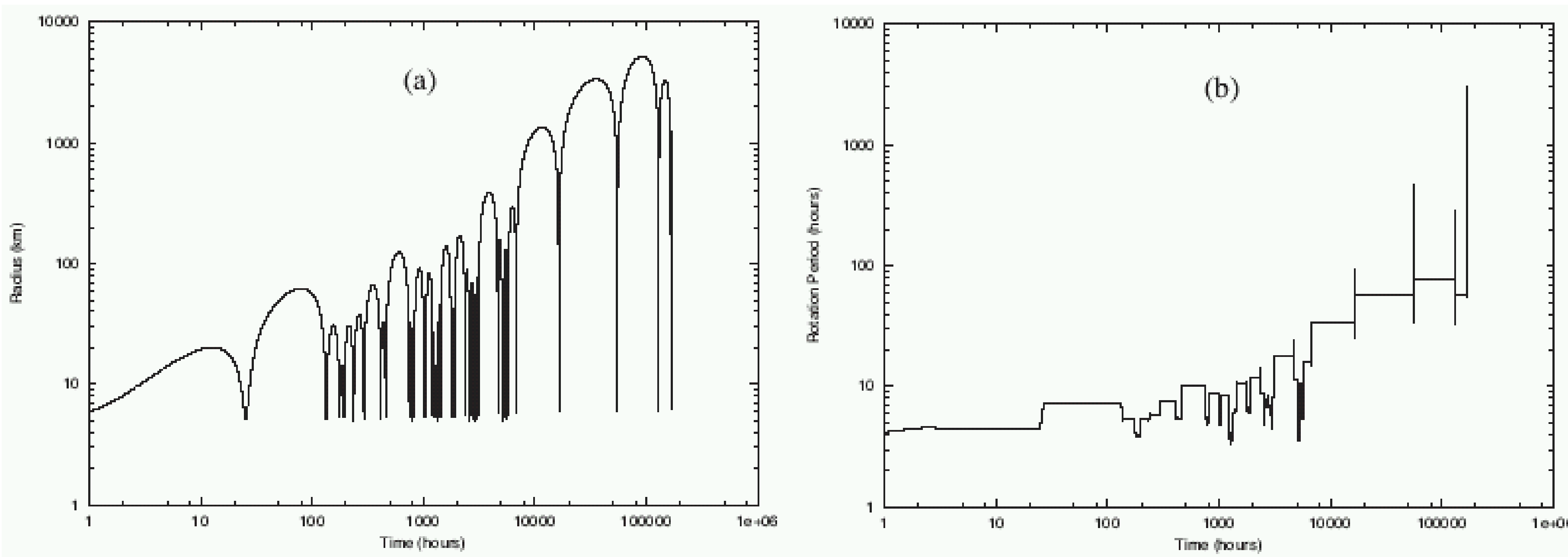
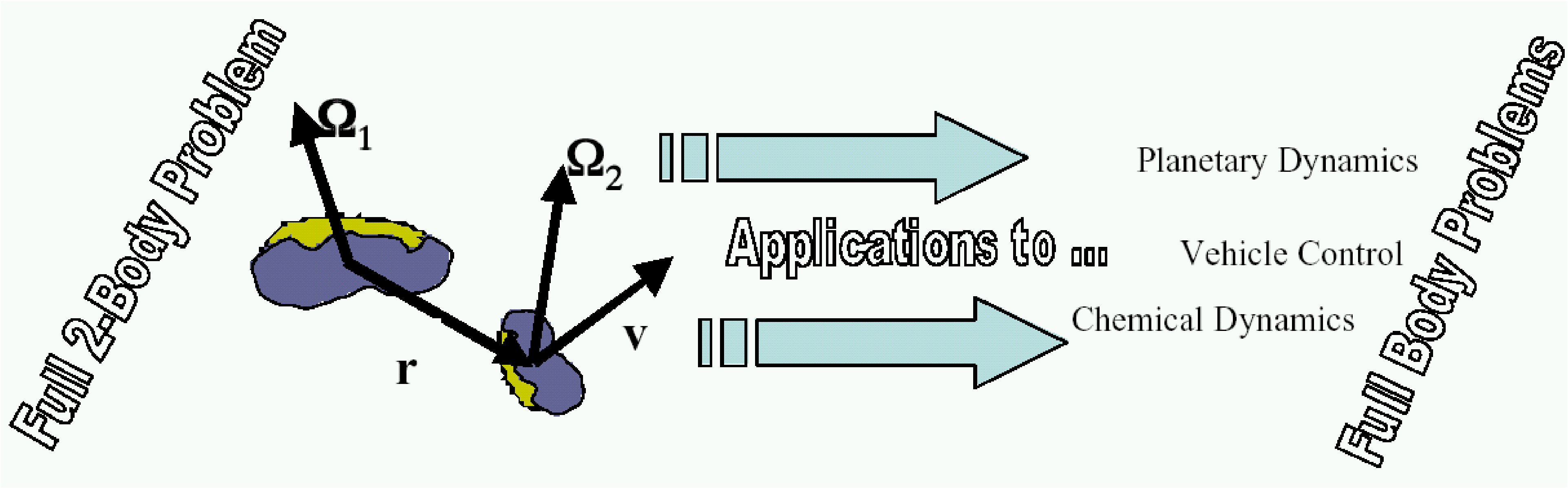


Time history of the orbit radius (a) and rotation period (b) for a gravitationally interacting sphere and tri-axial ellipsoid of equal mass.

Full Body Problem

- *Full Body Problem (N Bodies)*
- *begin with Full Two Body Problem*



Full Body Problem

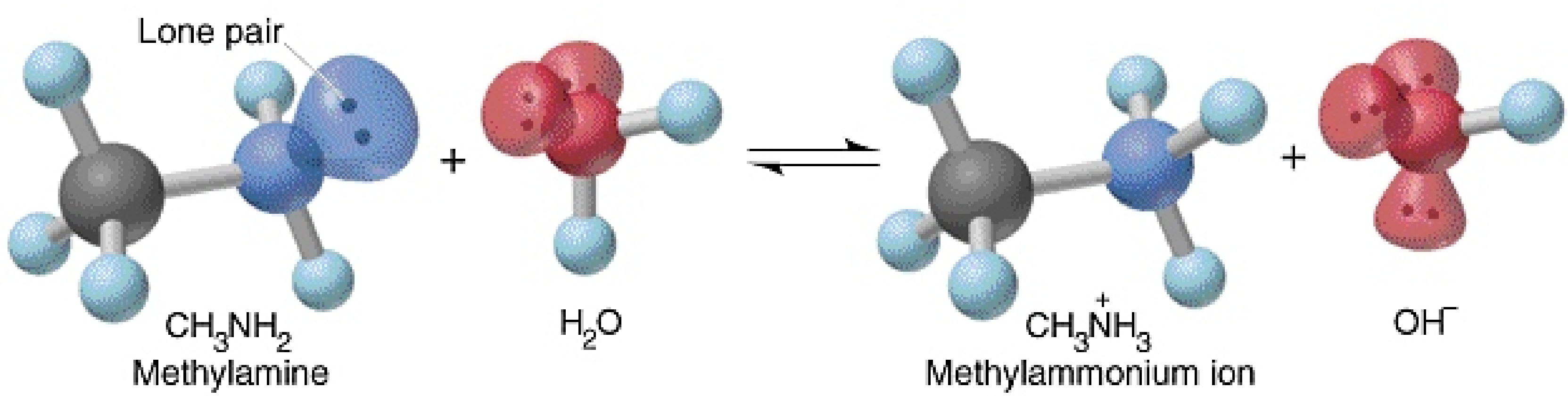
- ***Relevant to***
  - asteroid and Kuiper belt binary evolution
  - variation of planetary obliquities
  - comet nucleus evolution due to outgassing
  - close approaches of galaxies

Full Body Problem

galaxy collision

Full Body Problem

- ***Furthermore***
  - The mathematical description of the FBP and phase space transport phenomena applies to a wide range of physical systems across many scales (chemistry, biology, fluid dynamics, ...)



