Periodic Orbits and Transport: Some Interesting Dynamics in the Three-Body Problem

Shane Ross

Martin Lo (JPL), Wang Sang Koon and Jerrold Marsden (Caltech)

CDS 280, January 8, 2001

shane@cds.caltech.edu
http://www.cds.caltech.edu/~shane/

Control and Dynamical Systems

Outline

- ***Important dynamical objects***: Equilibria, periodic orbits, stable and unstable manifolds, bottlenecks
- ***Context***: Three-body problem (Hamiltonian)
- ***Equilibria***: Collinear libration points have saddle $\times$ center structure
- ***Periodic orbits***: Stable and unstable invariant manifolds divide energy surface, channeling flow in phase space
- ***Classification***: Interesting orbits can be classified and constructed using Poincaré sections and symbolic dynamics
- ***Theorem***: Near home/heteroclinic orbits, "horseshoe"-like dynamics exists
- ***Application***: Actual space missions

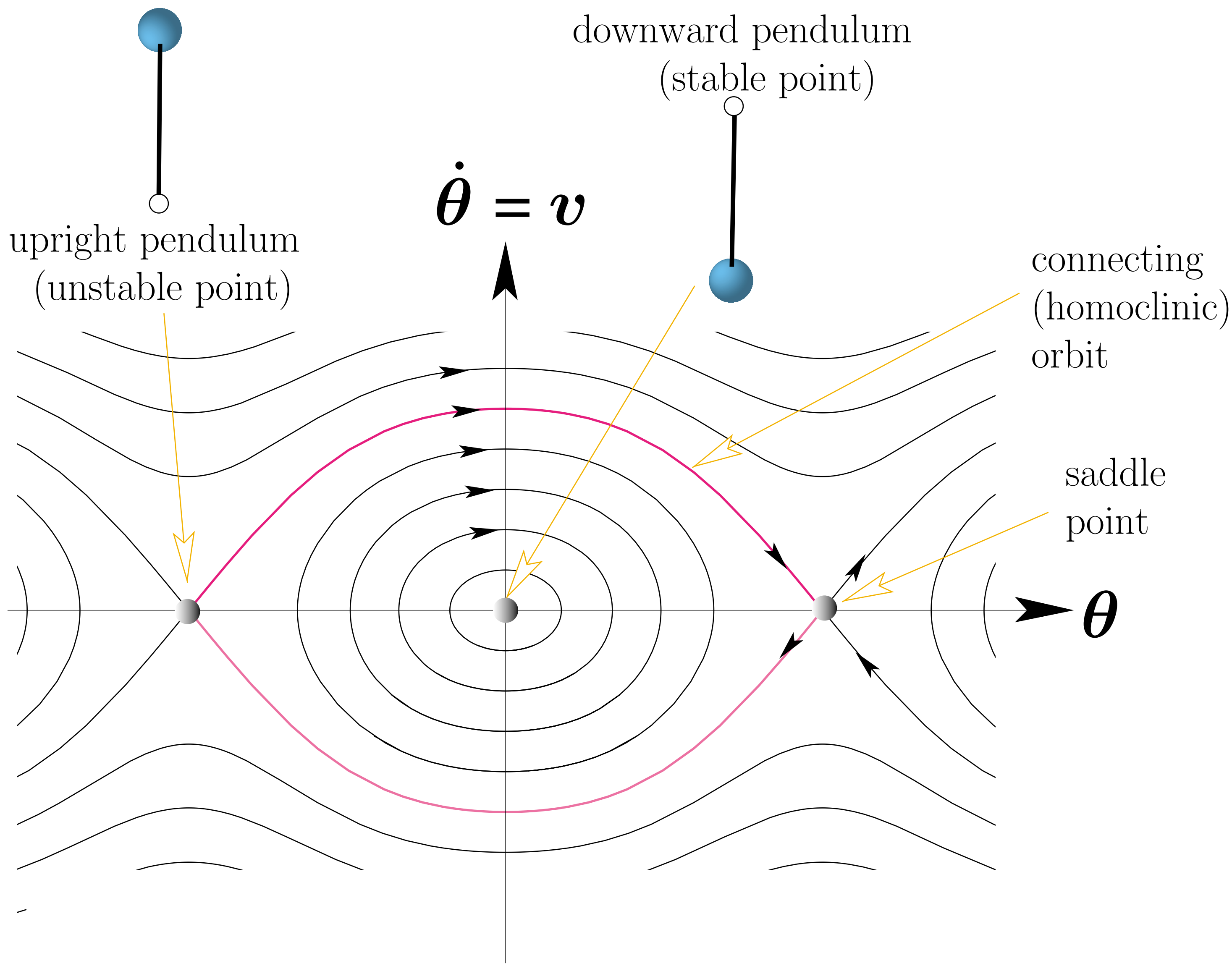
Connecting Orbits

Simple Pendulum

- Equations of a ***simple pendulum*** are $\ddot{\theta} + \sin\theta = 0$.
- Write as a ***system in the plane***;

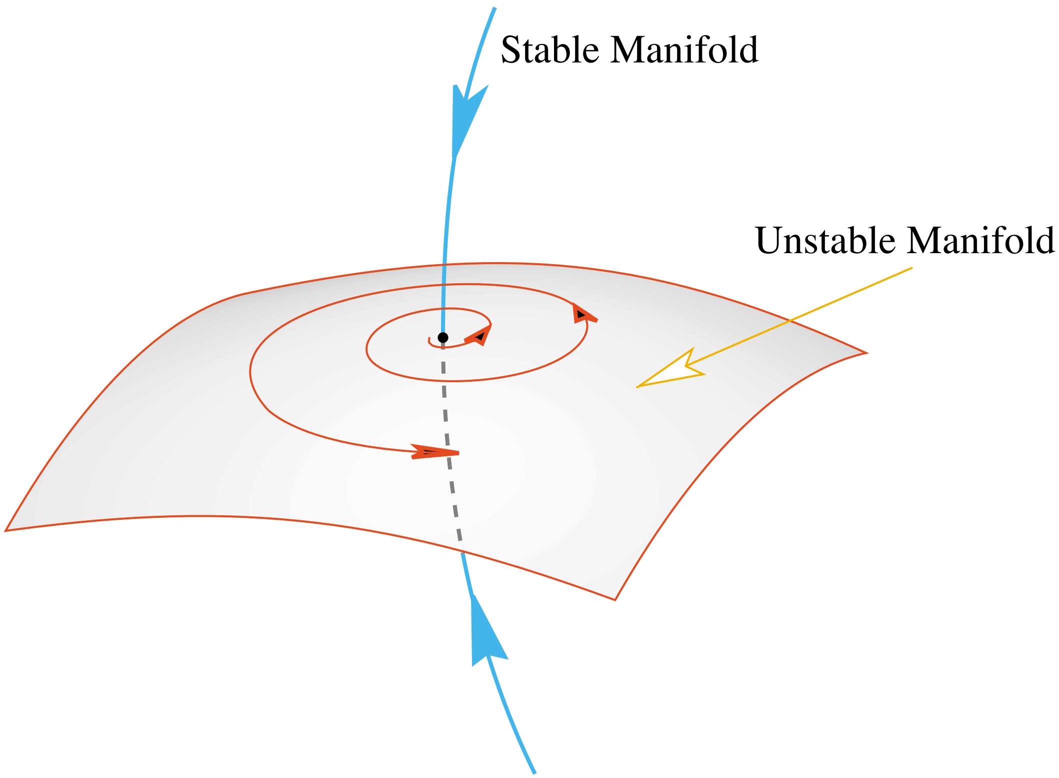
$$\frac{d\theta}{dt} = v$$
$$\frac{dv}{dt} = -\sin\theta$$

- Solutions are trajectories in the plane.
- The resulting ***phase portrait*** shows some important basic features:



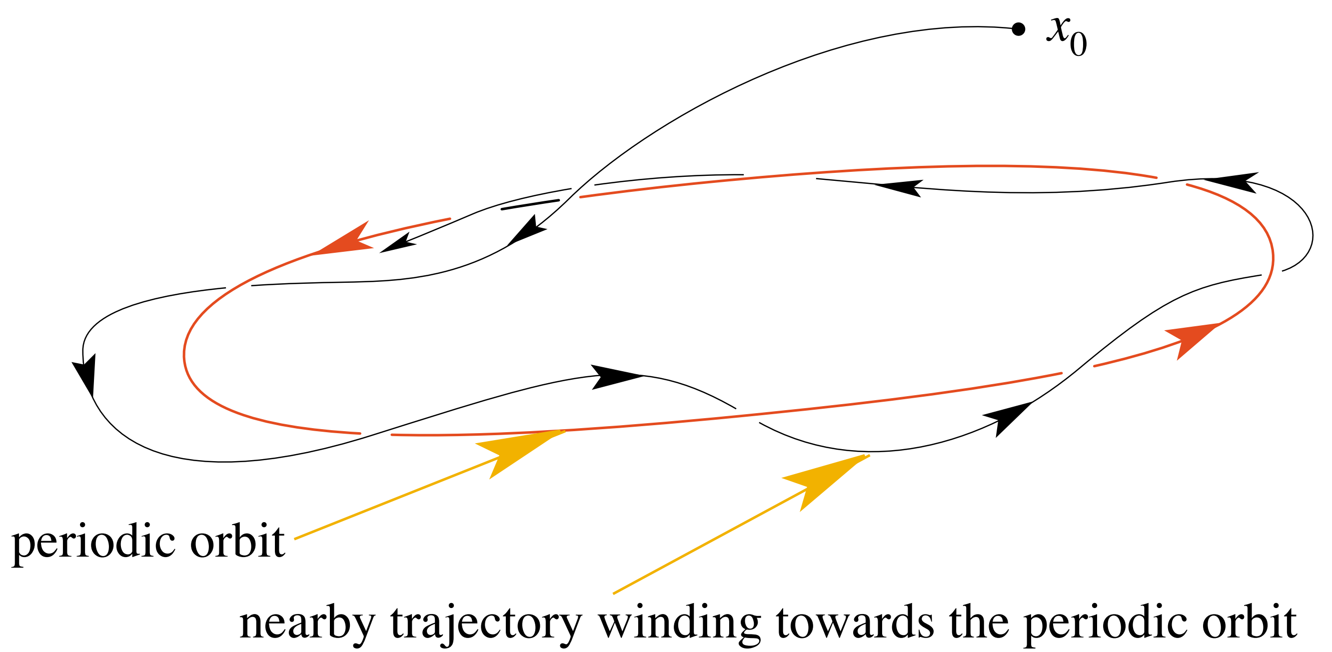
upright pendulum
(unstable point)
downward pendulum
(stable point)
$\dot{\theta} = v$
connecting
(homoclinic)
orbit
saddle
point
θ

Higher Dimensional Versions are Invariant Manifolds



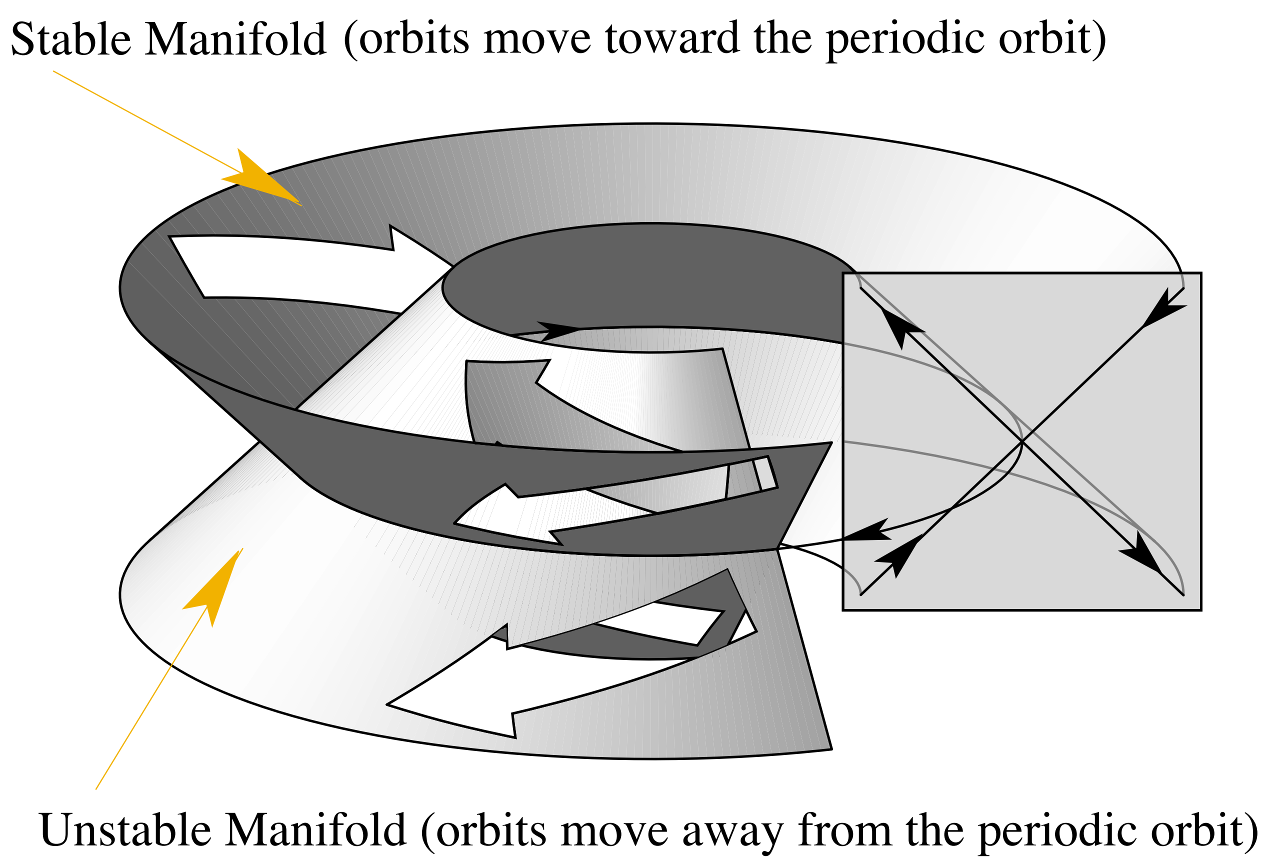
Periodic Orbits

- Can replace fixed points by ***periodic orbits*** and do similar things. For example, stability means nearby orbits stay nearby.

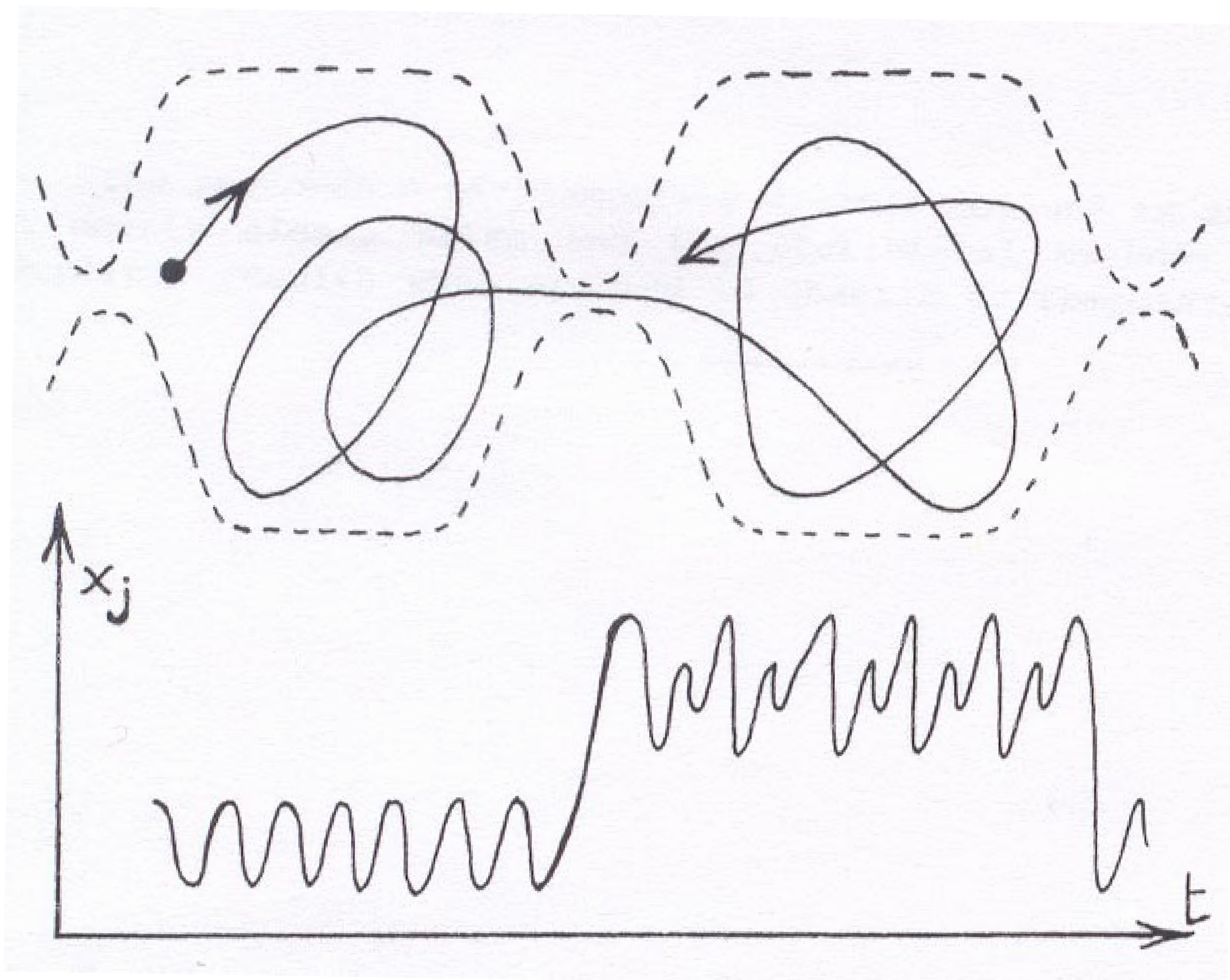


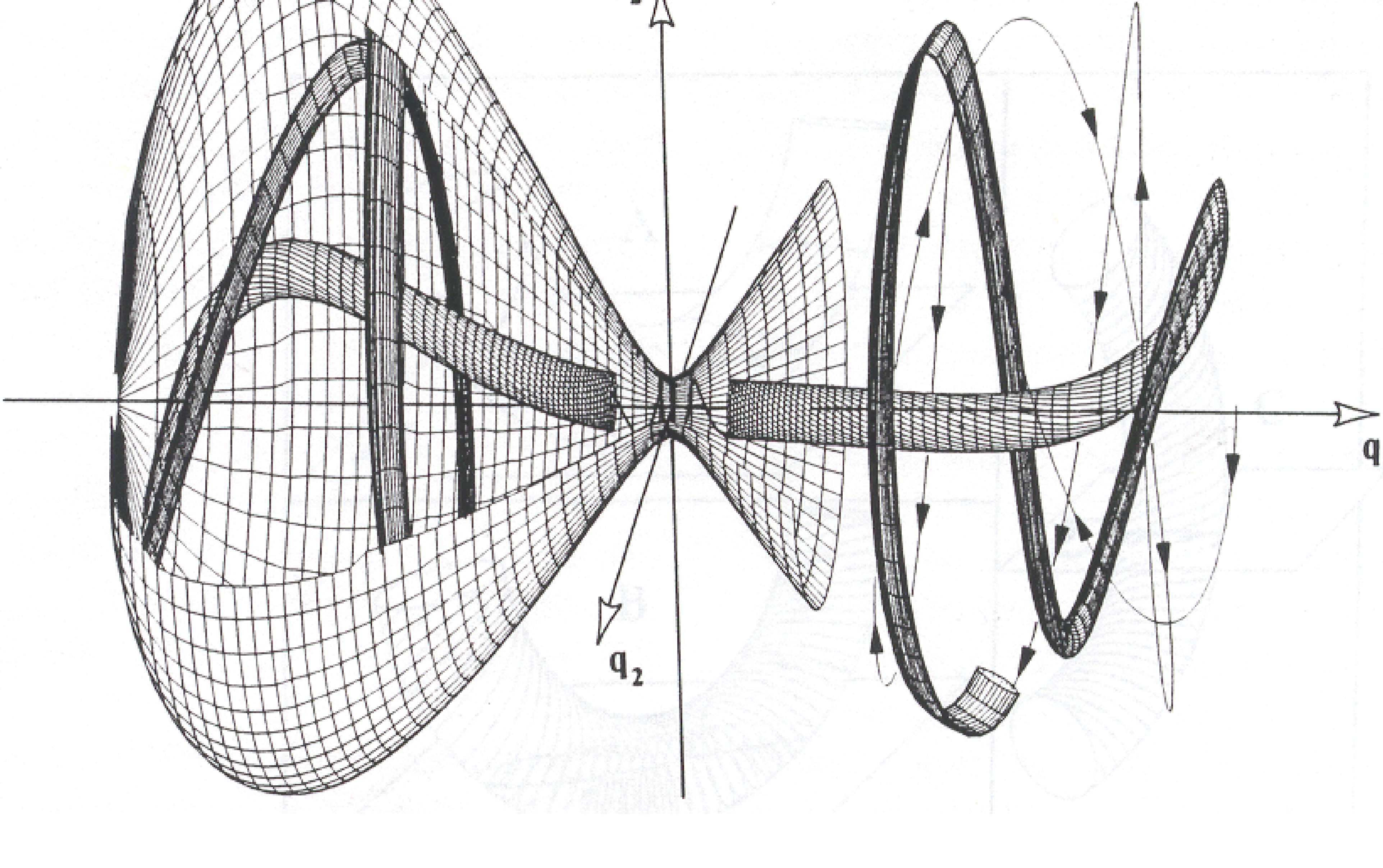
Invariant Manifolds for Periodic Orbits

- Periodic orbits have ***stable and unstable manifolds***.



Chaotic Motion and Intermittency



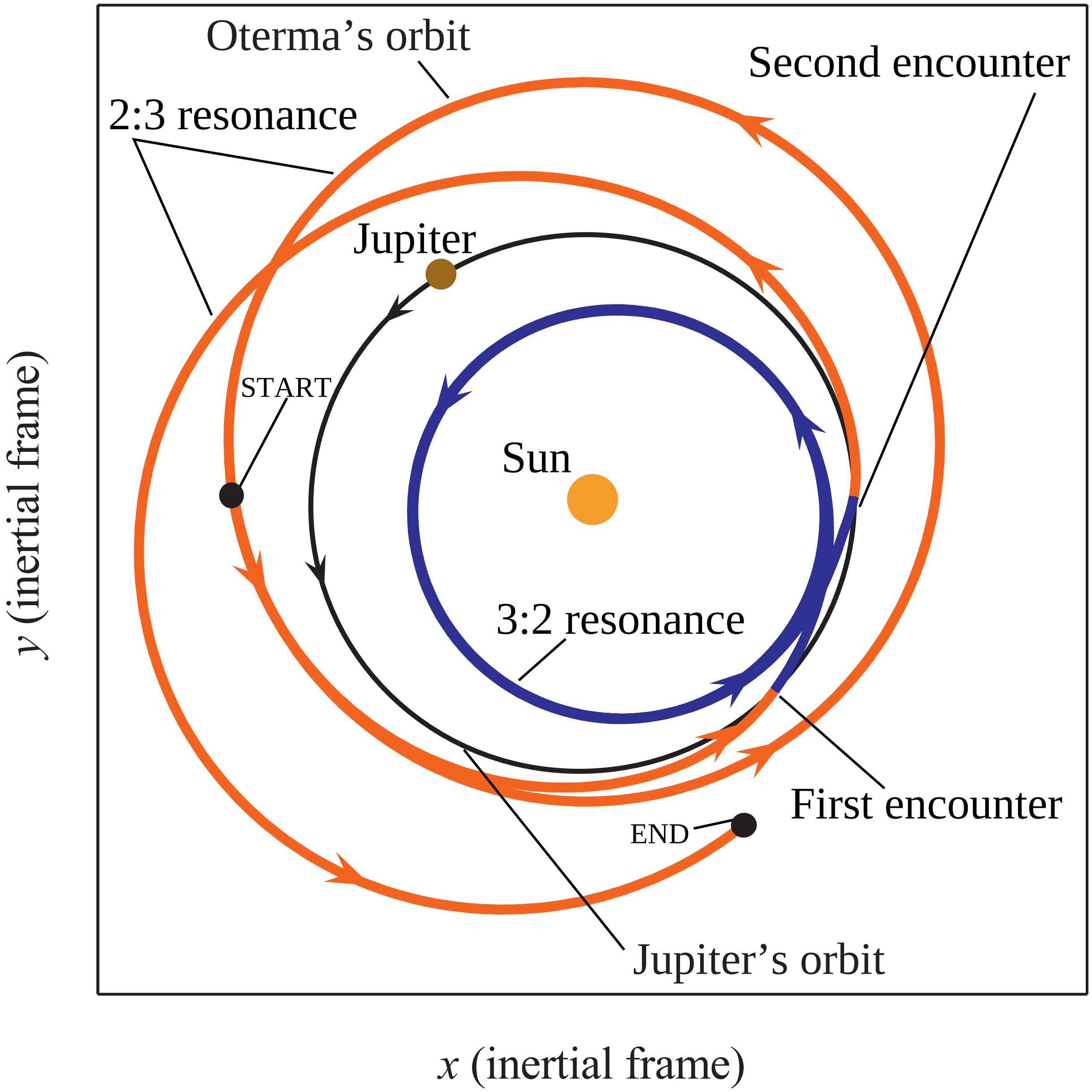


p_2
q_1
q_2

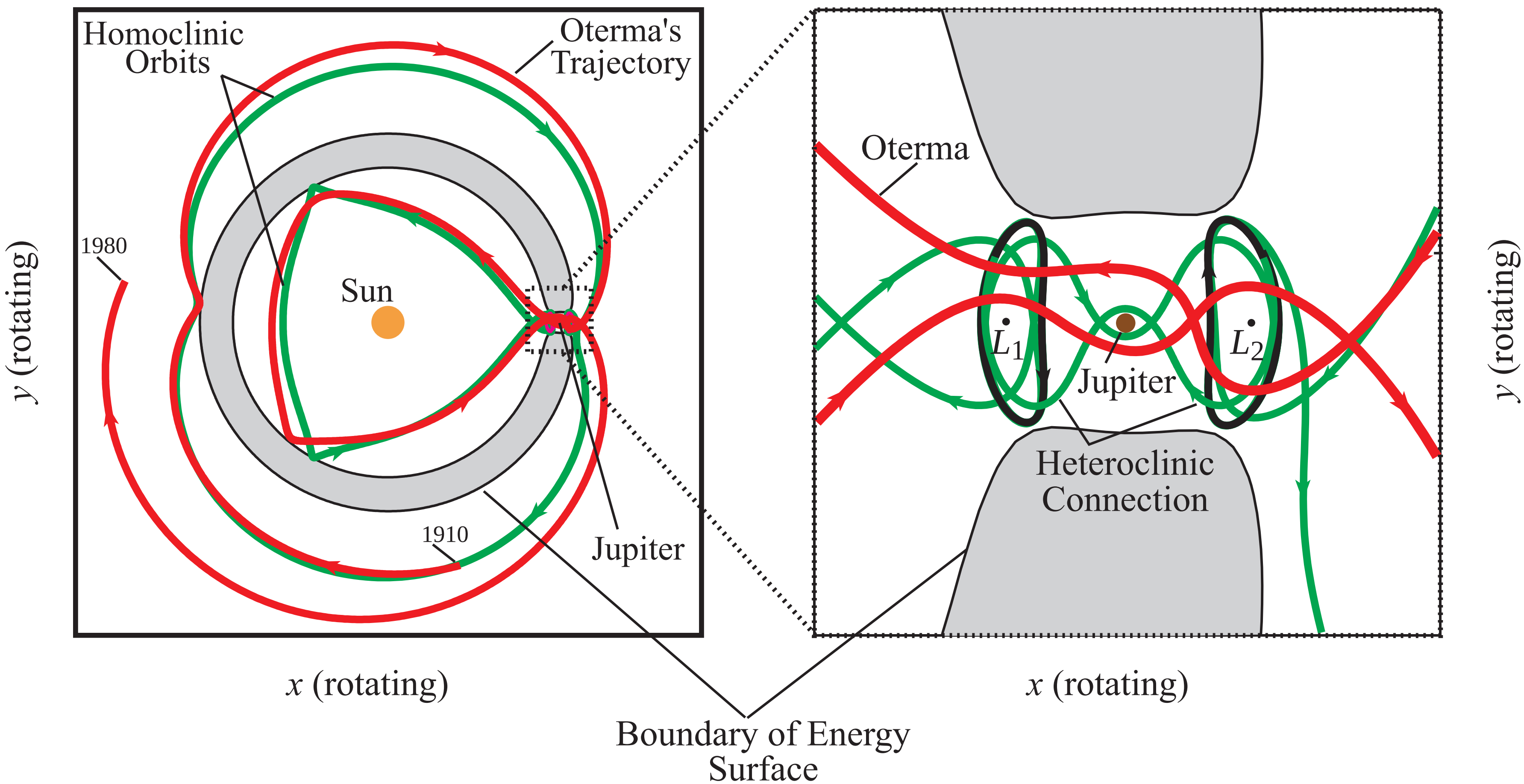
Motivation: Comet Transitions

Jupiter Comets–such as Oterma

- Comets moving in the vicinity of Jupiter do so mainly under the influence of Jupiter and the Sun–*i.e.*, in a three body problem.
- These comets sometimes make a ***rapid transition*** from **outside** to **inside** Jupiter's orbit.
- ***Captured temporarily*** by Jupiter during transition.
- **Exterior** (2:3 resonance) $\longrightarrow$ **Interior** (3:2 resonance).
- The next figure shows the orbit of Oterma (AD 1915–1980) in an inertial frame


Oterma's orbit
Second encounter
2:3 resonance
Jupiter
START
Sun
3:2 resonance
First encounter
END
Jupiter's orbit
y (inertial frame)
x (inertial frame)

- Next figure shows Oterma's orbit in a ***rotating frame*** (so Jupiter looks like it is standing still) and with some invariant manifolds in the ***three body problem*** superimposed.


Homoclinic Orbits
Oterma's Trajectory
1980
Sun
1910
Jupiter
x (rotating)
y (rotating)
Oterma
L1
L2
Jupiter
Heteroclinic Connection
x (rotating)
y (rotating)
Boundary of Energy Surface

Movie: Oterma in a rotating frame

Planar Circular Restricted 3-Body Problem–PCR3BP

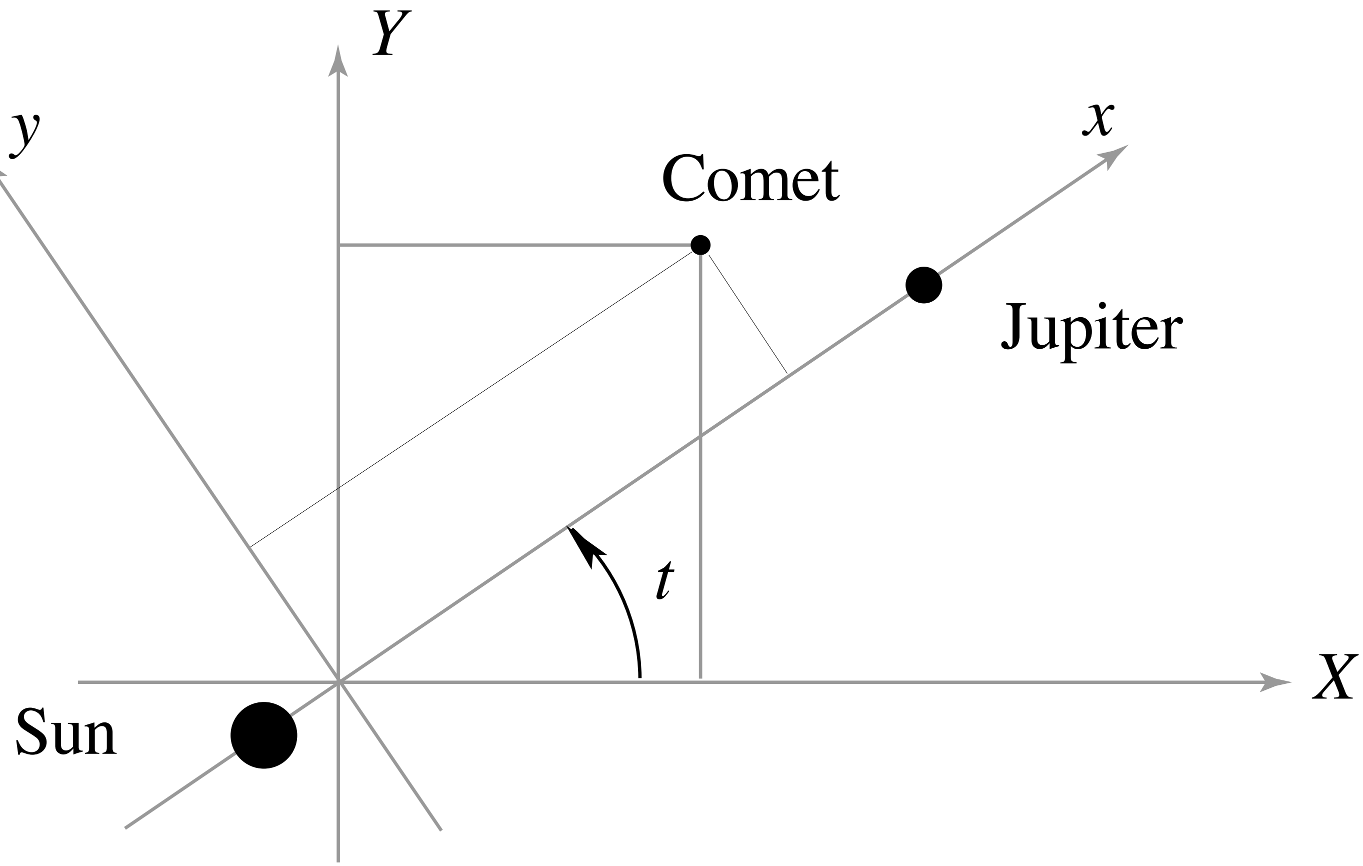
General Comments

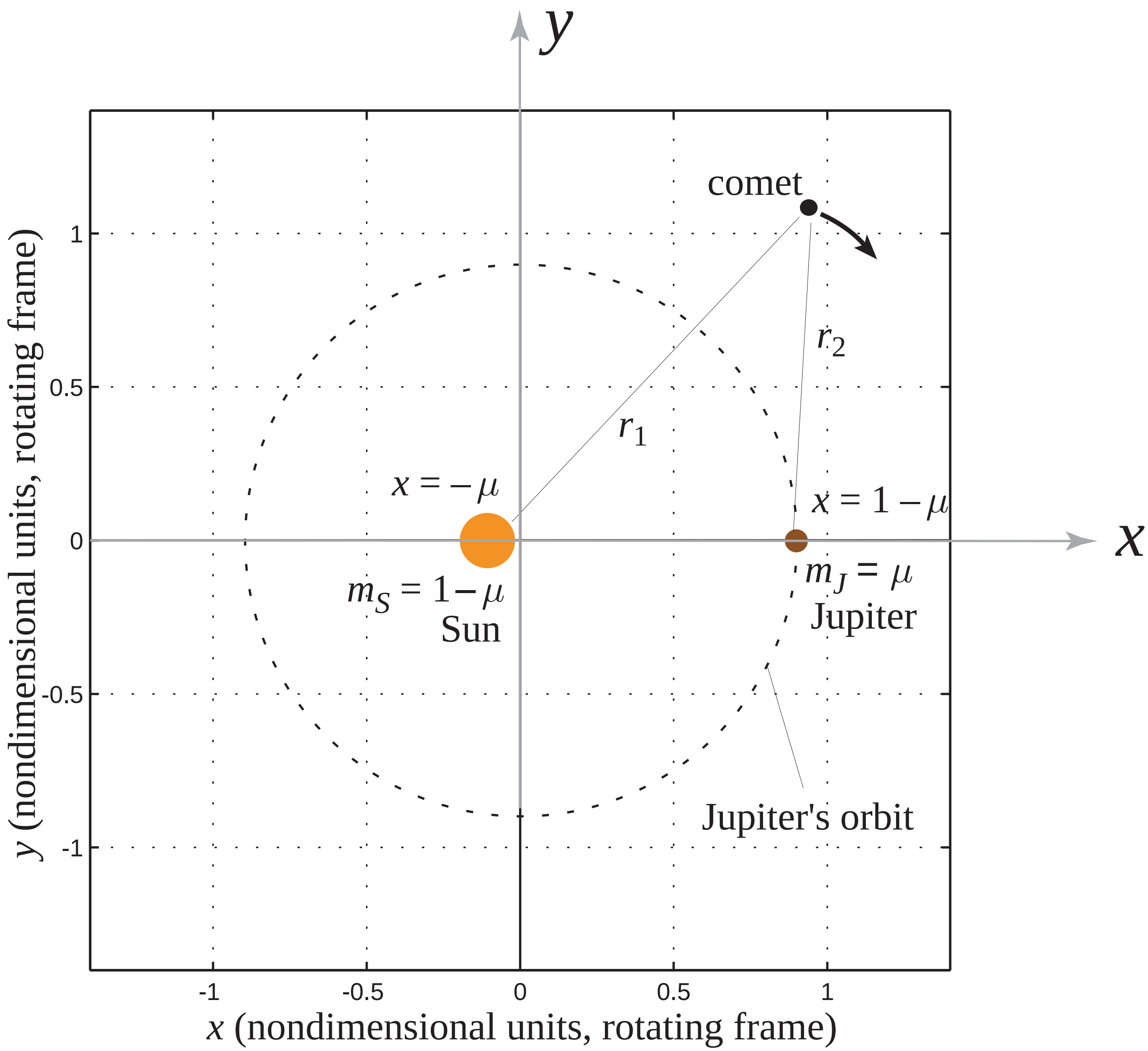
- The two main bodies could be the ***Sun and Jupiter***, or the ***Sun and Earth***, etc. The total mass is normalized to 1; they are denoted $m_S = 1 - \mu$ and $m_J = \mu$, so $0 < \mu \leq \frac{1}{2}$.
  - The two main bodies rotate in the plane in circles counterclockwise about their common center of mass and with angular velocity normalized to 1.
  - The third body, the ***comet or the spacecraft***, has mass zero and is free to move in the plane.
- The ***planar*** restricted three-body problem is used for simplicity. Generalization to the ***three-dimensional problem*** is of course important, but many of the effects can be described well with the planar model.