Important Tools

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- **Asynchronous variational integrators:** Symplectic integrators allowing different time steps at different spatial points.
- **Phase space transport:** For chaotic regimes of motion, the phase space has structures mediating transport (tube and lobe dynamics, ...).
- Approximate statistical models may be appropriate under certain conditions, e.g., mixing assumptions in chemical and celestial mechanics, etc.

F2BP: Models To Use

- Consider two masses, m_1 and m_2.
- A widely used model: m_1 is a sphere.
- The normalized and symmetry reduced equations are

$$\ddot{\mathrm{r}} + 2\boldsymbol{\omega} \times \dot{\mathrm{r}} + \dot{\boldsymbol{\omega}} \times \mathrm{r} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathrm{r}) = \frac{\partial \mathcal{U}}{\partial \mathrm{r}}$$

$$\mathcal{I} \cdot \dot{\boldsymbol{\omega}} + \boldsymbol{\omega} \times \mathcal{I} \cdot \boldsymbol{\omega} = -\mu \mathrm{r} \times \frac{\partial \mathcal{U}}{\partial \mathrm{r}},$$

where

$\omega =$ rotational velocity vector in the body-fixed frame,

$\mathrm{r} =$ relative position vector in the body-fixed frame,

$\mathrm{A} =$ attitude tensor of the non-spherical body,

$\mathcal{I} =$ specific inertia tensor of the non-spherical body,

$\mathcal{U} =$ gravitational potential of the non-spherical body.

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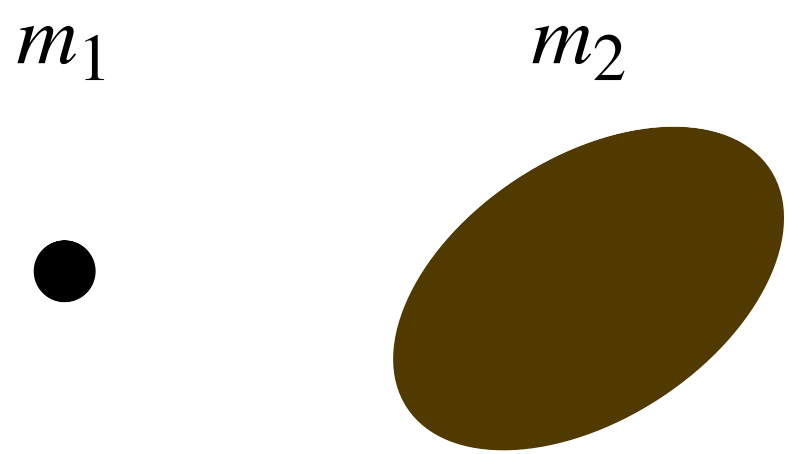
  Particle around asteroid

  restricted F2BP (RF2BP)



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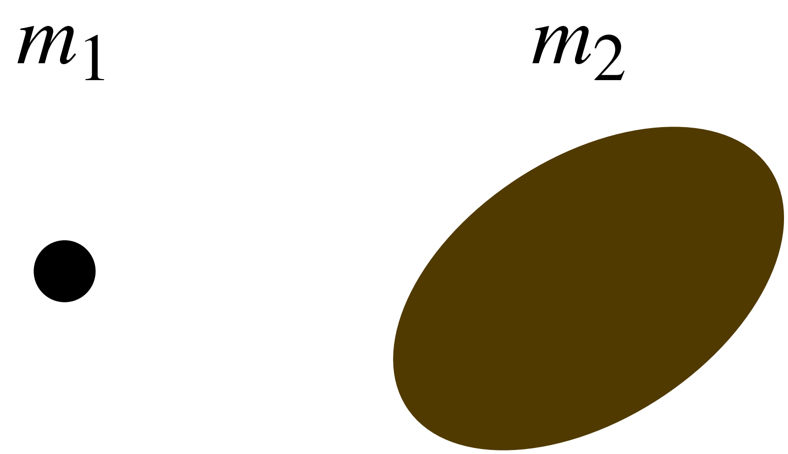
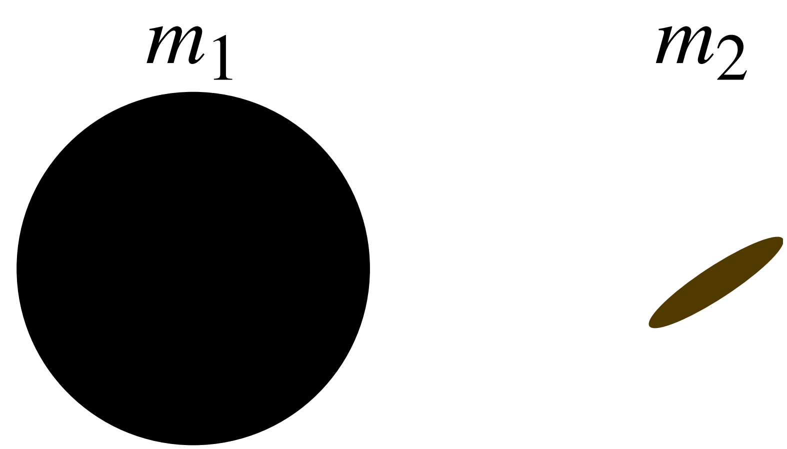


- $\mu \rightarrow 1$
  Spacecraft around planet



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- The full 2-body problem has a $SE(3)$ symmetry and corresponding conserved quantities
- **Phase space**: $Q = SE(3) \times SE(3)$
- Reduce: **shape space** Q/G gives the **system shape**.
- All the power of geometric mechanics can be brought to bear: symmetry reduction, relative equilibria, energy-momentum method (and its converse), phases (translational and rotational drift and coupling),...

Reduction for the FB2P

- For the F2BP, $Q = SE(3) \times SE(3)$.
- Material points in a reference configuration X_i,
- Points in the current configuration x_i.
- Given $((A_1, r_1), (A_2, r_2)) \in SE(3) \times SE(3)$, related by $x_1 = A_1 X_1 + r_1$ and $x_2 = A_2 X_2 + r_2$
- Lagrangian equals kinetic minus potential energy:

$$L(A_1, r_1, A_2, r_2) = \frac{1}{2}\int_{\mathcal{B}_1} \|\dot{x}_1\|^2 d\mu_1(X_1) + \frac{1}{2}\int_{\mathcal{B}_2} \|\dot{x}_2\|^2 d\mu_2(X_2) + \int_{\mathcal{B}_1}\int_{\mathcal{B}_2} \frac{G d\mu_1(X_1) d\mu_2(X_2)}{\|x_1 - x_2\|}$$

$$= \frac{m_1}{2}\|\dot{r}_1\|^2 + \frac{1}{2}\langle \Omega_1, I_1 \Omega_1 \rangle + \frac{m_2}{2}\|\dot{r}_2\|^2 + \frac{1}{2}\langle \Omega_2, I_2 \Omega_2 \rangle + \int_{\mathcal{B}_1}\int_{\mathcal{B}_2} \frac{G d\mu_1(X_1) d\mu_2(X_2)}{\|A_1 X_1 - A_2 X_2 + r_1 - r_2\|}.$$

Reduction for the FB2P

- Reduce by overall translations and rotations.
- $SE(3)$ acts by the diagonal left action on Q:
  $(A, r) \cdot (A_1, r_1, A_2, r_2) = (AA_1, Ar_1 + r, AA_2, Ar_2 + r)$.
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- **Momentum map** is the total linear momentum and the total angular momentum.
- **Shape space** Q/G: one copy of $SE(3)$; coordinatized by the **relative attitude** $A_2^T A_1 = A^T$ and **relative position** $A_2^T(r_1 - r_2) = R$.
- General reduction theory says that the reducted equations of motion are in $T(Q/G) \times \mathfrak{g}$ (for velocities) or $(T^*Q)/G \times \mathfrak{g}^*$ (for momenta).