Equations of Motion

- **Notation:** Choose a ***rotating coordinate system*** so that
  - the origin is at the center of mass
  - the Sun and Jupiter are on the x-axis at the points $(-\mu, 0)$ and $(1-\mu, 0)$ respectively–i.e., the distance from the Sun to Jupiter is normalized to be 1.
  - Let (x, y) be the position of the comet in the plane relative to the positions of the Sun and Jupiter.
  - distances to the Sun and Jupiter:

$$r_1 = \sqrt{(x+\mu)^2 + y^2} \quad \text{and} \quad r_2 = \sqrt{(x-1+\mu)^2 + y^2}.$$


Y
y
x
Comet
Jupiter
t
X
Sun


y
comet
r_2
r_1
x = −μ
x = 1 − μ
x
m_S = 1−μ
Sun
m_J = μ
Jupiter
Jupiter's orbit
1
0.5
0
-0.5
-1
-1
-0.5
0
0.5
1
y (nondimensional units, rotating frame)
x (nondimensional units, rotating frame)

- **Lagrangian approach–rotating frame**: In the rotating frame, the Lagrangian L is given by

$$L(x, y, \dot{x}, \dot{y}) = \frac{1}{2}((\dot{x} - y)^2 + (x + \dot{y})^2) - U(x, y)$$

where the ***gravitational potential*** in rotating coordinates is

$$U = -\frac{1-\mu}{r_1} - \frac{\mu}{r_2}.$$

Reason:

$$\begin{aligned} \dot{X} &= (\dot{x} - y)\cos t - (x + \dot{y})\sin t, \\ \dot{Y} &= (x + \dot{y})\cos t - (\dot{x} - y)\sin t \end{aligned}$$

which yields kinetic energy (wrt inertial frame)
$\dot{X}^2 + \dot{Y}^2 = (\dot{x} - y)^2 + (x + \dot{y})^2$.

Also, since both the distances r_1 and r_2 are invariant under rotation, we have

$$
\begin{aligned}
r_1^2 &= (x+\mu)^2 + y^2, \\
r_2^2 &= (x-(1-\mu))^2 + y^2.
\end{aligned}
$$

- The theory of ***moving systems*** says that one can simply write down the Euler-Lagrange equations in the rotating frame and one will get the correct equations. It is a very efficient general method for computing equations for either moving systems or for systems seen from rotating frames (see Marsden & Ratiu, 1999).
- In the present case, the Euler-Lagrange equations are given by

$$
\begin{aligned}
\frac{d}{dt}(\dot{x} - y) &= x + \dot{y} - U_x, \\
\frac{d}{dt}(x + \dot{y}) &= -\dot{x} + y - U_y.
\end{aligned}
$$

- After simplification, we have the ***equations of motion:***

$$\boxed{\ddot{x} - 2\dot{y} = -U_x^{\text{eff}}, \qquad \ddot{y} + 2\dot{x} = -U_y^{\text{eff}}}$$

where

$$U^{\text{eff}} = -\frac{(x^2 + y^2)}{2} - \frac{1-\mu}{r_1} - \frac{\mu}{r_2}.$$

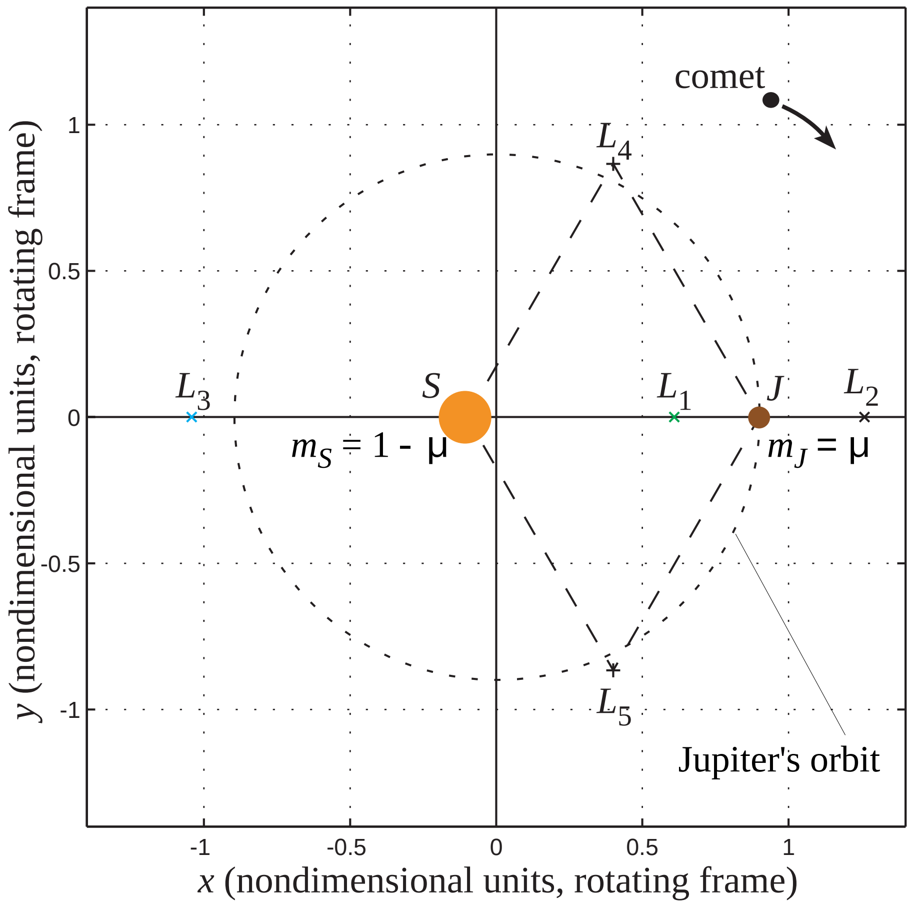
- They have a first integral, the ***Hamiltonian energy***, given by

$$E(x, y, \dot{x}, \dot{y}) = \frac{1}{2}(\dot{x}^2 + \dot{y}^2) + U^{\text{eff}}(x, y).$$

- ***Energy manifolds*** are 3-dimensional surfaces foliating the 4-dimensional phase space.
- For fixed energy, ***Poincaré sections*** are then 2-dimensional, making visualization of intersections between sets in the phase space particularly simple.

Five Equilibrium Points

- Three ***collinear*** (Euler, 1767) on the x-axis— L_1, L_2, L_3
- Two ***equilateral points*** (Lagrange, 1772)— L_4, L_5.



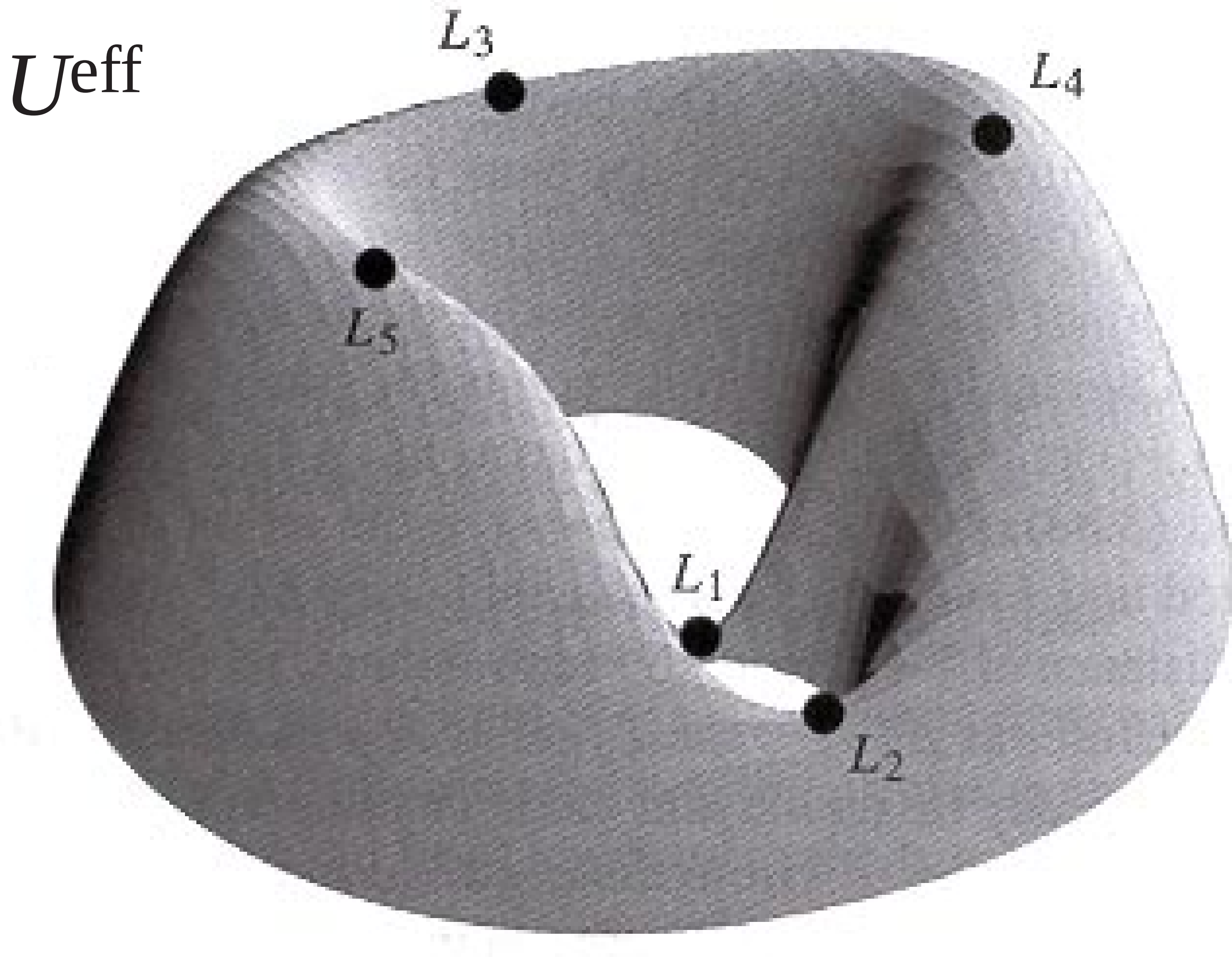
Energy Manifold

- The energy E is given by

$$
\begin{aligned}
E(x, y, \dot{x}, \dot{y}) &= \frac{1}{2}(\dot{x}^2 + \dot{y}^2) + U^{\text{eff}}(x, y) \\
&= \frac{1}{2}(\dot{x}^2 + \dot{y}^2) - \frac{1}{2}(x^2 + y^2) - \frac{1-\mu}{r_1} - \frac{\mu}{r_2}.
\end{aligned}
$$

  This energy integral will help us determine the region of ***possible motion***, i.e., the region in which the comet can possibly move along and the region which it is forbidden to move. The first step is to look at the surface of the effective potential U^{eff}.

- Note that the energy manifold is 3-dimensional.


U^{eff}
L_3
L_4
L_5
L_1
L_2

- Near either the Sun or Jupiter, we have a potential well.
- Far away from the Sun-Jupiter system, the term that corresponds to the centrifugal force dominates, we have another potential well.
- Moreover, by applying multivariable calculus, one finds that there are 3 saddle points at L_1, L_2, L_3 and 2 maxima at L_4 and L_5.
- Let E_i be the energy at L_i, then $E_5 = E_4 > E_3 > E_2 > E_1$.

- Let $\mathcal{M}$ be the ***energy surface*** given by setting the energy integral equal to a constant, i.e.,

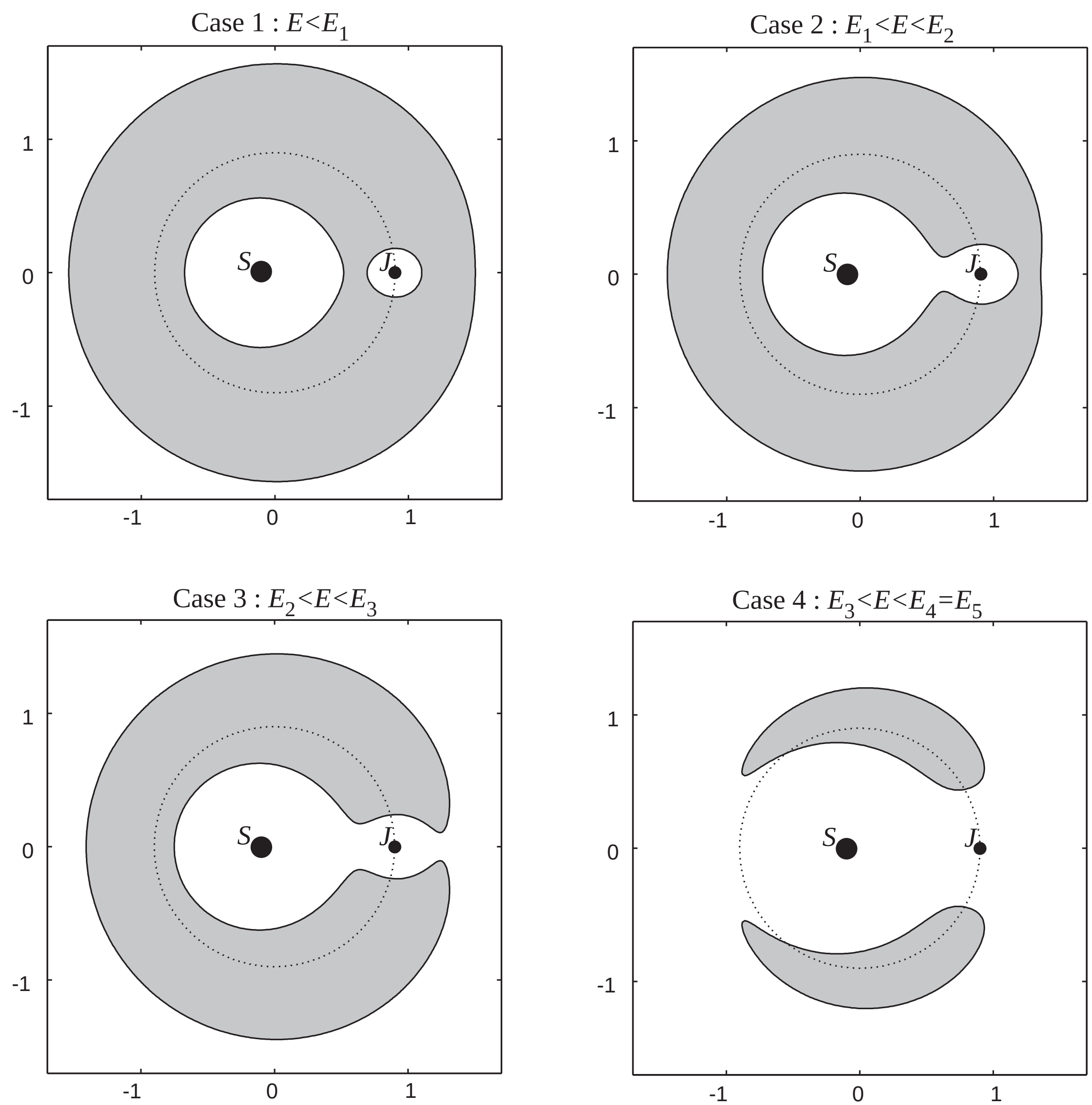
$$\mathcal{M}(\mu, e) = \{(x, y, \dot{x}, \dot{y}) \mid E(x, y, \dot{x}, \dot{y}) = e\} \tag{1}$$

where e is a constant.