Reduction for the FB2P

- **Equations of motion** in $T(Q/G)$ (resp. $T^*(Q/G)$) involve A, R, and their velocities (resp. conjugate momenta Γ, P).
- Coupled to equations in $\mathfrak{se}(3)^*$, identified with equations for the body angular and linear momenta of the second rigid body, Γ_2, P_2.

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- **Shape space**: key to **geometric phases** that are important for rotational and translational drifts.
- **Reduced Lagrangian**: rewrite L in variables:

$$A = A_2^T A_1, \quad R = A_2^T(r_1 - r_2),$$
$$\hat{\Omega} = A_2^T \dot{A}_1, \quad V = A_2^T(\dot{r}_1 - \dot{r}_2),$$

Reduction for the FB2P

which are coordinates on $T(Q/G)$, as well as

$$\hat{\Omega}_2 = A_2^T \dot{A}_2, \; V_2 = A_2^T \dot{r}_2,$$

which are coordinates on $\mathfrak{se}(3)$.

- **Hamilton's variational principle** on $T(SE(3) \times SE(3))$ is equivalent to the **reduced variational principle**,

$$\delta \int_a^b l(A, R, \hat{\Omega}, V, \hat{\Omega}_2, V_2) dt = 0,$$

on $\mathbb{R}^{18}$ where the variations are of the form,

$$\delta A = -\hat{\Sigma}_2 A + \hat{\Sigma}, \quad \delta R = -\hat{\Sigma}_2 R + S, \quad \widehat{\delta\Omega} = \dot{\hat{\Sigma}} - \hat{\Sigma}_2 \hat{\Omega} + \hat{\Omega}_2 \hat{\Sigma},$$
$$\delta V = \dot{S} - \hat{\Sigma}_2 V + \hat{\Omega}_2 S, \quad \delta \Omega_2 = \dot{\Sigma}_2 + \Omega_2 \times \Sigma_2, \quad \delta V_2 = \dot{S} - \hat{\Sigma}_2 V_2 + \hat{\Omega}_2 S_2.$$

Systematic Structures

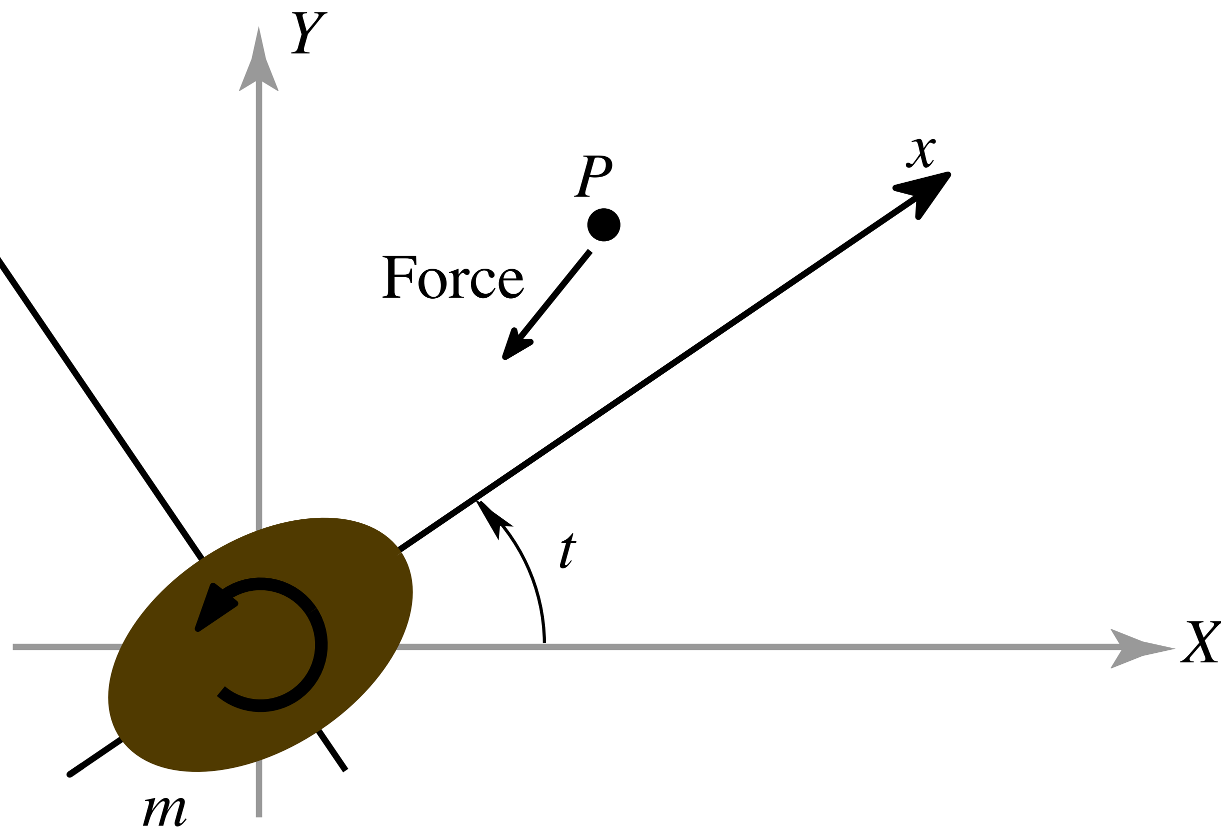
- For numerics as well as analysis of stability of relative equilibria (analog of the libration points), the variational and Hamiltonian structures are useful.
- Previous works **guessed** these structures and missed the variational structure altogether. Using reduction, one **derives** them in a simple and natural way, one gets the Jacobi integrals naturally, etc.
- Extra symmetries give extra conserved quantities and further reductions (e.g., cylindrical symmetry of one of the bodies).
- Special cases (such as an ellipsoid and a sphere).

Restricted Simpler Case

- ***Let's look at an example problem***
  - Restricted (as in restricted 3-body problem) simple case exhibits the basic capture, ejection, collision dynamics.

Restricted Simpler Case

- Point mass P moving in the x-y plane under the gravitational field of a uniformly rotating elliptical body m, without affecting its uniform rotation.



The rotating (x-y) and inertial (X-Y) frames.

Restricted Simpler Case

- **Equations of motion** relative to a rotating Cartesian coordinate frame and appropriately normalized:

$$\ddot{x} - 2\dot{y} = \frac{\partial U}{\partial x} \quad \text{and} \quad \ddot{y} + 2\dot{x} = \frac{\partial U}{\partial y},$$

  where

$$U(x, y) = -\frac{1}{r} - \frac{1}{2}r^2 - \frac{3C_{22}\left(x^2 - y^2\right)}{r^5},$$

  and

$$r = \sqrt{x^2 + y^2}.$$

- Gravity field coefficient C_{22}, the **ellipticity**, typically varies between 0 and 0.05.
- **Jacobi integral:** $E = \frac{1}{2}\left(\dot{x}^2 + \dot{y}^2\right) - U(x, y)$.

Restricted Simpler Case

- **Moving systems approach** gives the Lagrangian and Hamiltonian structure and Jacobi integral.
- Lagrangian (kinetic minus potential energy) written in the rotating system and with angular velocity normalized to unity, is
$$L = \frac{1}{2}[(\dot{x} - y)^2 + (x + \dot{y})^2] - V(x, y).$$
where
$$V(x, y) = -\frac{1}{r} - \frac{3C_{22}\left(x^2 - y^2\right)}{r^5}.$$
- Euler–Lagrange equations produce the previous equations and the Legendre transformation gives the Hamiltonian structure, the Jacobi integral, etc.

F2BP: Phase Space Structure

- The Jacobi integral (energy) is an indicator of the type of global dynamics possible.
- For energies above a threshold, $E > E_S$, corresponding to symmetric saddle points, movement between the **realm** near the asteroid (**interior realm**) and away from the asteroid (**exterior realm**) is possible. For energies $E \leq E_S$, no such movement is possible.

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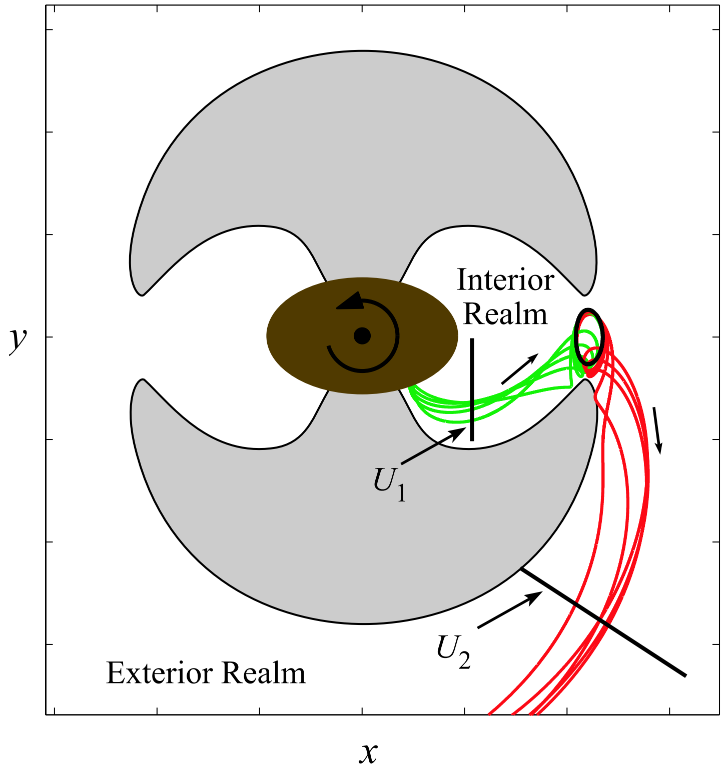
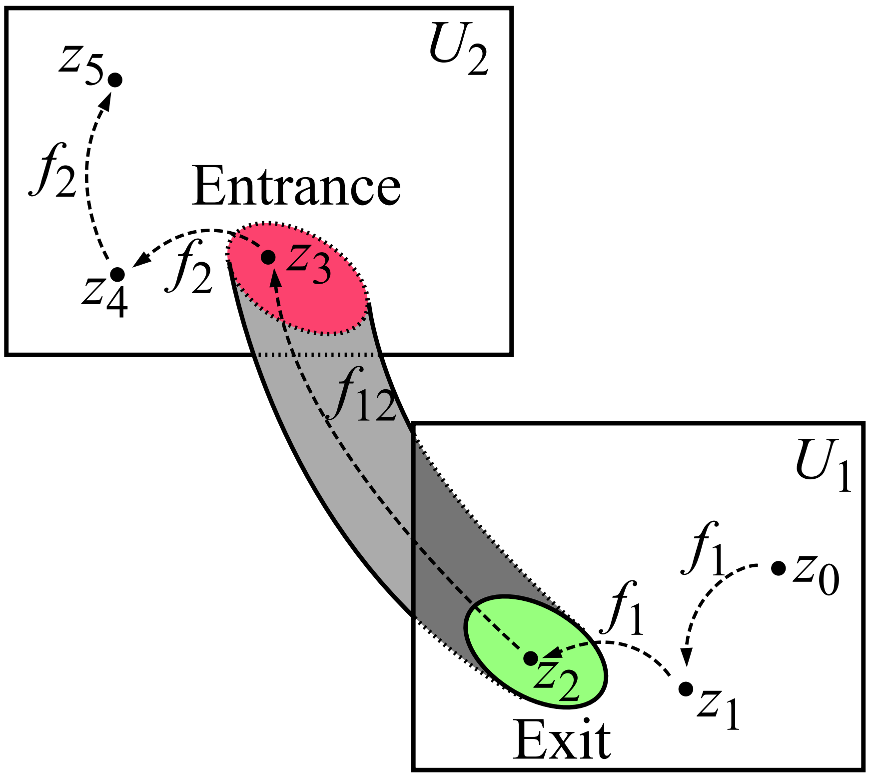
- **Multi-scale dynamics** : for chaotic regimes of motion, the phase space has structures mediating transport.
- **tube dynamics** : On the largest scale, phase space is organized into realms, connected via **tubes**
- **lobe dynamics** : In each realm, phase space is organized further into different **resonance regions**, connected via **lobes**.

F2BP: Phase Space Structure

- Slices of energy surface: Poincaré sections U_i
- Lobe dynamics: evolution **on** U_i
- Tube dynamics: evolution **between** U_i





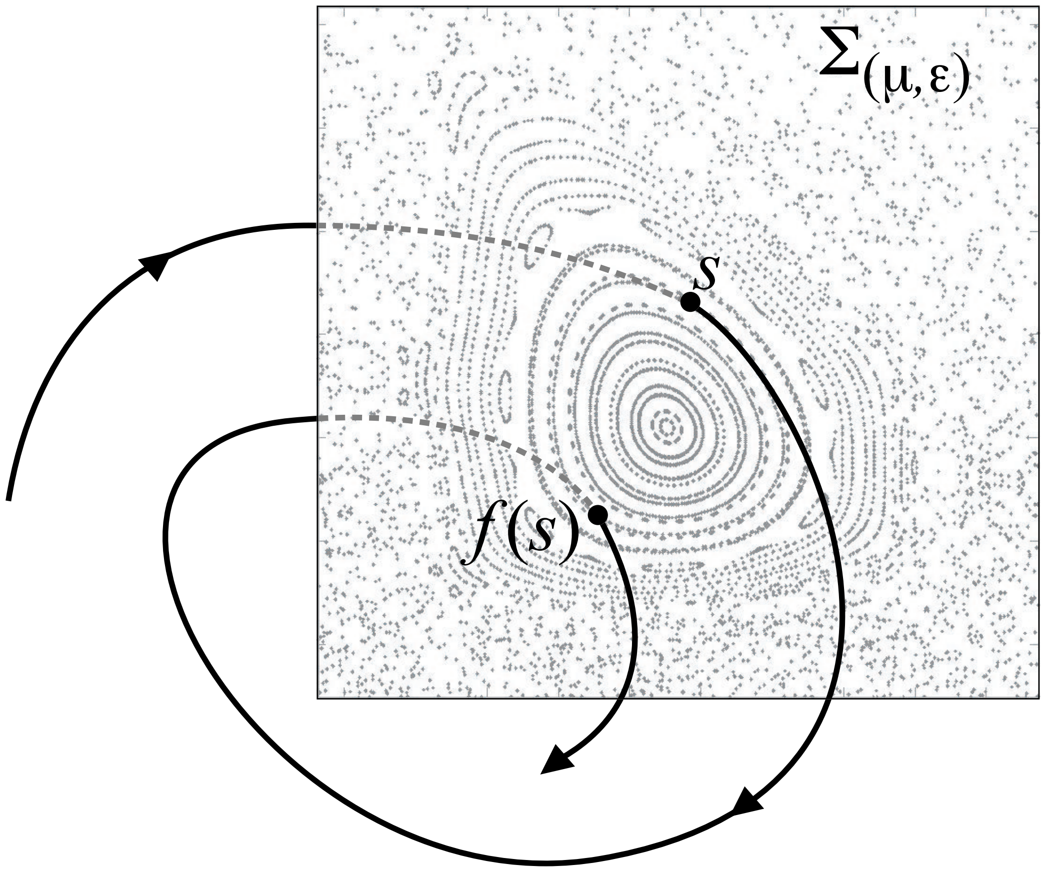
Poincaré Surface of Section

- Study **Poincaré surface of section** on energy surface:

$$U_i = \Sigma_{(\mu,E)} = \{(x,\dot{x}) | y = 0, \dot{y} = g(x,\dot{x};\mu,E) > 0\}$$