- The projection of this surface onto position space is called a ***Hill's region***

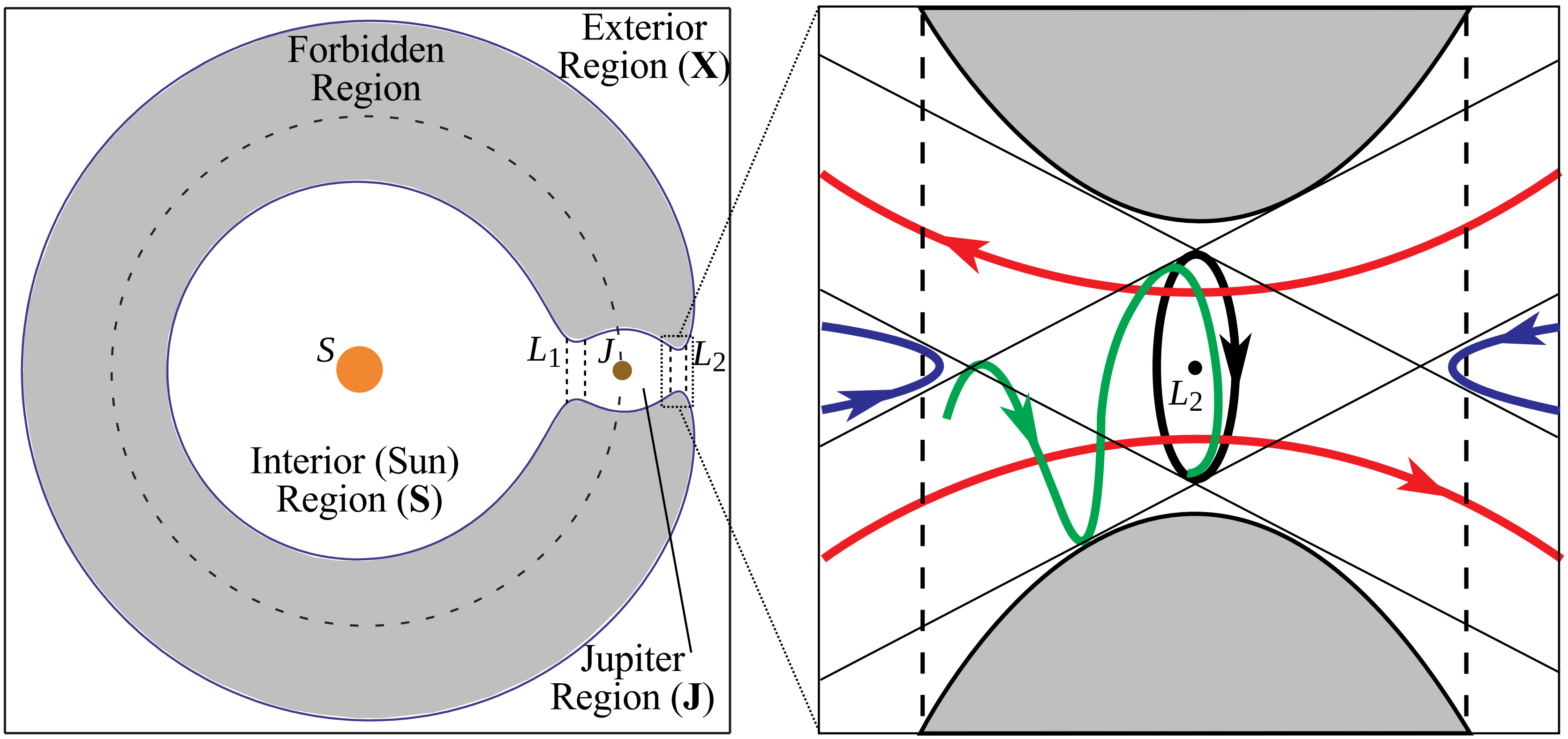
$$M(\mu, e) = \{(x, y) \mid U^{\text{eff}}(x, y) \leq e\}. \tag{2}$$

The boundary of $M(\mu, e)$ is the ***zero velocity curve***. The comet can move only within this region in the (x, y)-plane. For a given μ there are five basic configurations for the Hill's region, the first four of which are shown in the following figure.



Case 1 : $E<E_1$
Case 2 : $E_1<E<E_2$
Case 3 : $E_2<E<E_3$
Case 4 : $E_3<E<E_4=E_5$
S
J
1
0
-1

- [Conley] Orbits with energy just above that of L_2 can be **transit orbits**, passing through the neck region between the ***exterior region*** (outside Jupiter's orbit) and the ***temporary capture region*** (bubble surrounding Jupiter). They can also be **non-transit orbits** or **asymptotic orbits**.



Flow in the L_1 and L_2 Bottlenecks: Linearization

- [Moser] All the qualitative results of the linearized equations carry over to the full nonlinear equations.
- Recall equations of PCR3BP:

$$\dot{x} = v_x, \qquad \dot{v}_x = 2v_y - U_x^{\text{eff}},$$
$$\dot{y} = v_y, \qquad \dot{v}_y = -2v_x - U_y^{\text{eff}}.$$

- After linearization,

$$\dot{x} = v_x, \qquad \dot{v}_x = 2v_y + ax,$$
$$\dot{y} = v_y, \qquad \dot{v}_y = -2v_x - by.$$

  - Eigenvalues have the form $\pm\lambda$ and $\pm i\nu$.

- Corresponding eigenvectors are

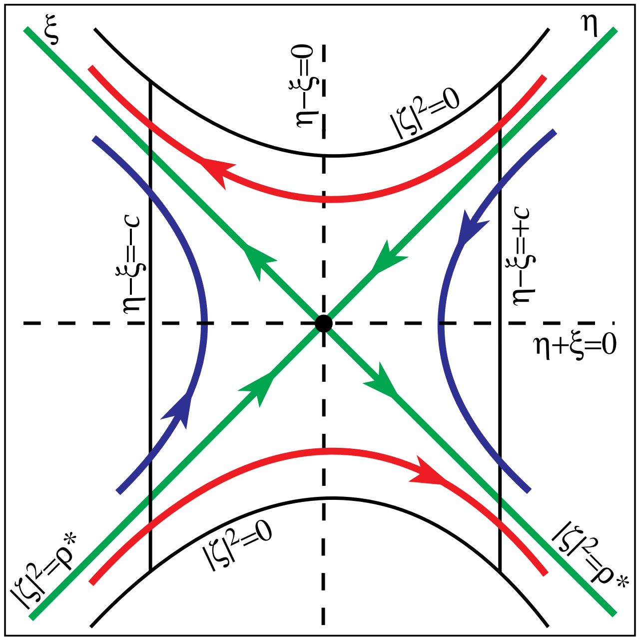
$$
\begin{aligned}
u_1 &= (1, -\sigma, \lambda, -\lambda\sigma), \\
u_2 &= (1, \sigma, -\lambda, -\lambda\sigma), \\
w_1 &= (1, -i\tau, i\nu, \nu\tau), \\
w_2 &= (1, i\tau, -i\nu, \nu\tau).
\end{aligned}
$$

- After ***linearization*** and making the ***eigenvectors*** the new coordinate axes, the equations of motion assume the simple form

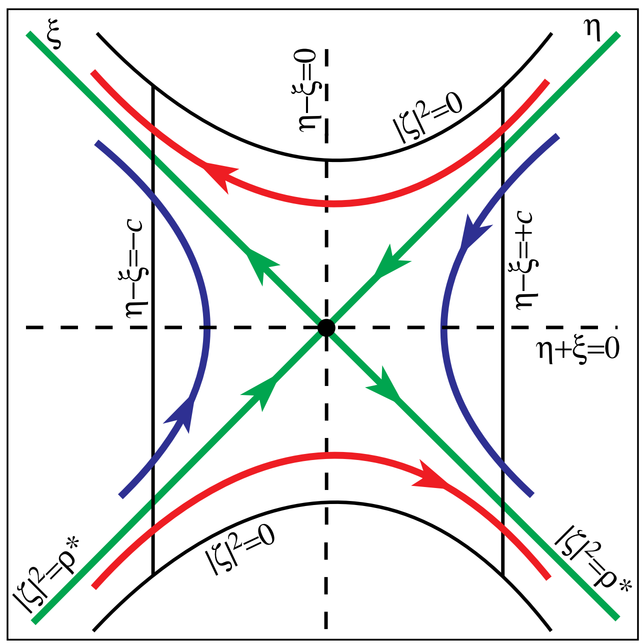
$$\dot{\xi} = \lambda\xi, \qquad \dot{\eta} = -\lambda\eta, \qquad \dot{\zeta}_1 = \nu\zeta_2, \qquad \dot{\zeta}_2 = -\nu\zeta_1,$$

with ***energy function*** $E_l = \lambda\eta\xi + \frac{\nu}{2}(\zeta_1^2 + \zeta_2^2)$.

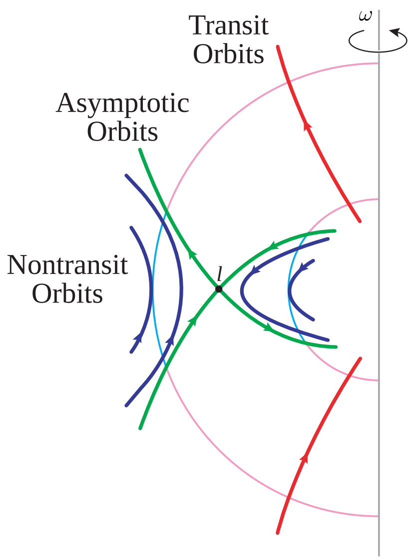
- The ***flow*** near L_1, L_2 has the form of a ***saddle***$\times$***center***.



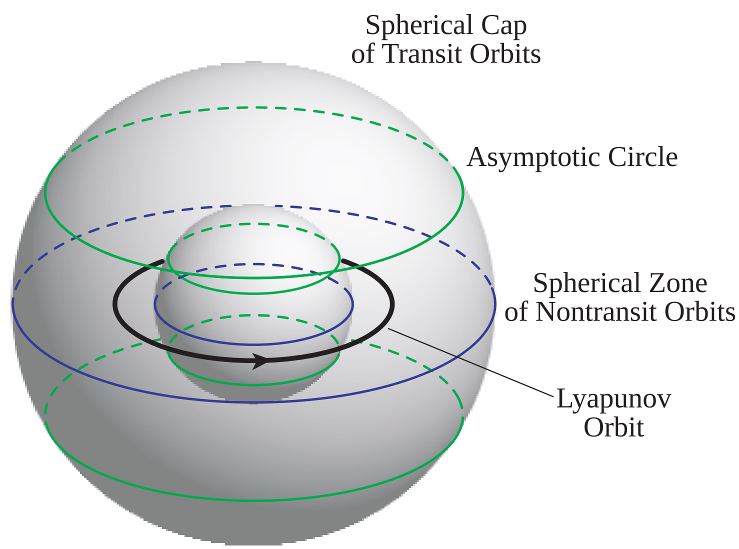
- For each fixed value of $\eta - \xi$ (vertical lines in figure below), $E_l = \mathcal{E}$ describes a ***2-sphere***.
- The ***equilibrium region*** $\mathcal{R}$ on the 3D energy manifold is homeomorphic to $S^2 \times I$.



- [McGehee] Can visualize **4 types** of orbits in $\mathcal{R} \simeq S^2 \times I$.
  - **Black** circle is the unstable **periodic** Lyapunov orbit.
  - 4 cylinders of **asymptotic** orbits form pieces of stable and unstable manifolds. They intersect the bounding spheres at asymptotic circles, separating ***spherical polar caps***, which contain **transit** orbits, from ***spherical equatorial zones***, which contain **nontransit** orbits.

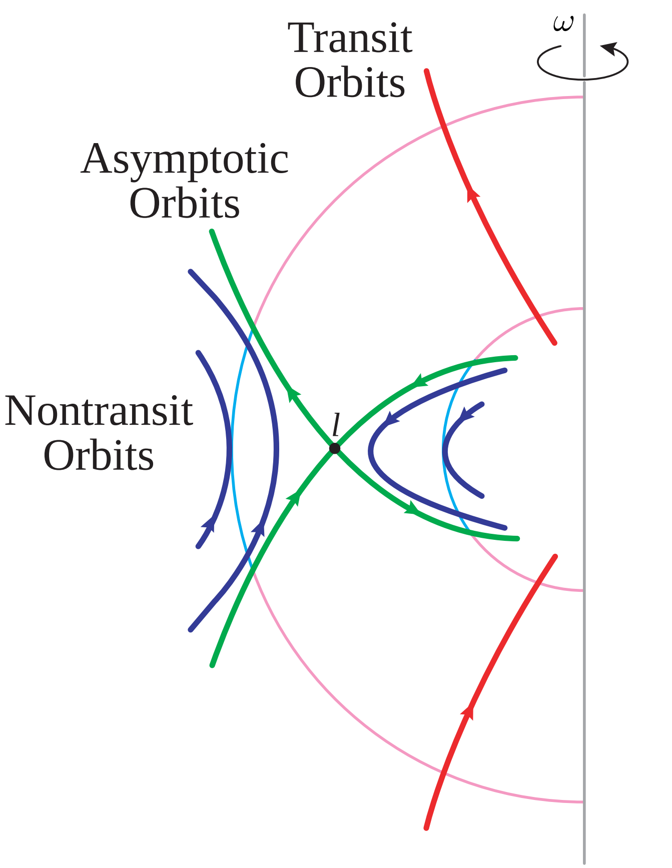


Cross-section of Equilibrium Region

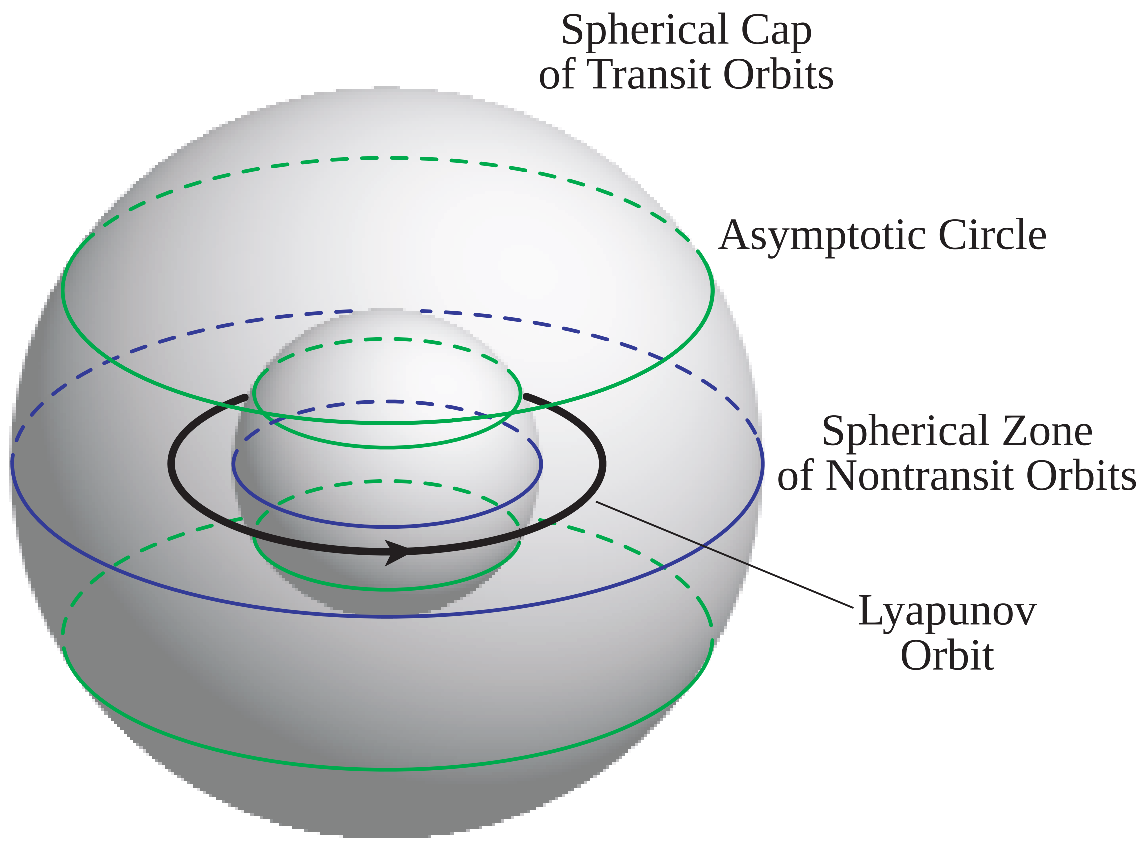


Equilibrium Region

- Roughly speaking, for fixed energy, the equilibrium region has the dynamics of a ***saddle*** $\times$ ***harmonic oscillator***.

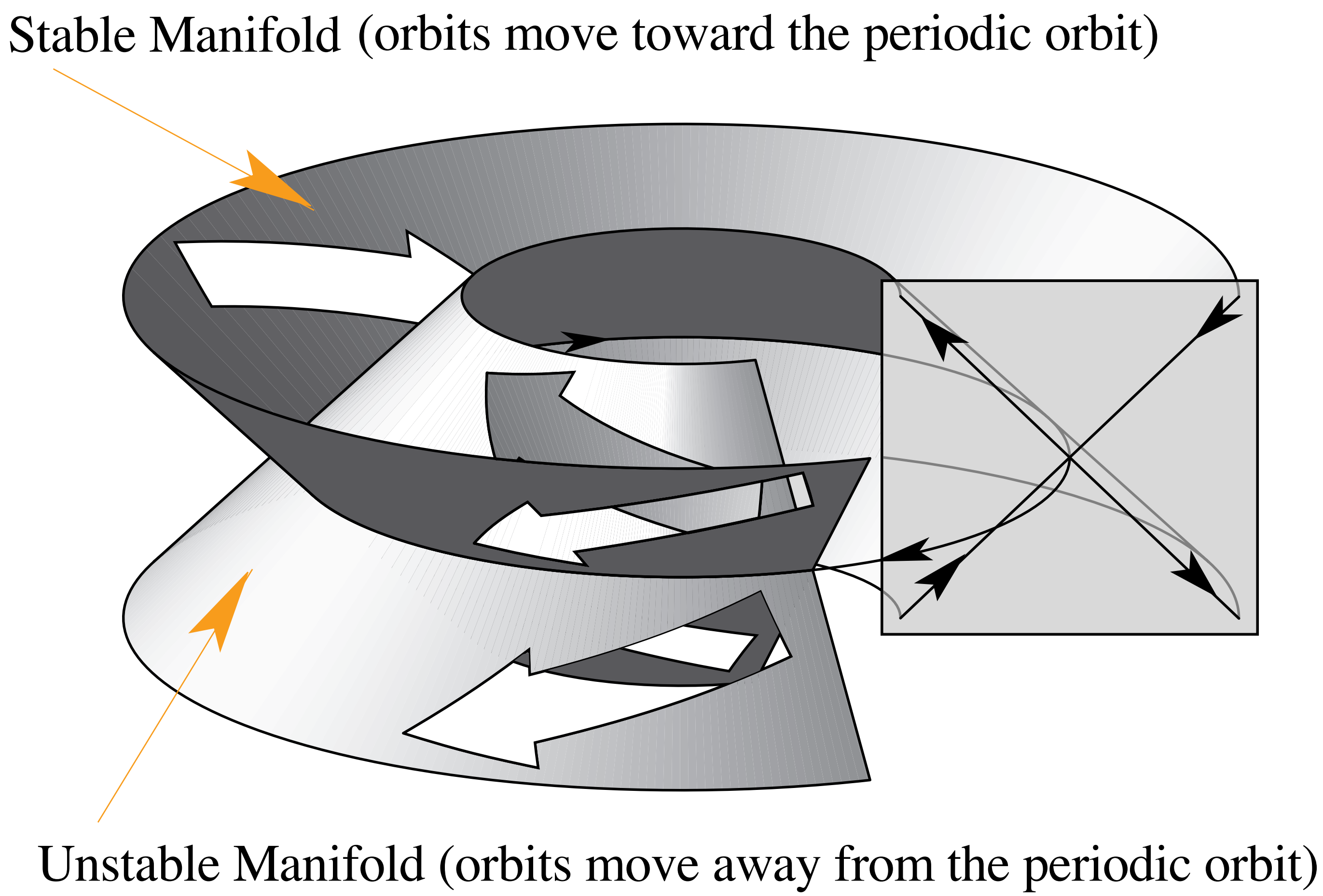


Cross-section of Equilibrium Region



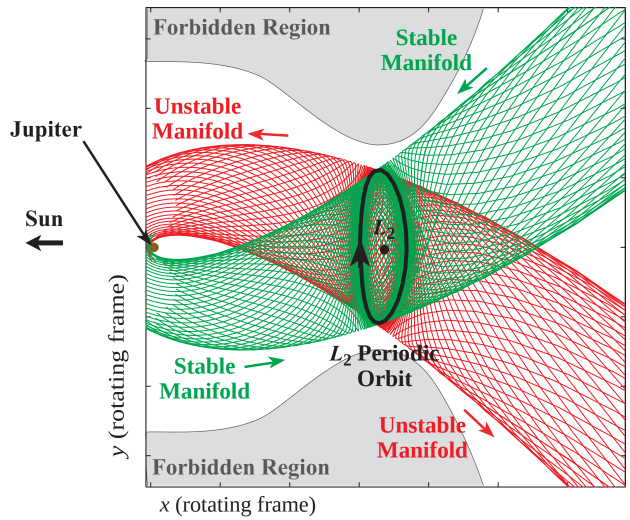
Equilibrium Region

- 4 cylinders of asymptotic orbits: ***stable and unstable manifolds***.