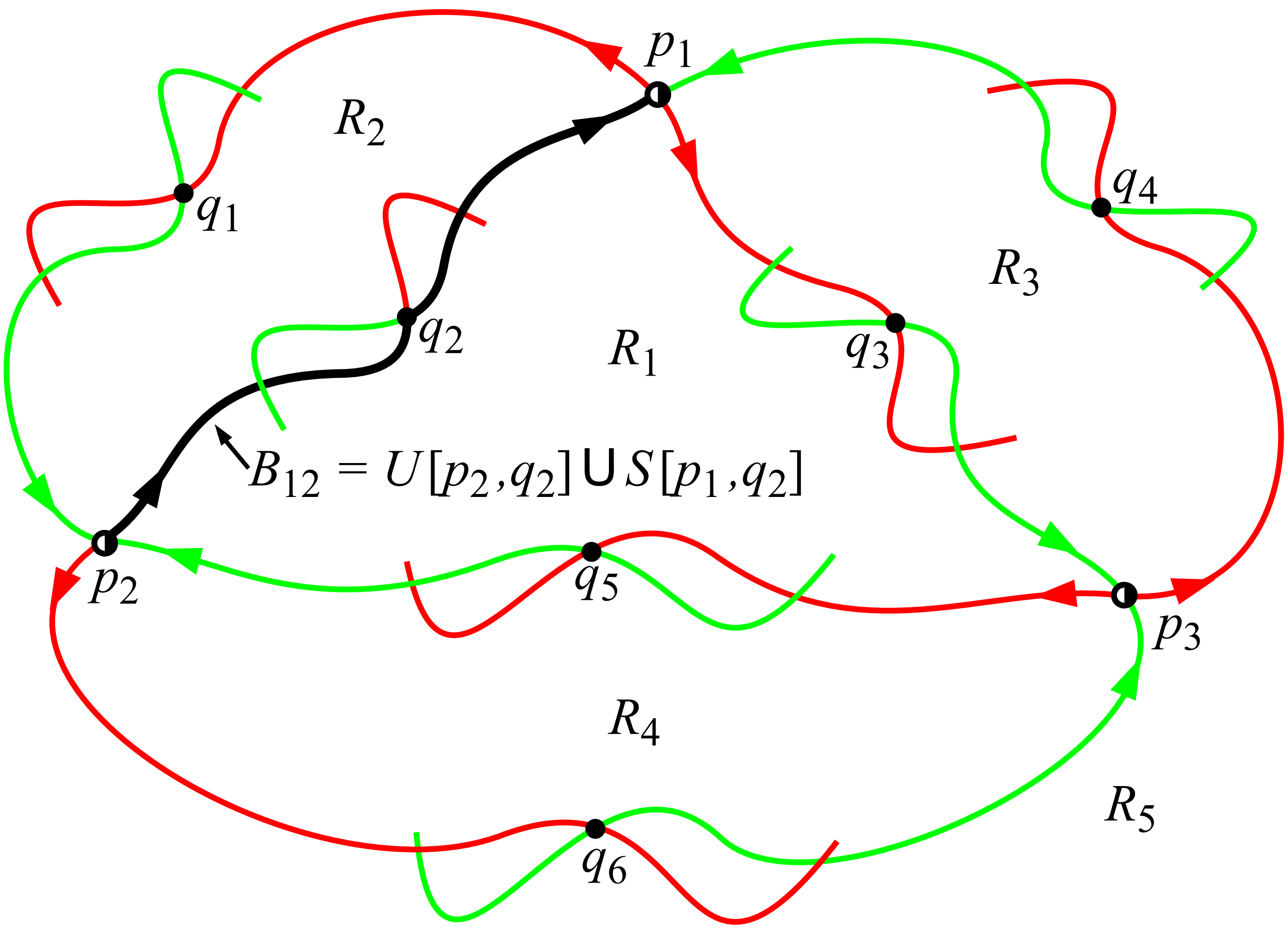
reducing the system to an area preserving map on the plane,

$$f_i : U_i \longrightarrow U_i,$$



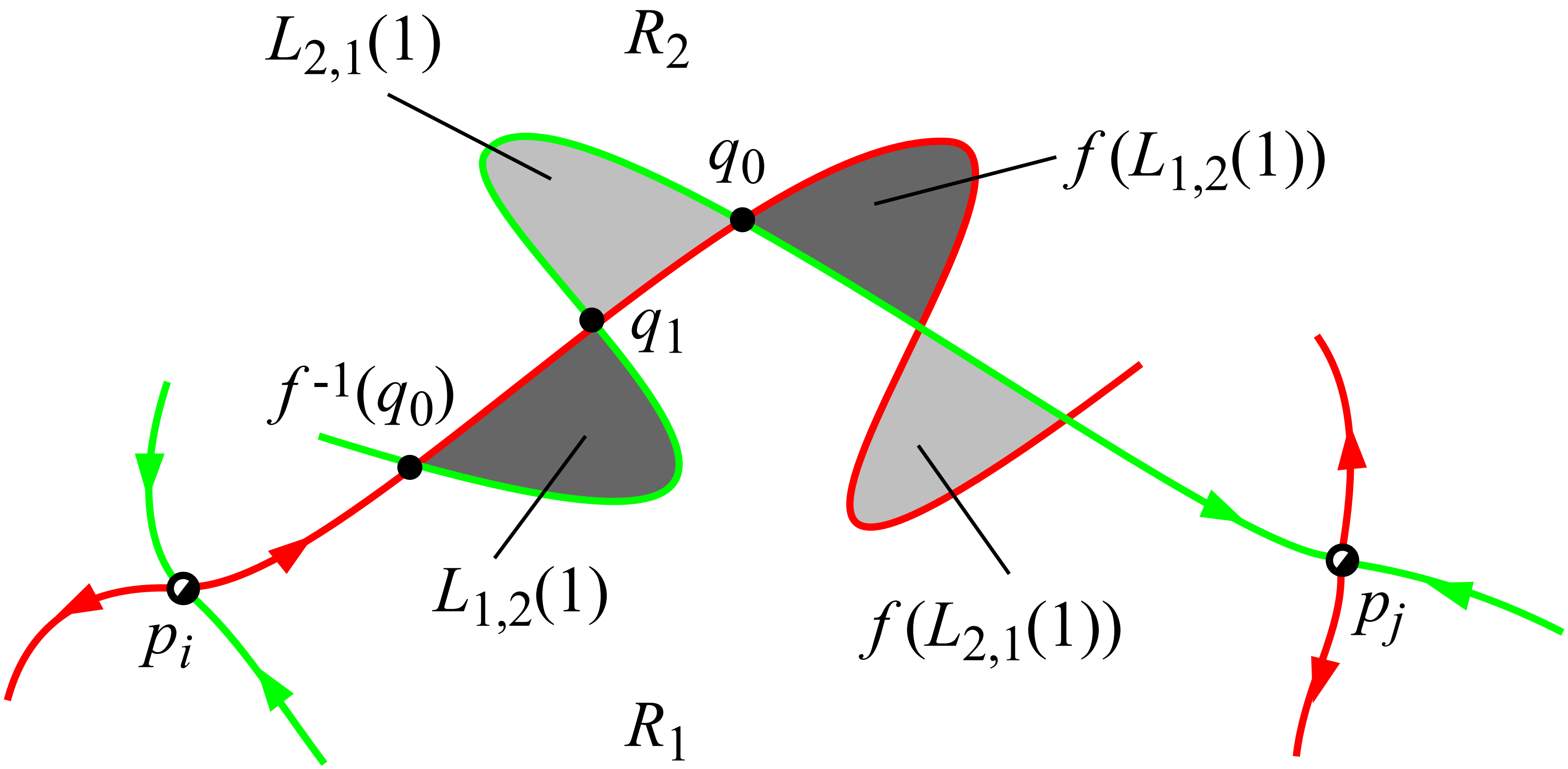
Transport in Poincaré Section

- Phase space divided into regions $R_i, i = 1, ..., N_R$ bounded by segments of stable and unstable manifolds of unstable fixed points.



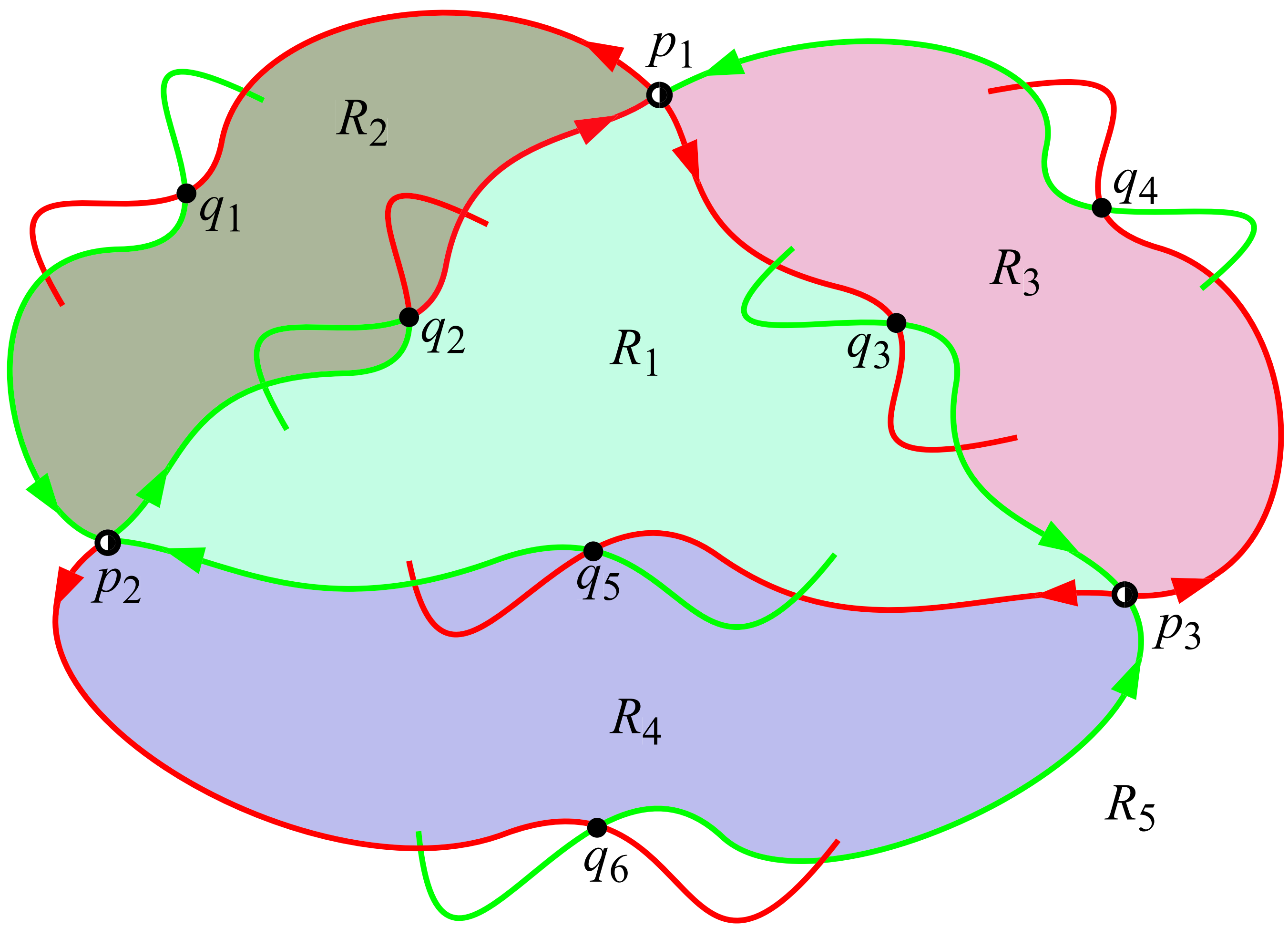
Lobe Dynamics

Transport between regions is computed via ***lobe dynamics.***



Movement Between Resonances

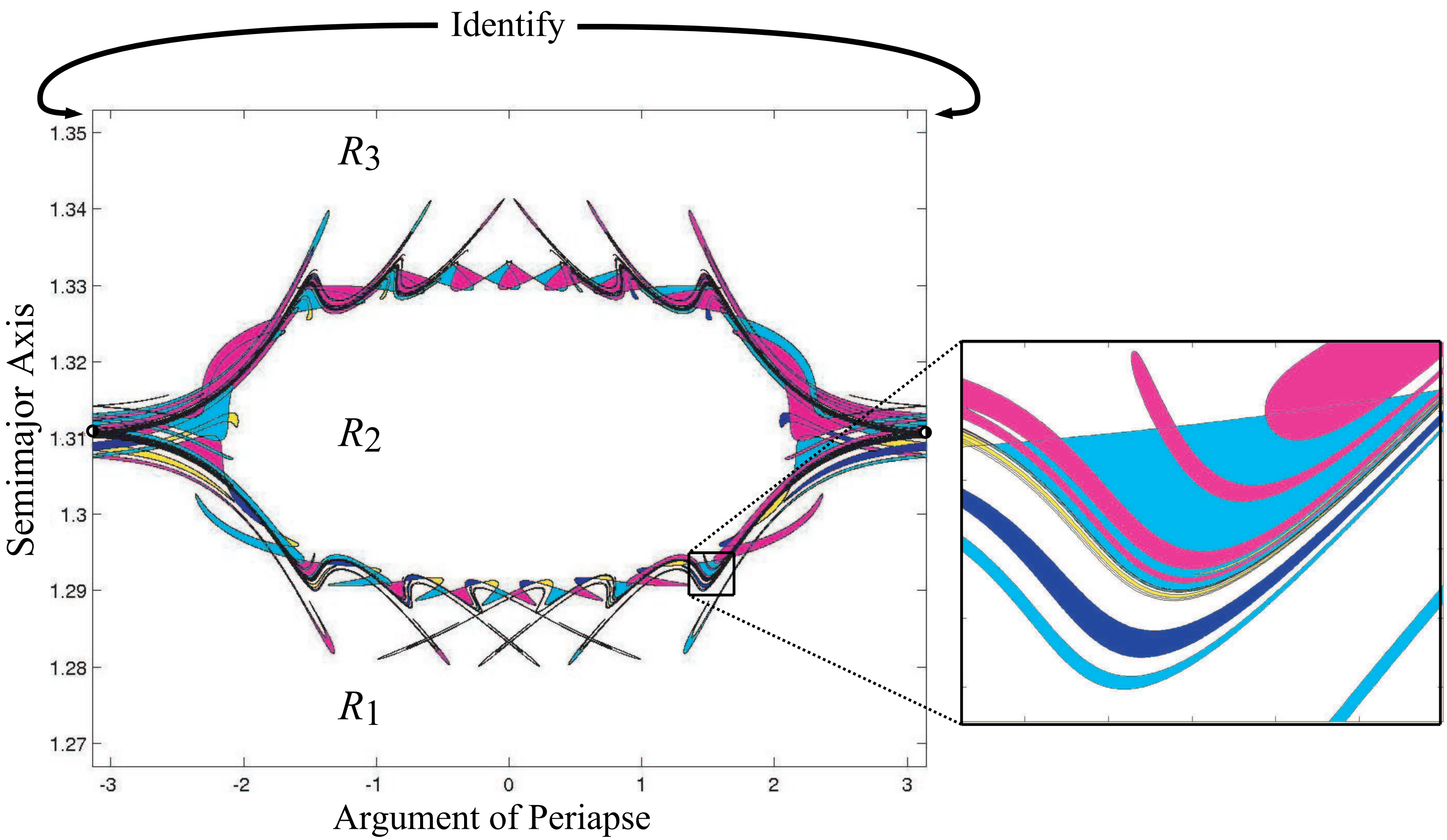
We can compute manifolds which naturally divide the phase space into ***resonance regions.***