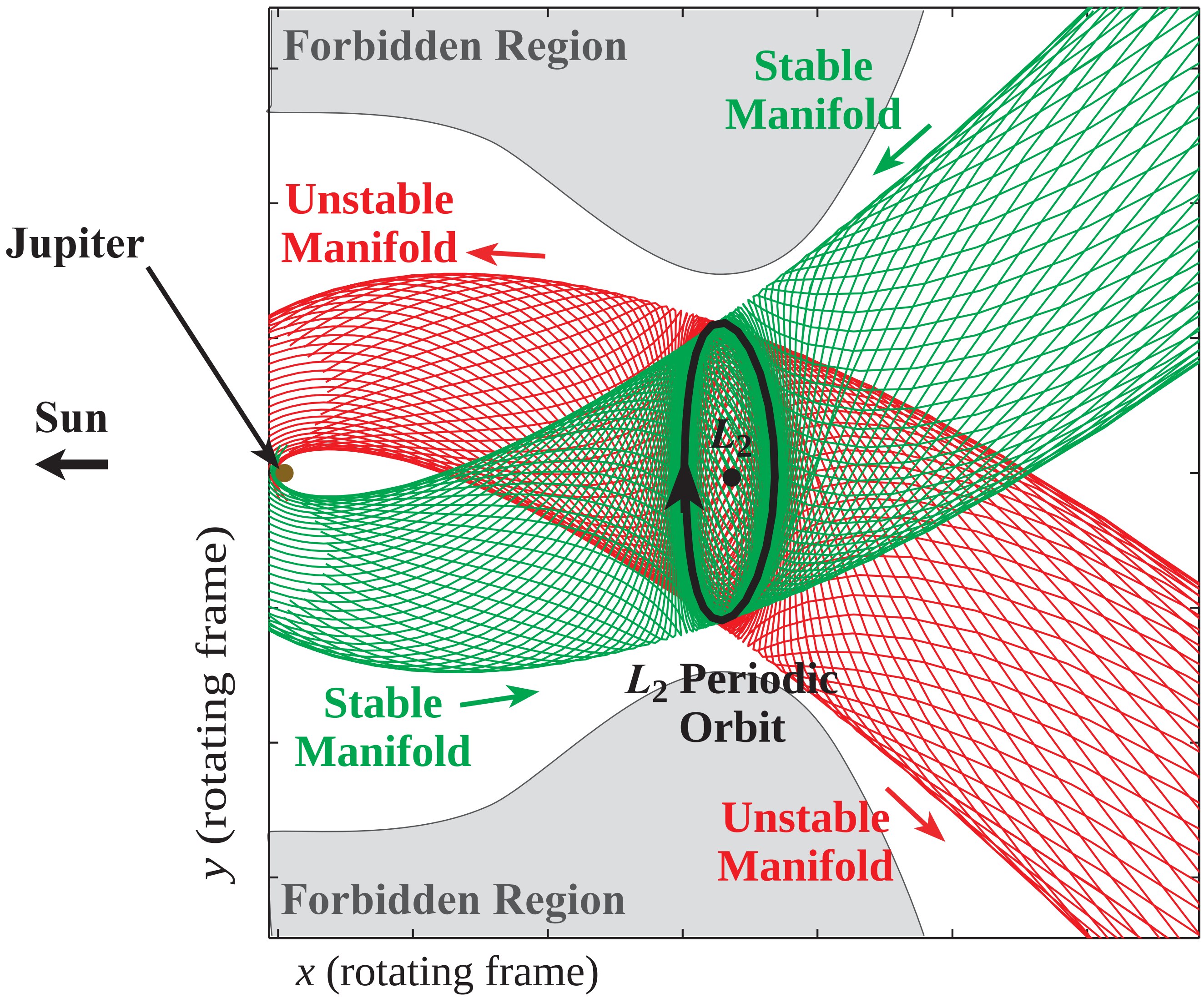


Invariant Manifold Tubes Partition the Energy Surface

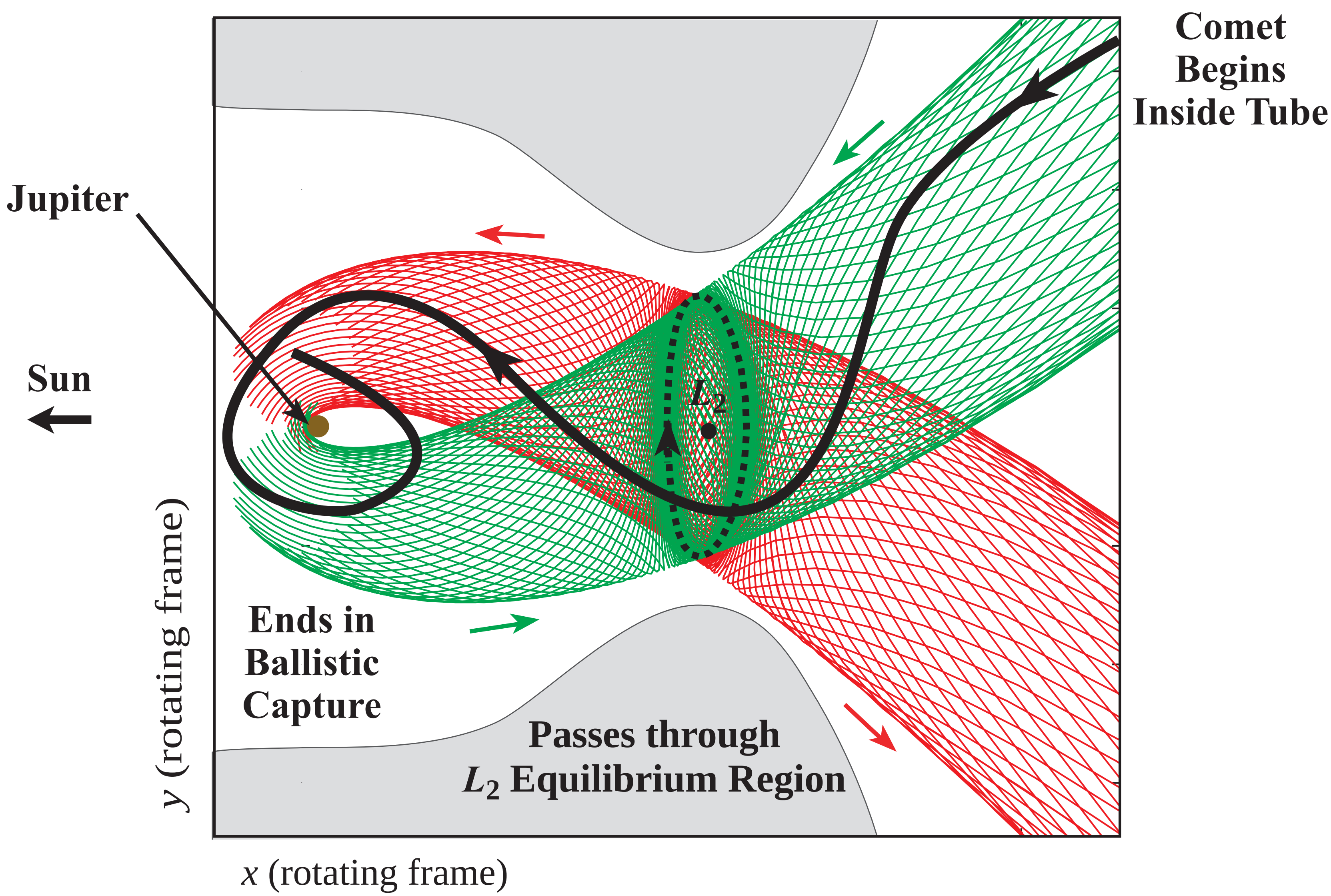
- **Stable** and **unstable** ***manifold tubes*** act as ***separatrices*** for the flow in the equilibrium region.
  - Those inside the tubes are ***transit*** orbits.
  - Those outside the tubes are ***nontransit*** orbits.
  - e.g., transit from outside Jupiter's orbit to Jupiter capture region possible ***only*** through L_2 periodic orbit stable tube.



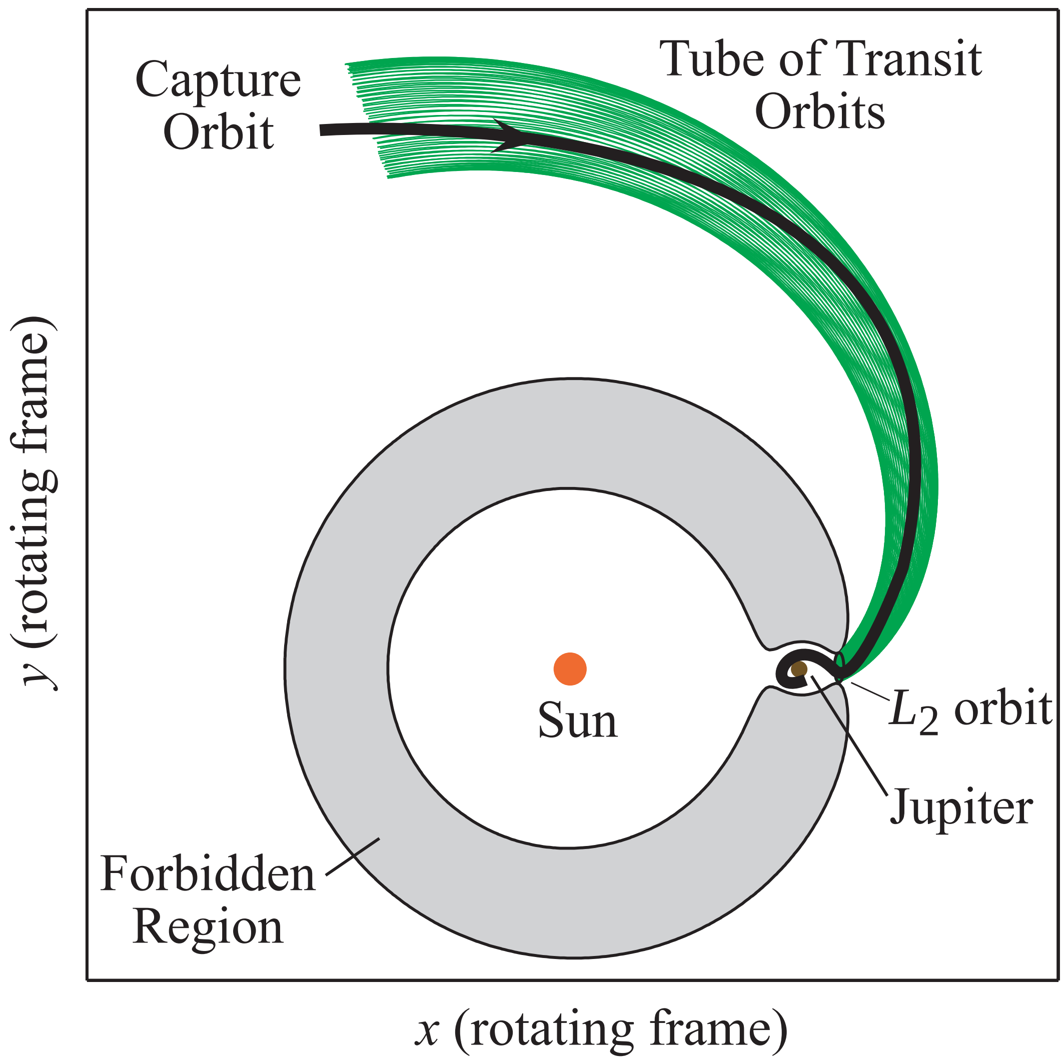
- **Stable** and **unstable** manifold tubes control the ***transport*** of material to and from the capture region.



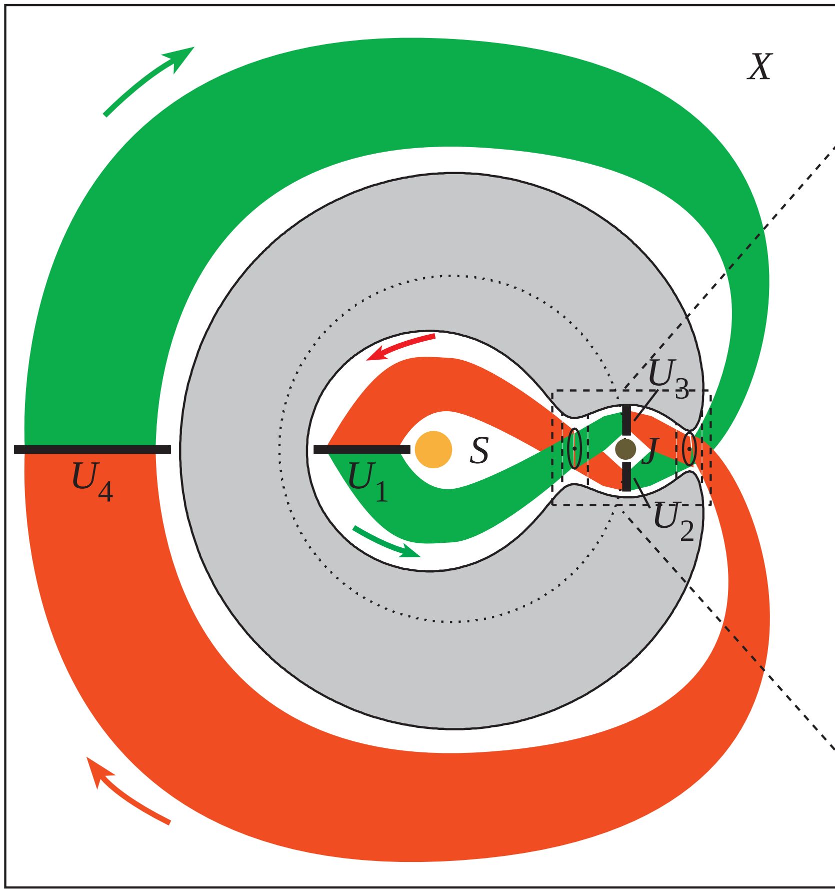
- Tubes of transit orbits contain ***ballistic capture*** orbits.

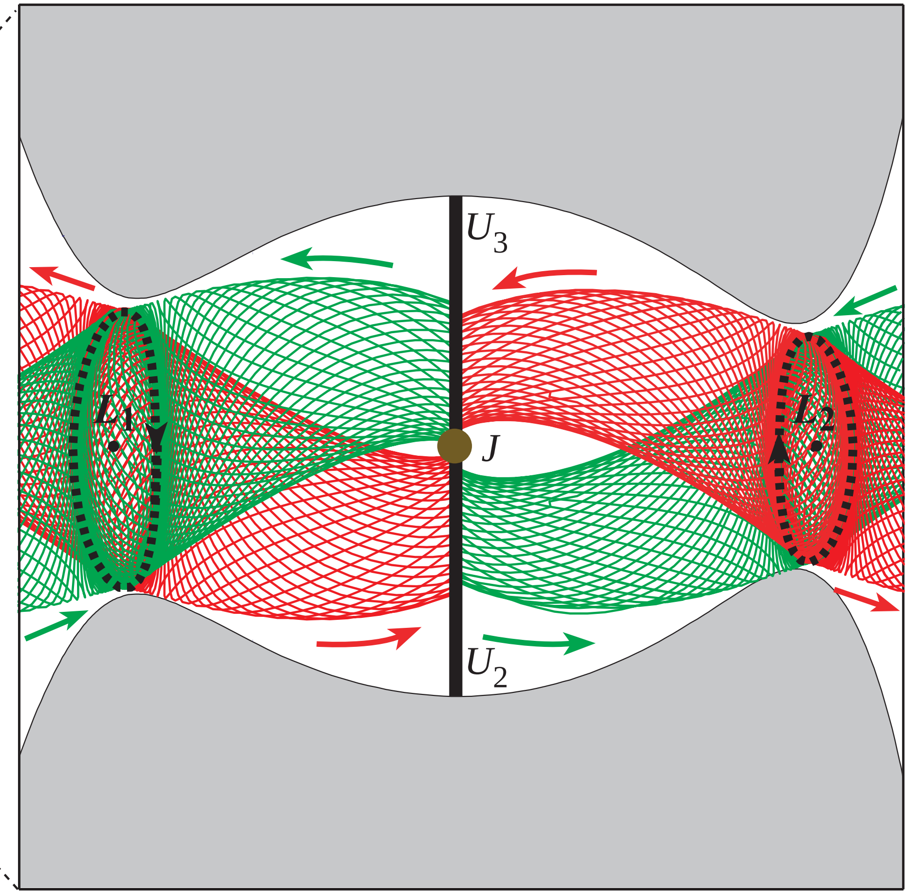


- Invariant manifold tubes are ***global objects*** — extend far beyond vicinity of libration points.

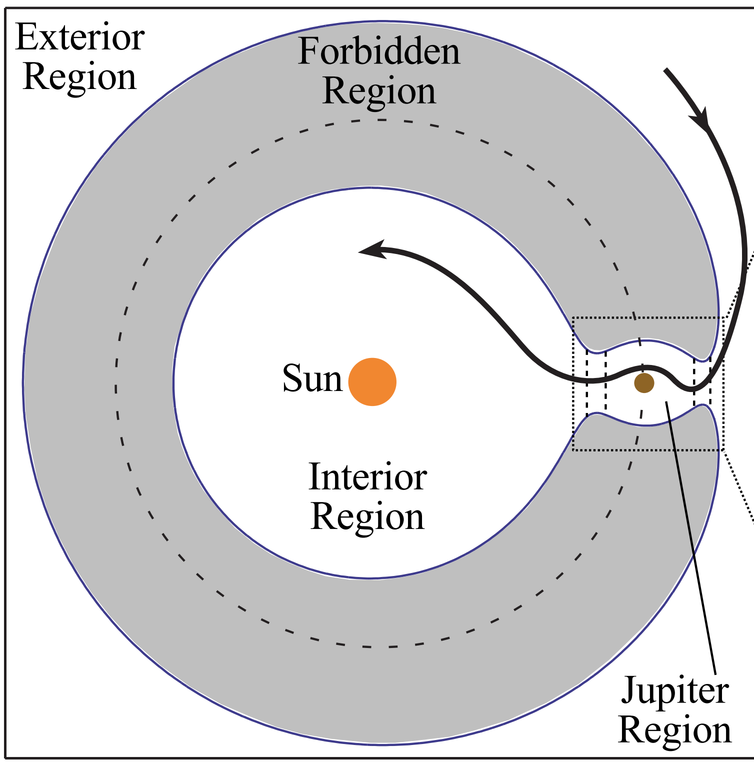


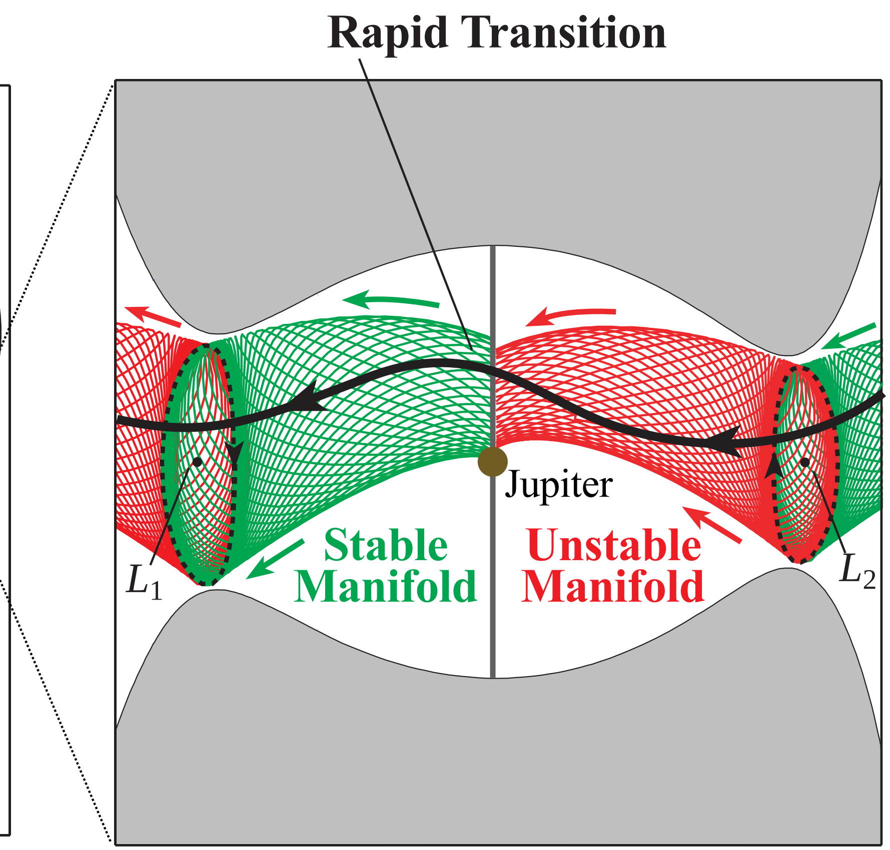
- Transport between all three regions (interior, Jupiter, exterior) is controlled by the intersection of stable and unstable manifold tubes.



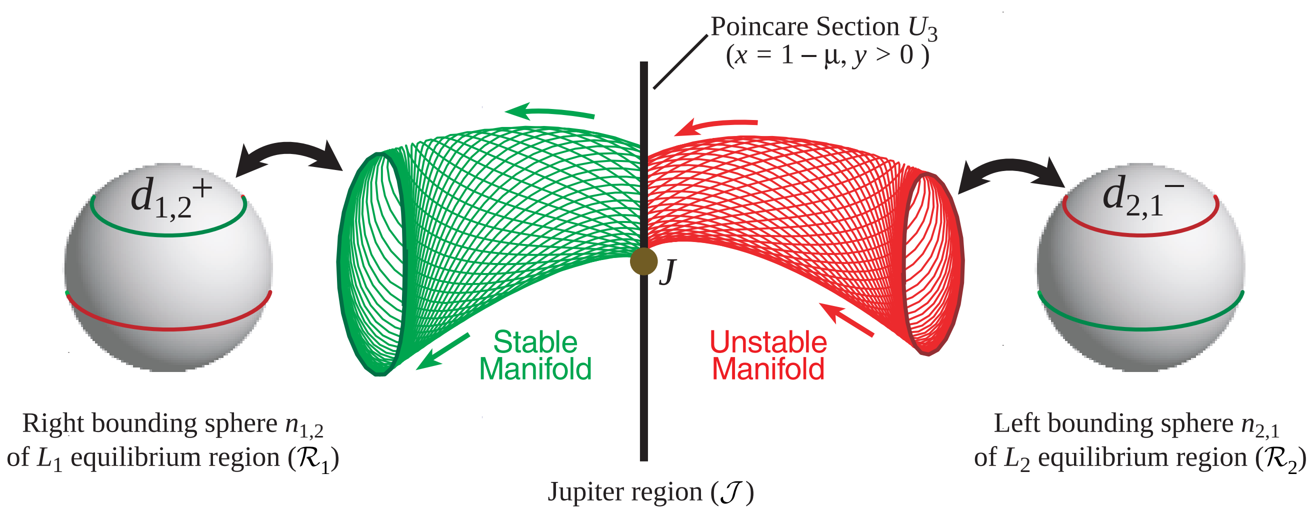


- In particular, rapid transport between outside and inside of Jupiter's orbit is possible.



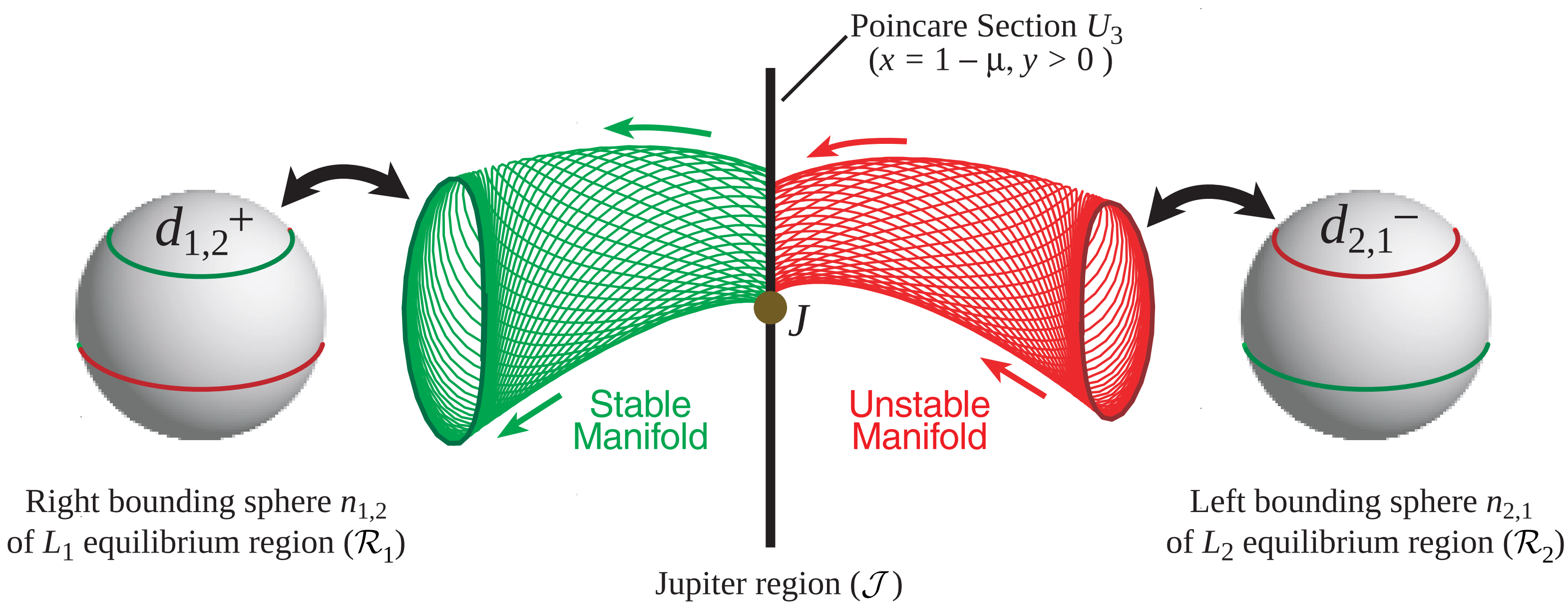


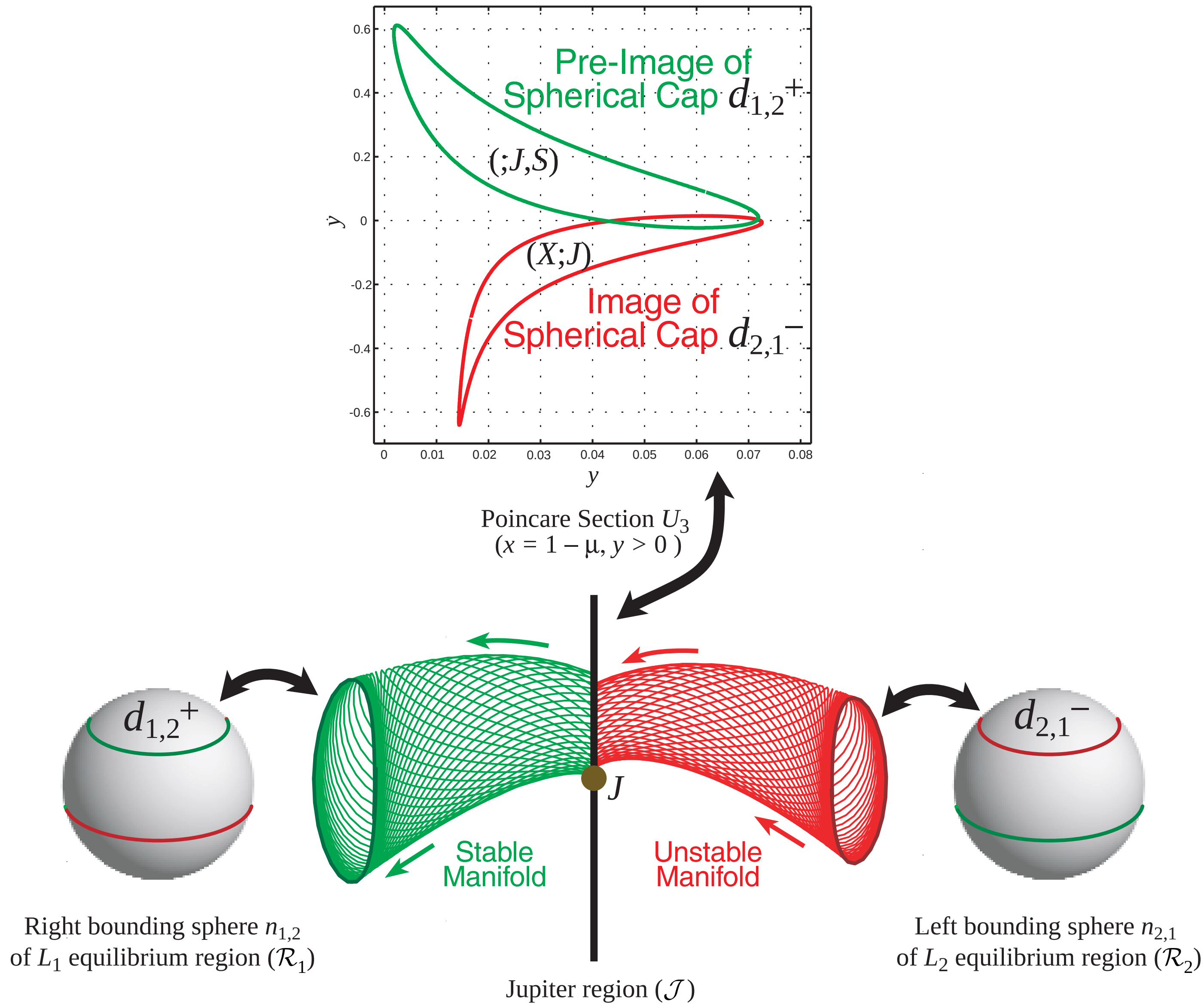
- This can be seen by recalling the bounding spheres for the equilibrium regions.
- We will look at the images and pre-images of the **spherical caps** of **transit orbits** on a suitable Poincaré section.
  - The images and pre-images of the spherical caps form the tubes that partition the energy surface.



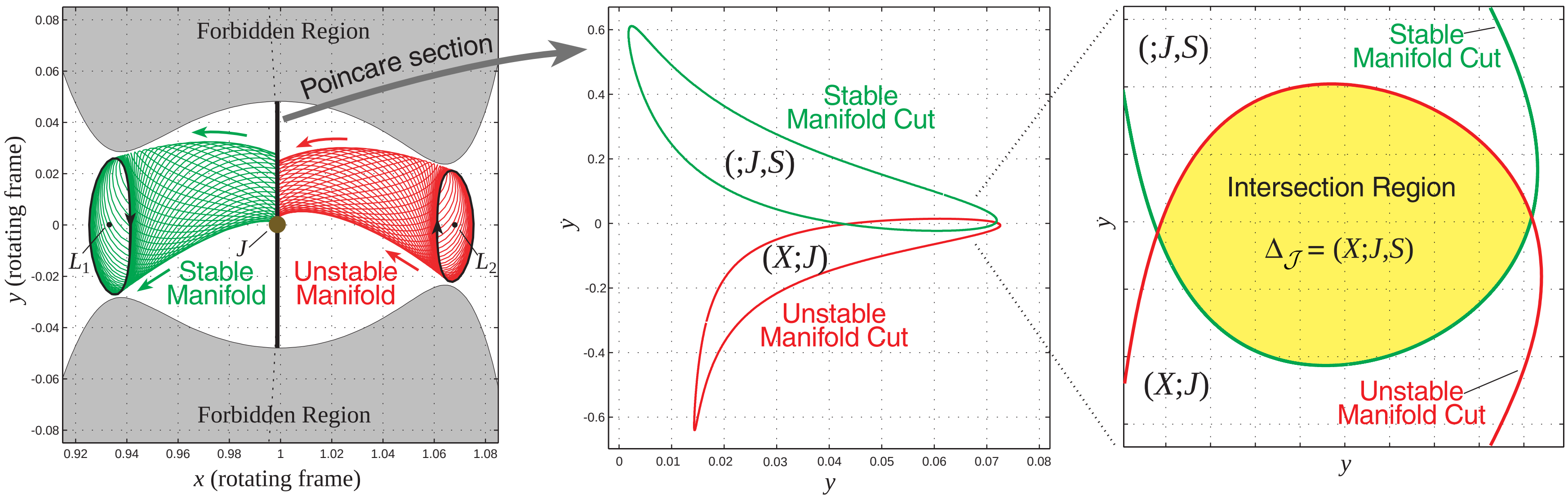
- For instance, on a Poincaré section between L_1 and L_2,
  - We look at the **image of the cap** on the left bounding sphere of the L_2 equilibrium region $\mathcal{R}_2$ containing **orbits leaving** $\mathcal{R}_2$.
  - We also look at the **pre-image of the cap** on the right bounding sphere of $\mathcal{R}_1$ containing **orbits entering** $\mathcal{R}_1$.

- The Poincaré cut of the **unstable manifold** of the L_2 periodic orbit forms the boundary of the **image of the cap** containing transit orbits leaving $\mathcal{R}_2$.
  - All of these orbits came from the exterior region and are now in the Jupiter region, so we label this region $(\mathbf{X}; \mathbf{J})$. Etc.

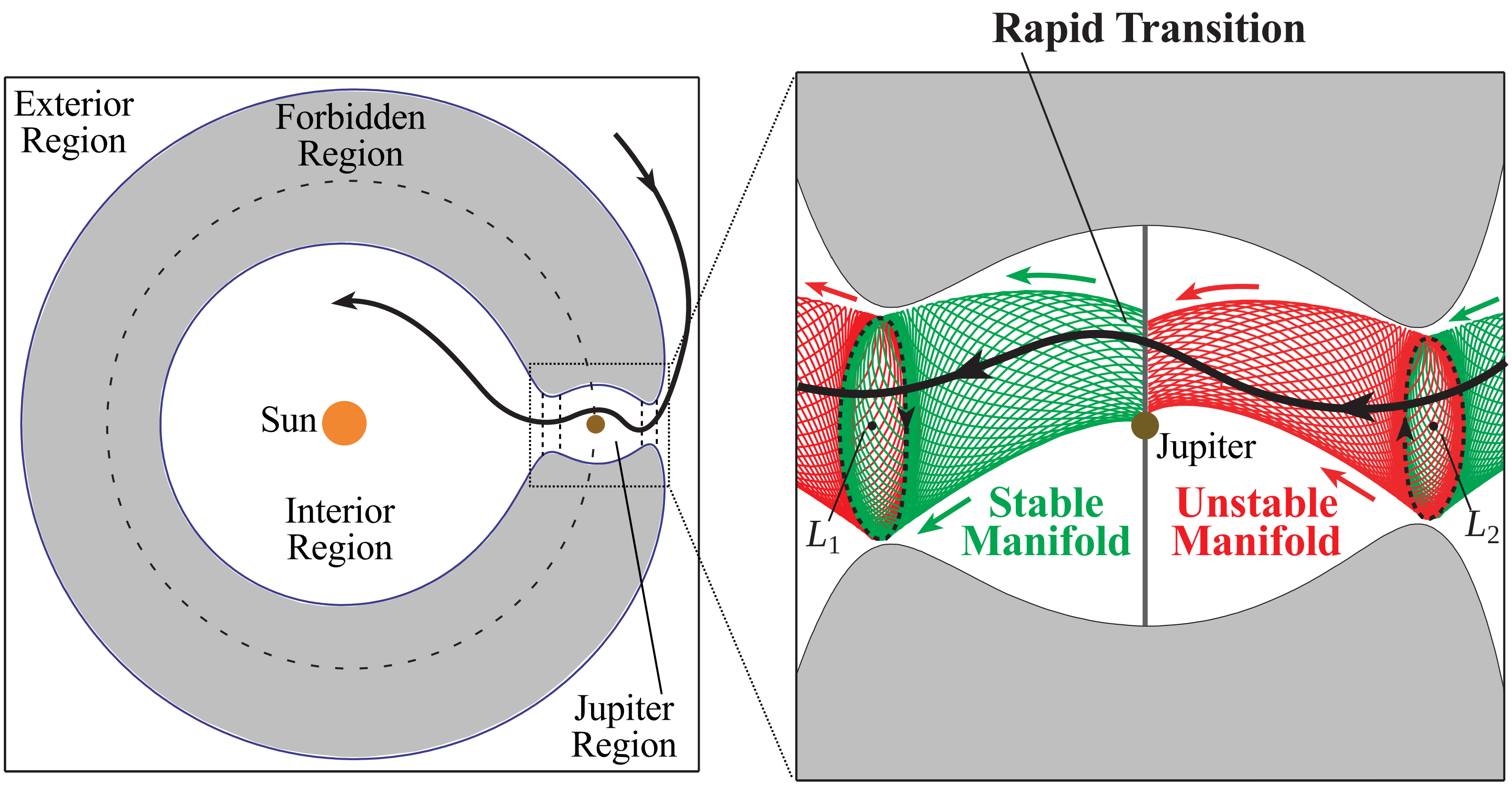



Pre-Image of
Spherical Cap $d_{1,2}^{+}$
(;J,S)
(X;J)
Image of
Spherical Cap $d_{2,1}^{-}$
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
0
0.01
0.02
0.03
0.04
0.05
0.06
0.07
0.08
y
y
Poincare Section U_3
($x = 1 - \mu, y > 0$)
$d_{1,2}^{+}$
$d_{2,1}^{-}$
J
Stable
Manifold
Unstable
Manifold
Right bounding sphere $n_{1,2}$
of L_1 equilibrium region ($\mathcal{R}_1$)
Jupiter region ($\mathcal{J}$)
Left bounding sphere $n_{2,1}$
of L_2 equilibrium region ($\mathcal{R}_2$)

- The dynamics of the invariant manifold tubes naturally suggest the ***itinerary*** representation.