Unstable and stable manifolds in **red** and **green**, resp.

Movement Between Resonances

Transport and mixing between regions can be computed.

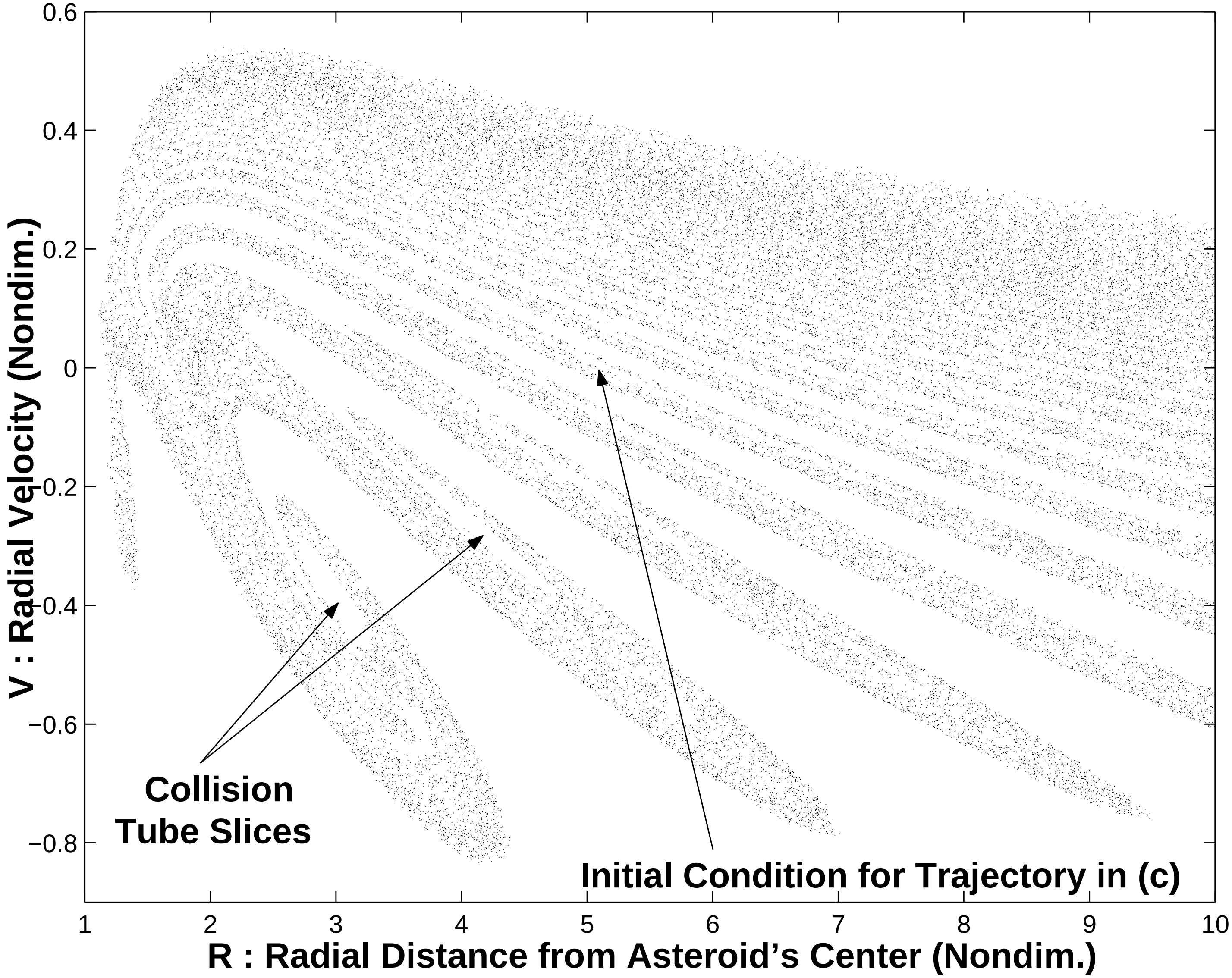


Four sequences of color coded lobes are shown.

Exterior Poincaré Sections

- A Poincaré section with $C_{22} = 0.05$ & fixed energy, illustrates the relevance of tube and lobe dynamics.
- Choose the section in the exterior region along the positive x-axis.
- Choose $E = -1.62$, slightly above energy of saddle points along the x-axis, such that particles beginning in the exterior region may be ejected from the system or collide with the asteroid.

Exterior Poincaré Sections



(a) Poincaré section for the exterior realm

Exterior Poincaré Sections

- Particles in the exterior region get captured by the asteroid if they lie within the phase space tubes associated with the unstable periodic orbits about either the left or right saddle points. Consider a captured particle to have "collided" with the asteroid if it enters the circle of radius 1 around the origin.