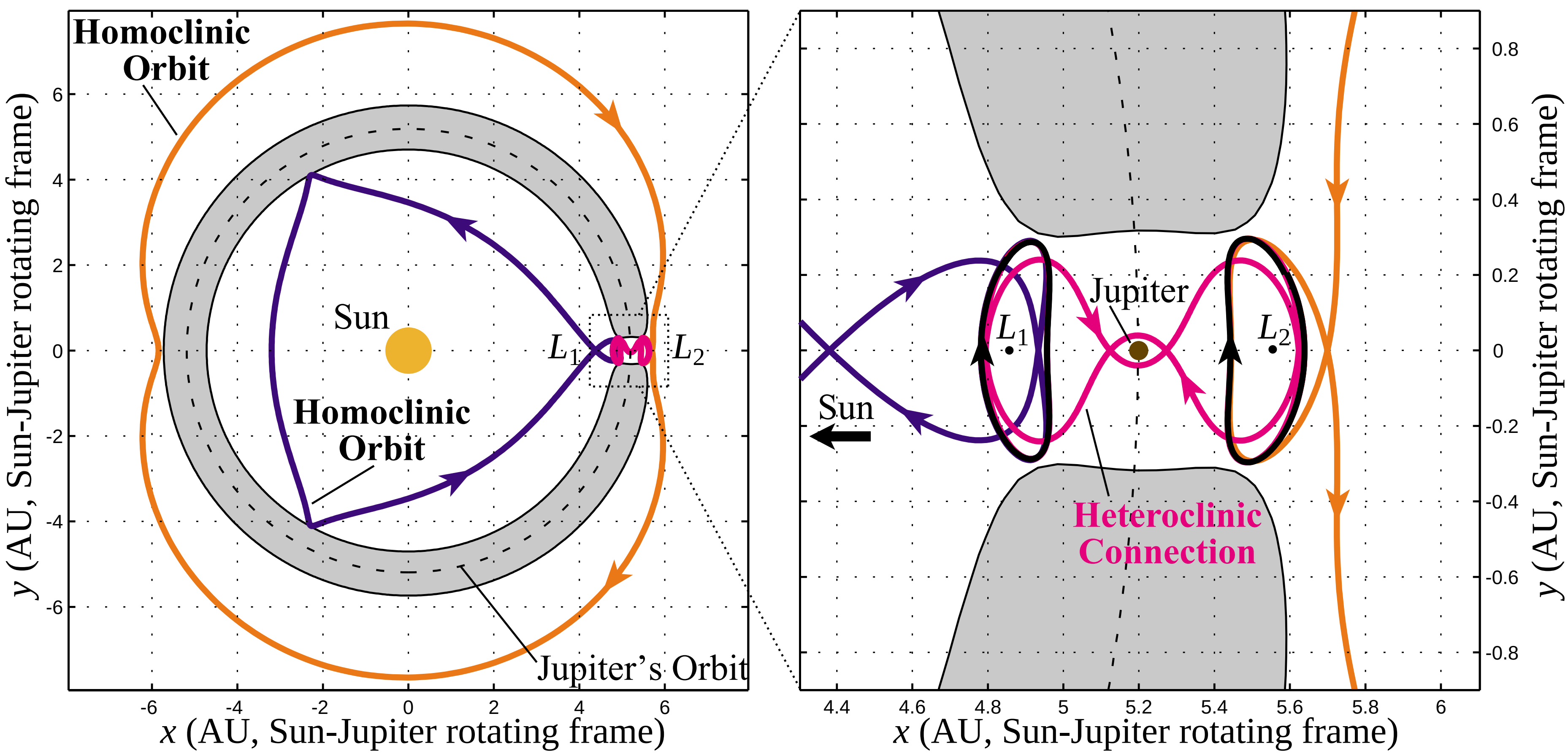


- Integrating an initial condition in the intersection region would give us an orbit with the desired itinerary $(\mathbf{X}, \mathbf{J}, \mathbf{S})$.



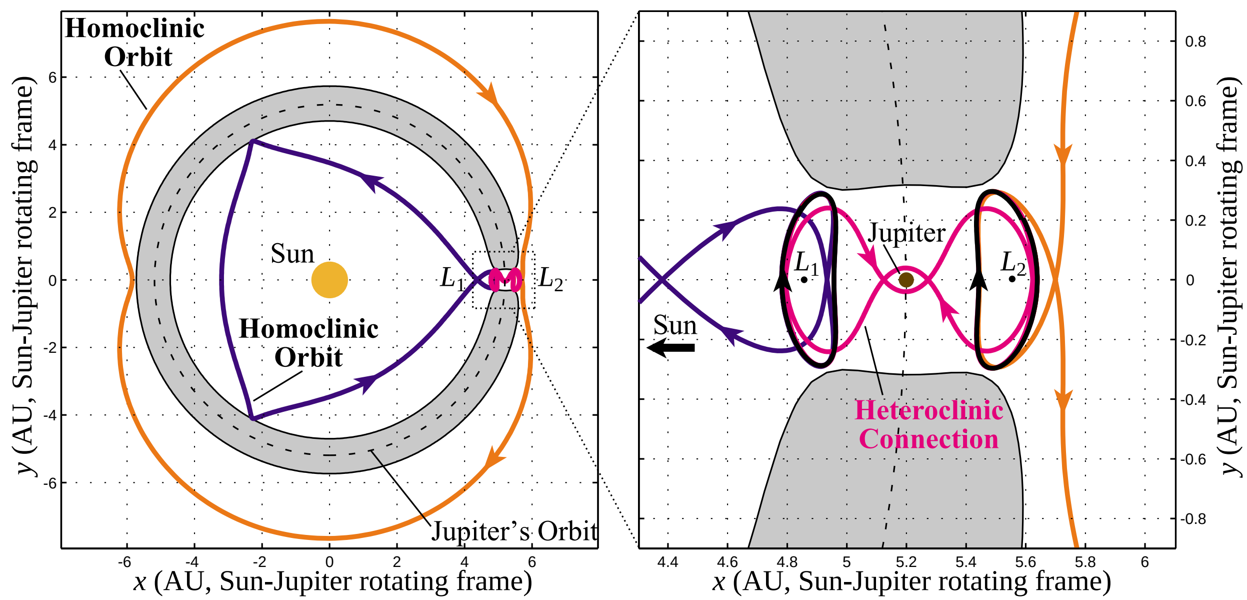
Global Orbit Structure: Overview

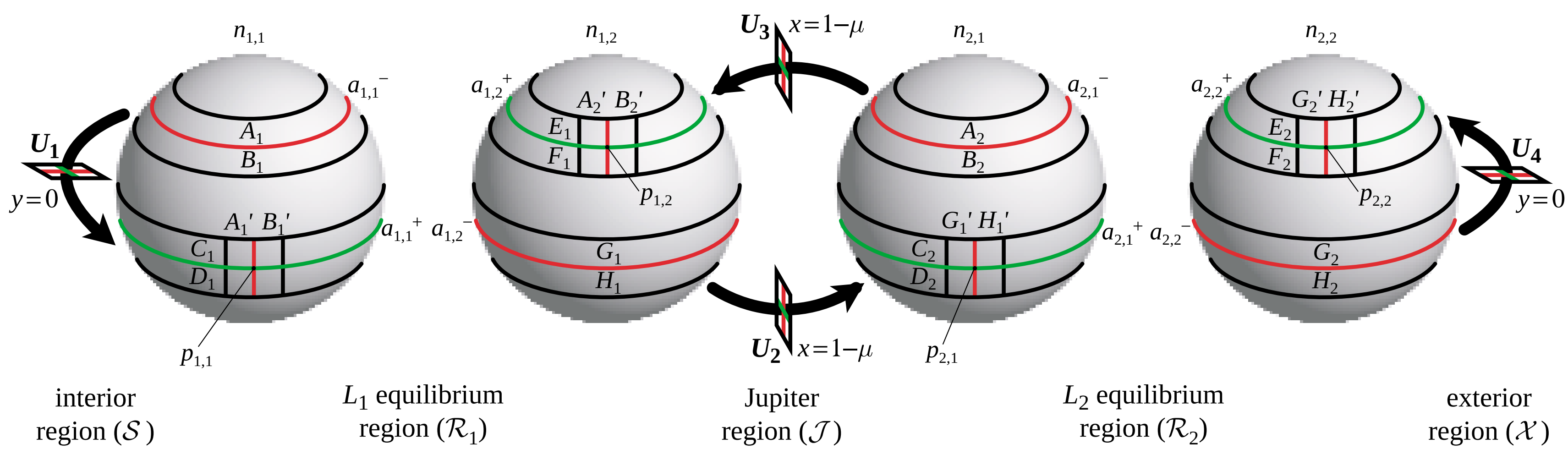
- Found **heteroclinic connection** between pair of periodic orbits.
- Find a large class of **orbits** near this (homo/heteroclinic) ***chain***.
- Comet can follow these ***channels*** in rapid transition.



Global Orbit Structure: Energy Manifold

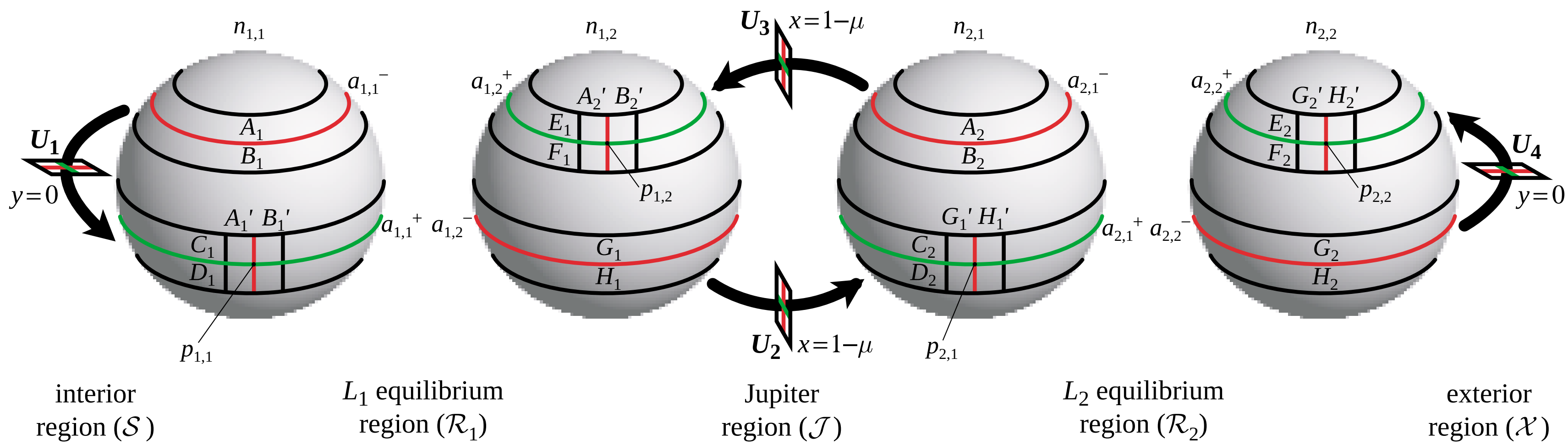
- Schematic view of energy manifold.

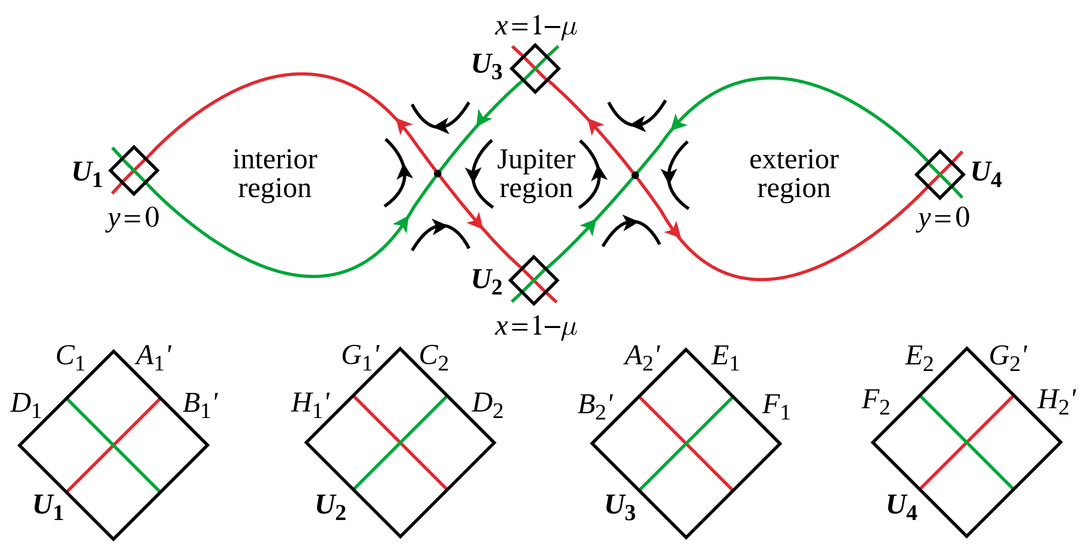




Global Orbit Structure: Poincaré Map

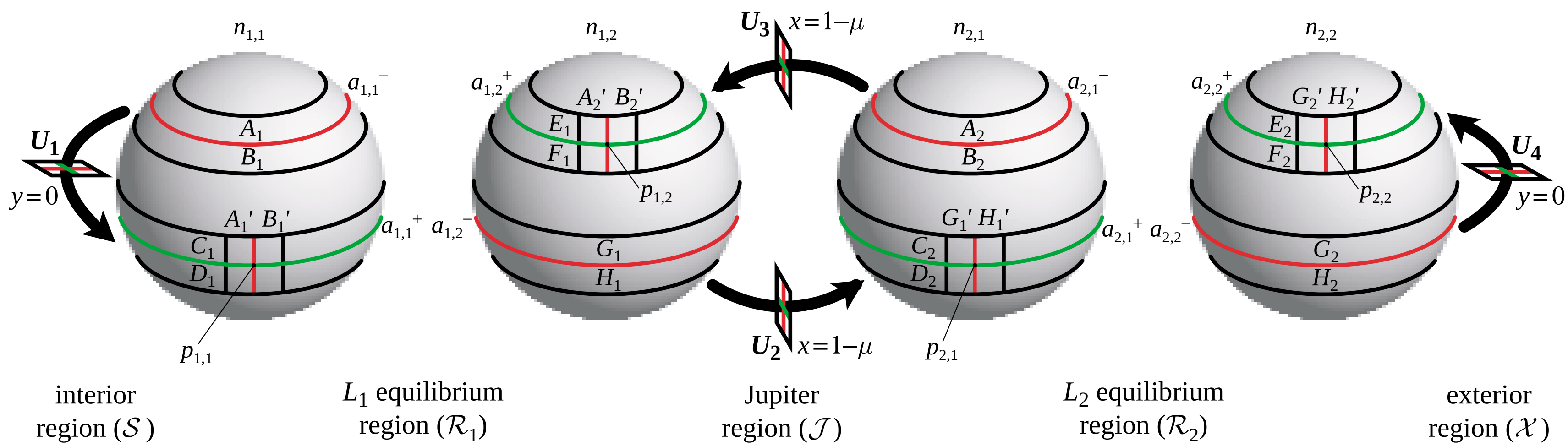
- Reducing study of global orbit structure to study of discrete map.





Construction of Poincaré Map

- Construct **Poincaré map** P (tranversal to the flow) whose domain U consists of 4 squares U_i.
- Squares U_1 and U_4 contained in $y = 0$, each centers around a transversal **homoclinic** point.
- Squares U_2 and U_3 contained in $x = 1 - \mu$, each centers around a transversal **heteroclinic** point.



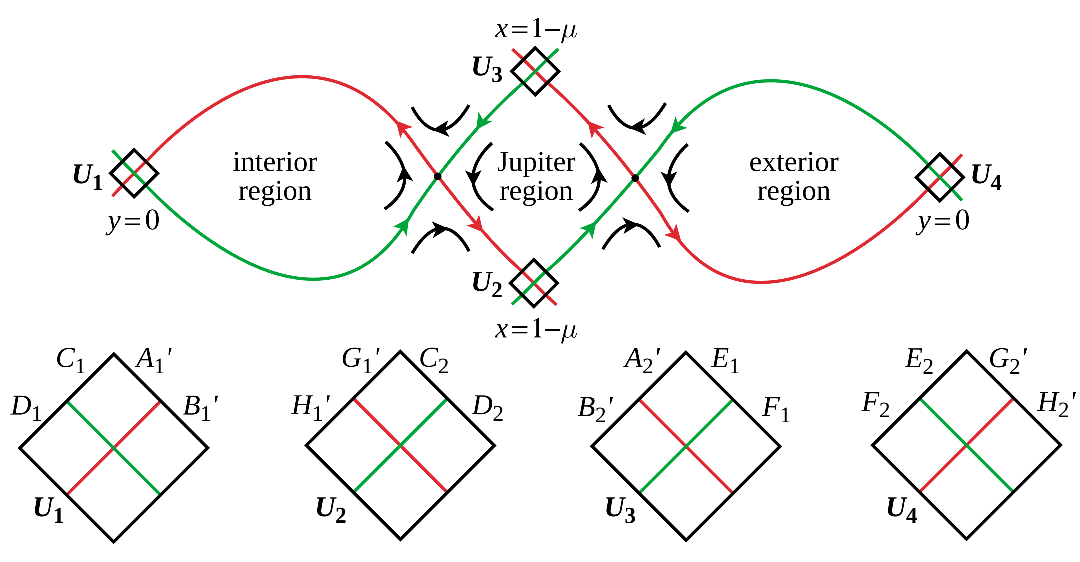
Global Orbit Structure near the Chain

- Consider **invariant set** Λ of points in U whose images and pre-images under all iterations of P remain in U.