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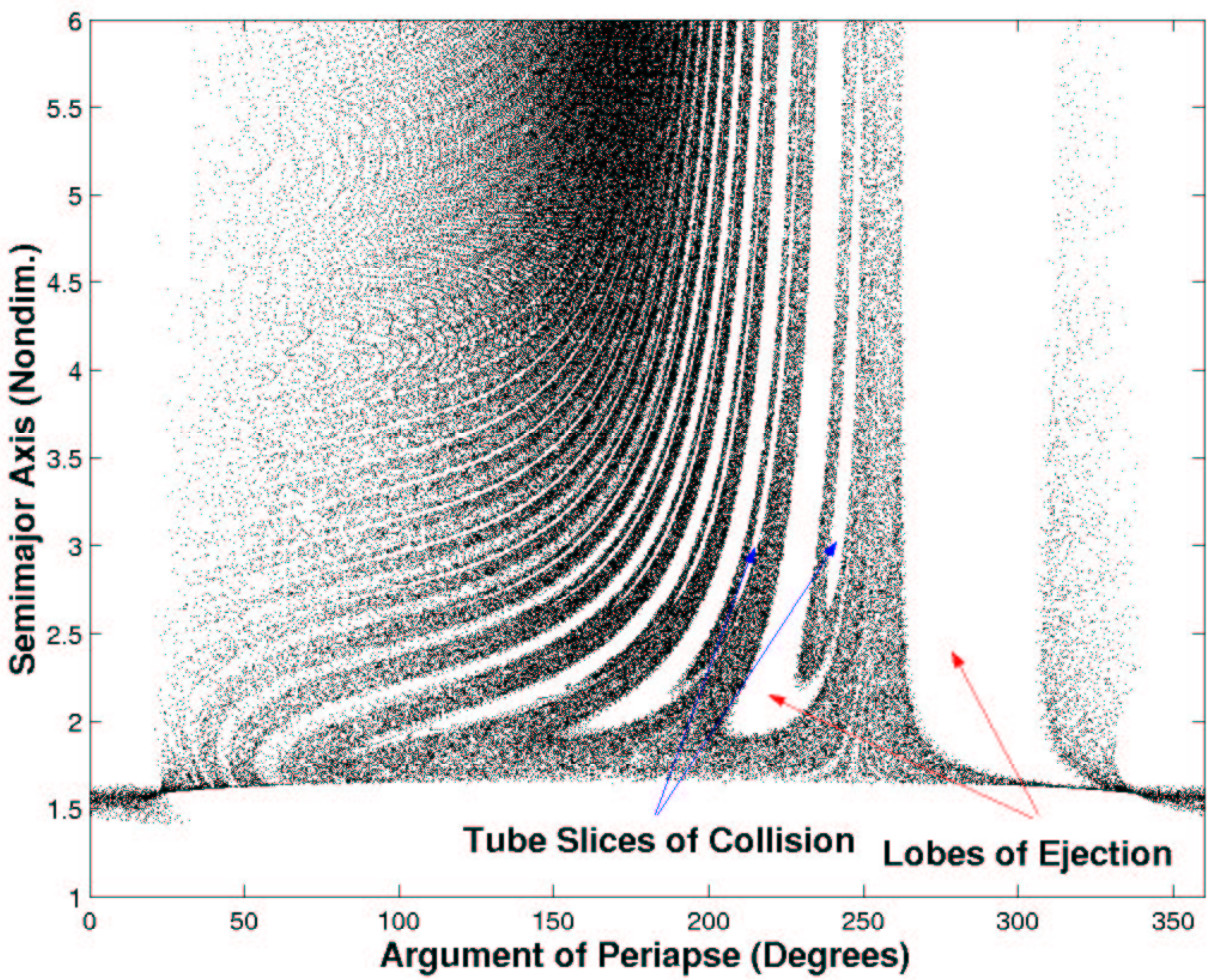
- Tube slices on this section: **tube slices of collision**.
- Particles are **ejected** if they lie within lobes enclosed by the stable and unstable manifolds of a hyperbolic fixed point at $(+\infty, 0)$—**lobes of ejection**.

Exterior Poincaré Sections

- ☐ Transform to **Delaunay variables**.



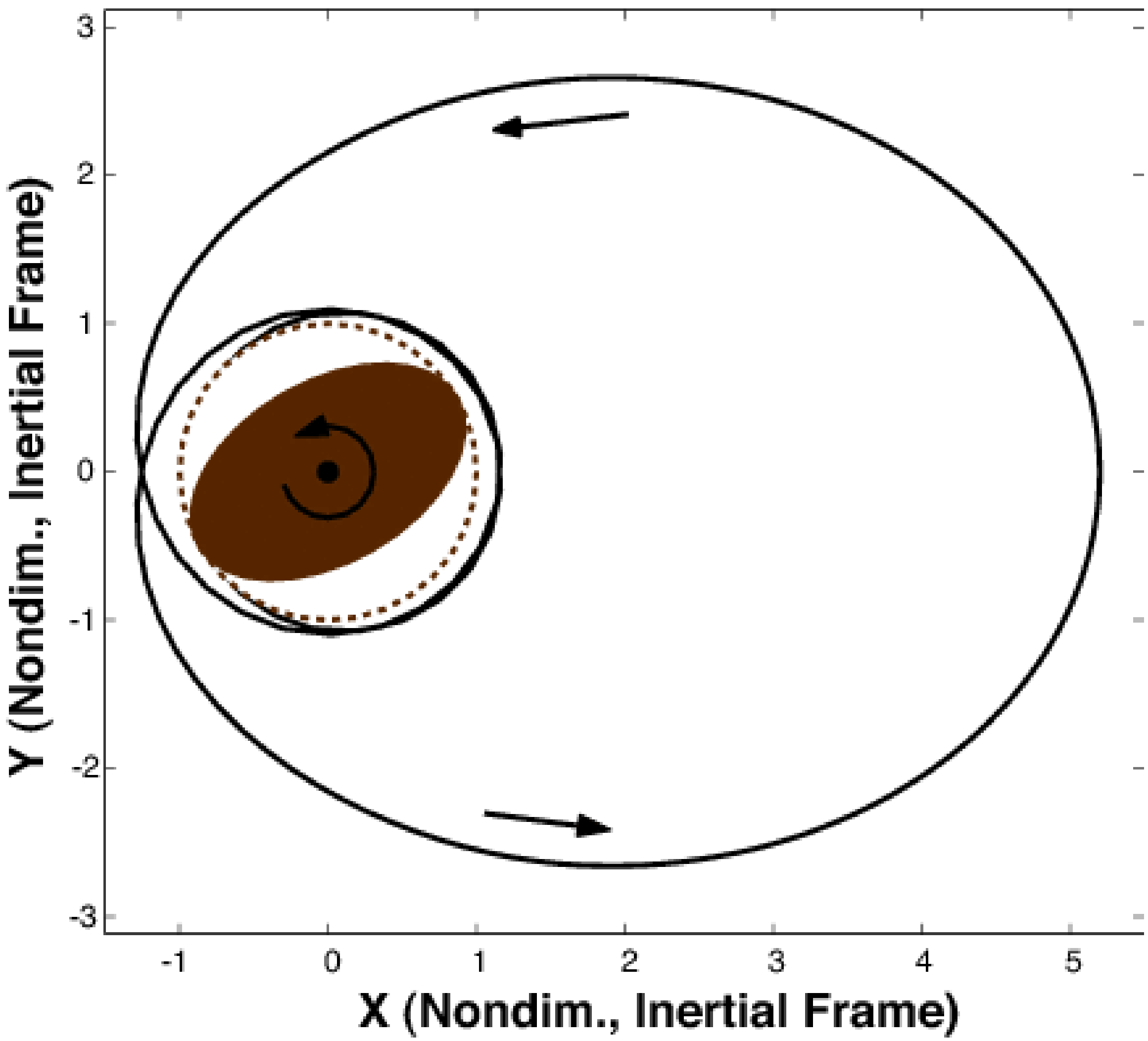
(b) Poincaré section in Delaunay variables

Exterior Poincaré Sections

- The semimajor axis is shown versus the argument of periapse with respect to the rotating asteroid (the body-fixed frame).
- Alternate fates of collision and ejection are intimately intermingled.
- The number of particles remaining in the fourth quadrant is smaller than that in the other three quadrants, in agreement with observations in Scheere's work.

Exterior Poincaré Sections

- Escape and re-capture.



(c)

Selected References

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For papers, movies, etc., visit the websites:

http://www-personal.engin.umich.edu/∼scheeres

http://www.cds.caltech.edu/∼shane

http://www.nast-group.caltech.edu/

The End

Typesetting Software: TEX, ***Textures***, LATEX, `hyperref`, `texpower`, Adobe Acrobat 4.05
Graphics Software: Adobe Illustrator 10.0.1
LATEX Slide Macro Packages: Wendy McKay, Ross Moore