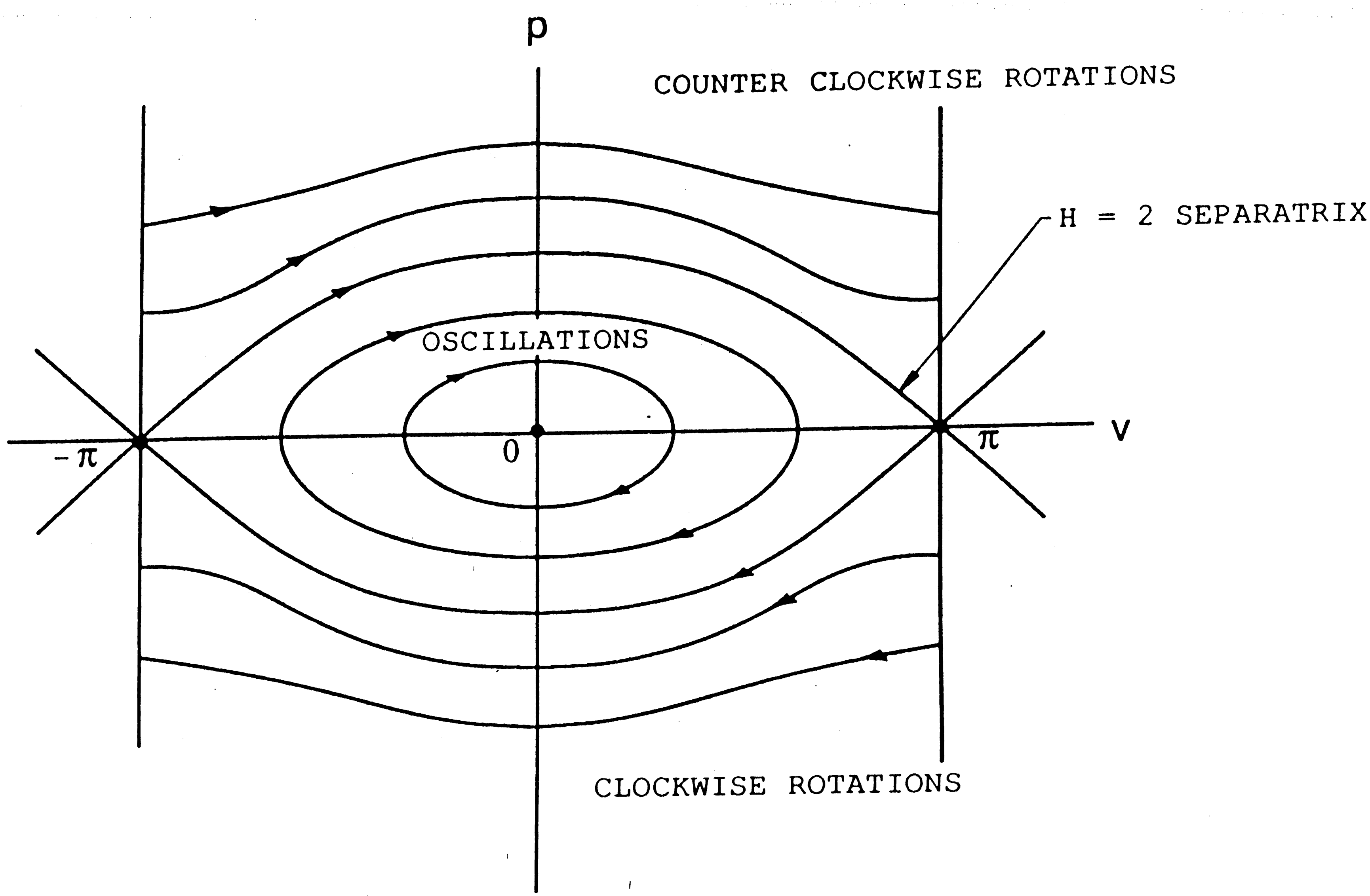
$$\Lambda = \cap_{n=-\infty}^{\infty} P^n(U).$$

- Invariant set Λ contains all **recurrent orbits** near the chain. It provides insight into the **global dynamics** around the chain.
- Chaos theory told us to first consider only the **first** forward and backward iterations:

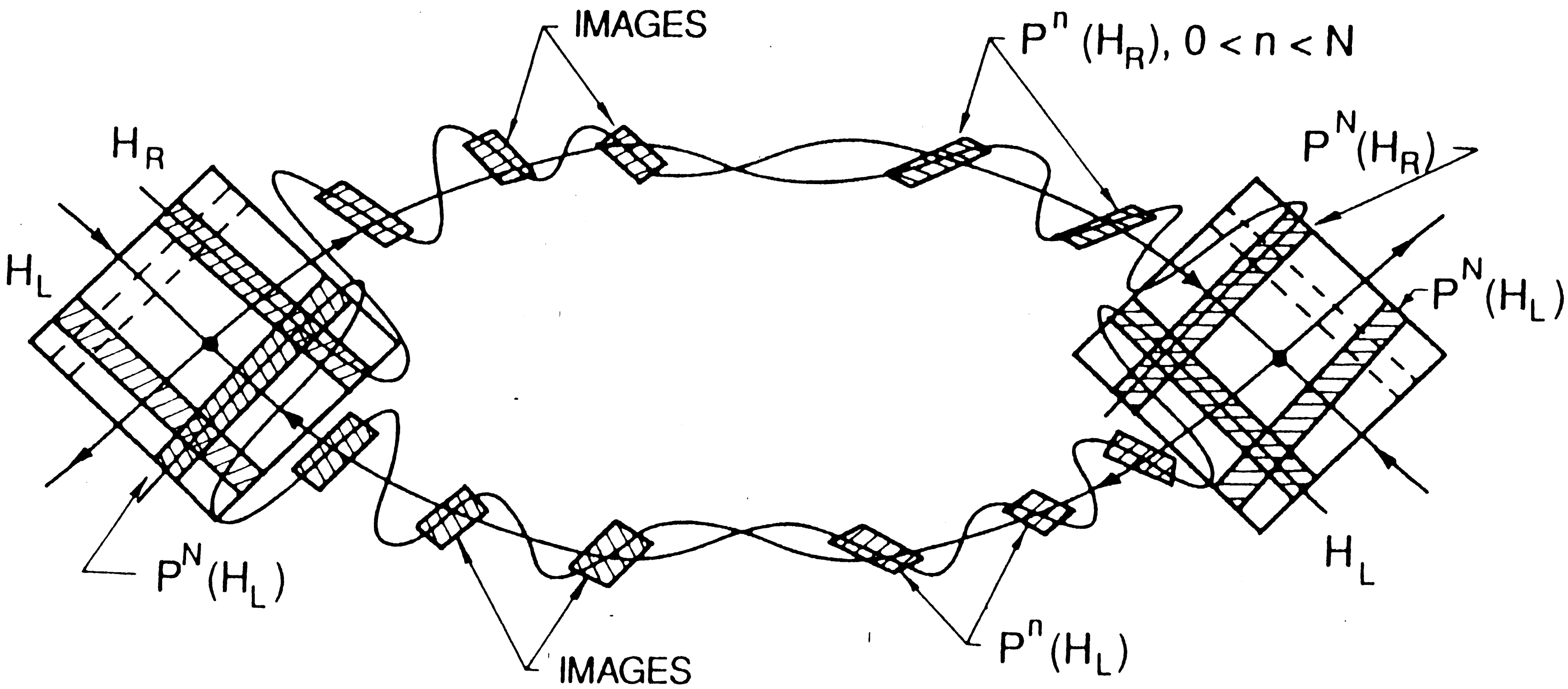
$$\Lambda^1 = P^{-1}(U) \cap U \cap P^1(U).$$



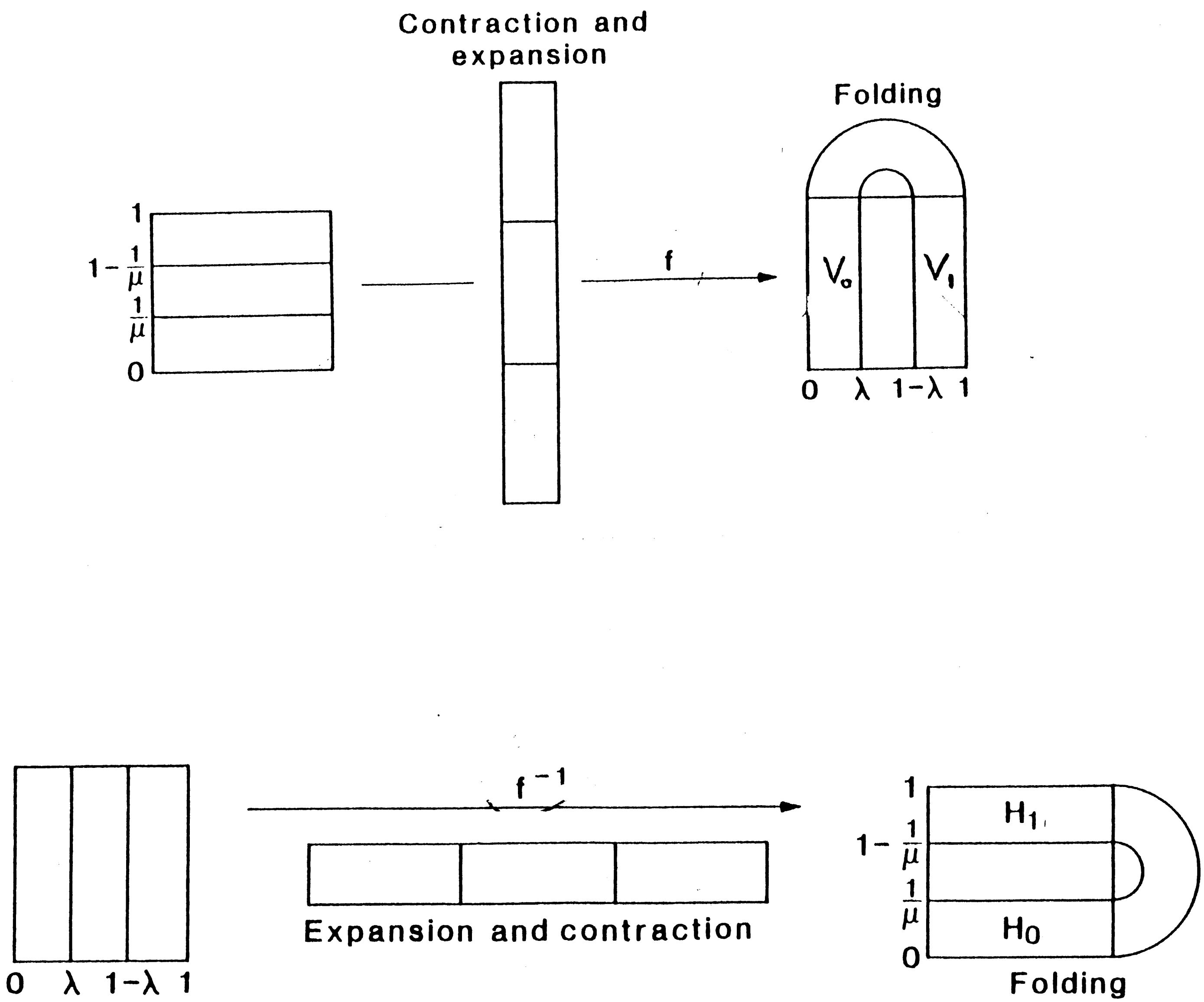
Review of Horseshoe Dynamics: Pendulum



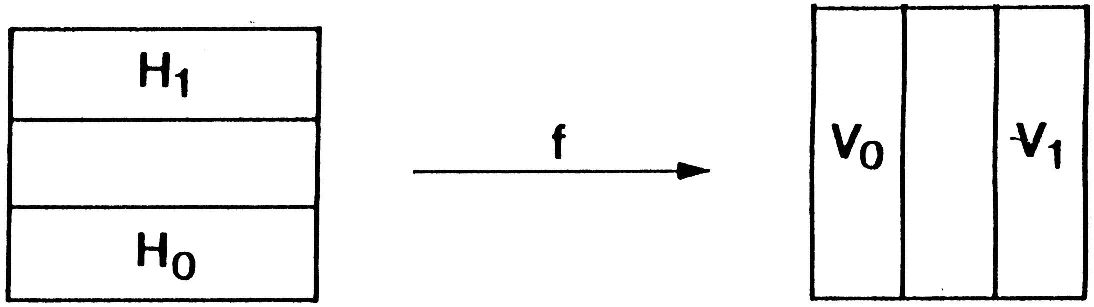
Review of Horseshoe Dynamics: Forced Pendulum

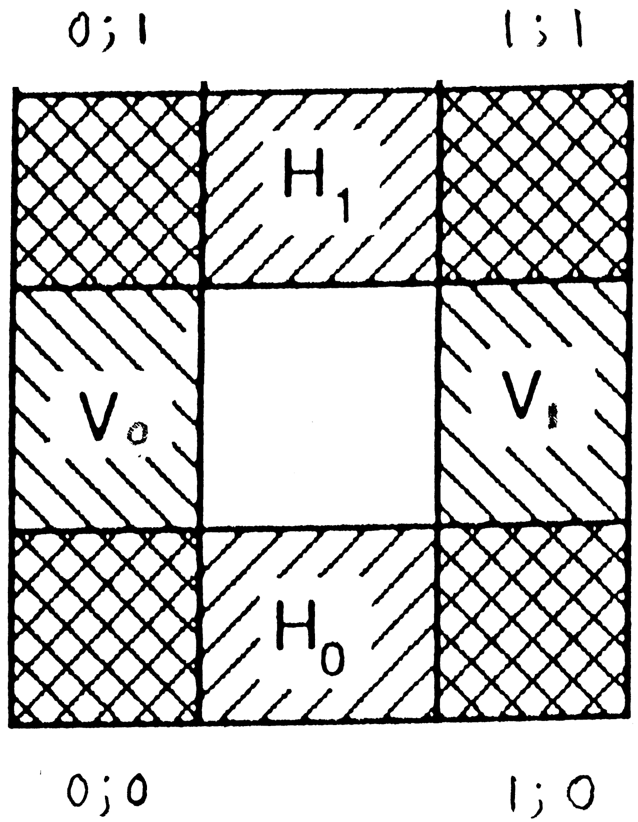


Review of Horseshoe Dynamics: First Iteration

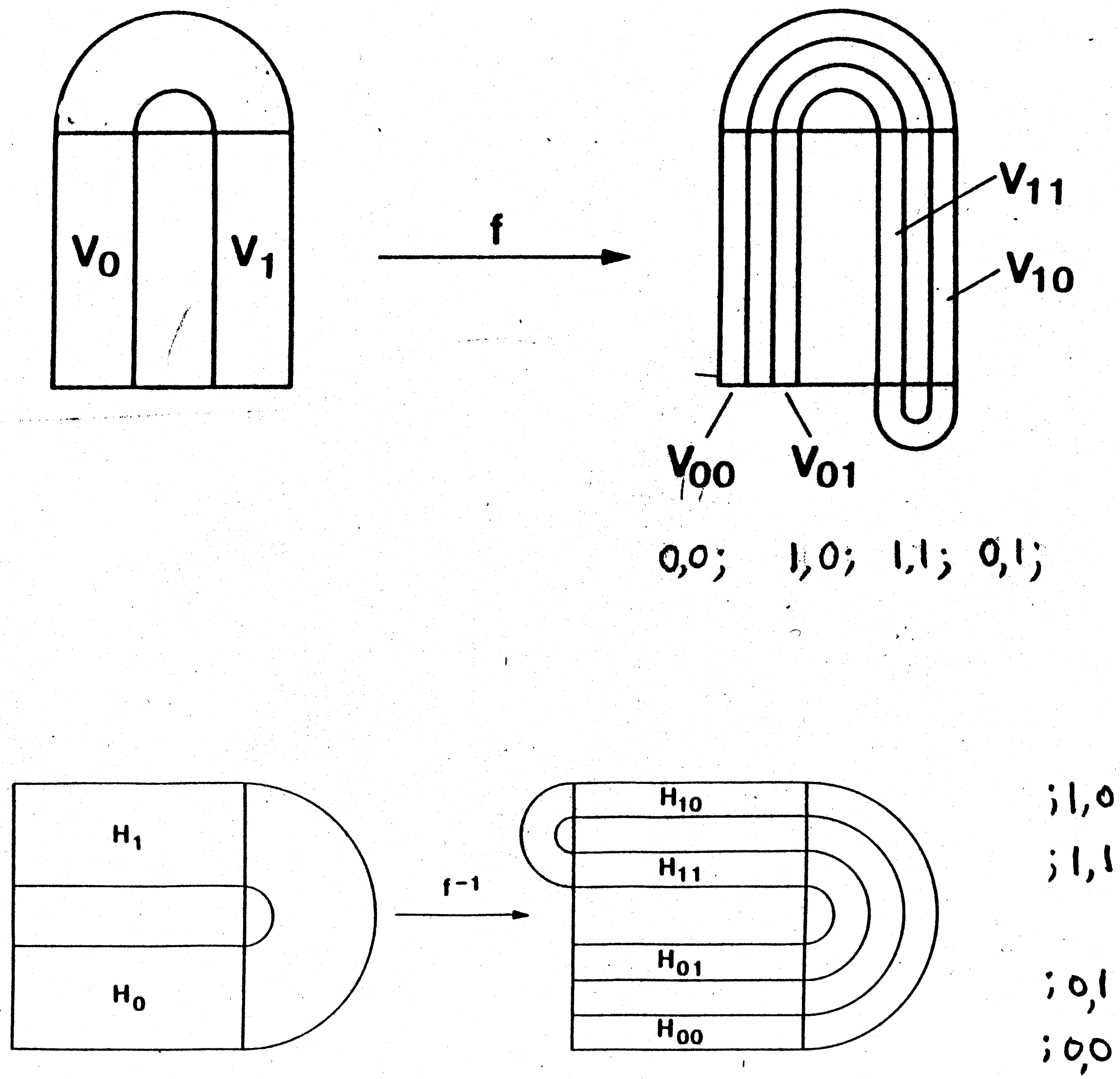


Review of Horseshoe Dynamics: First Iteration

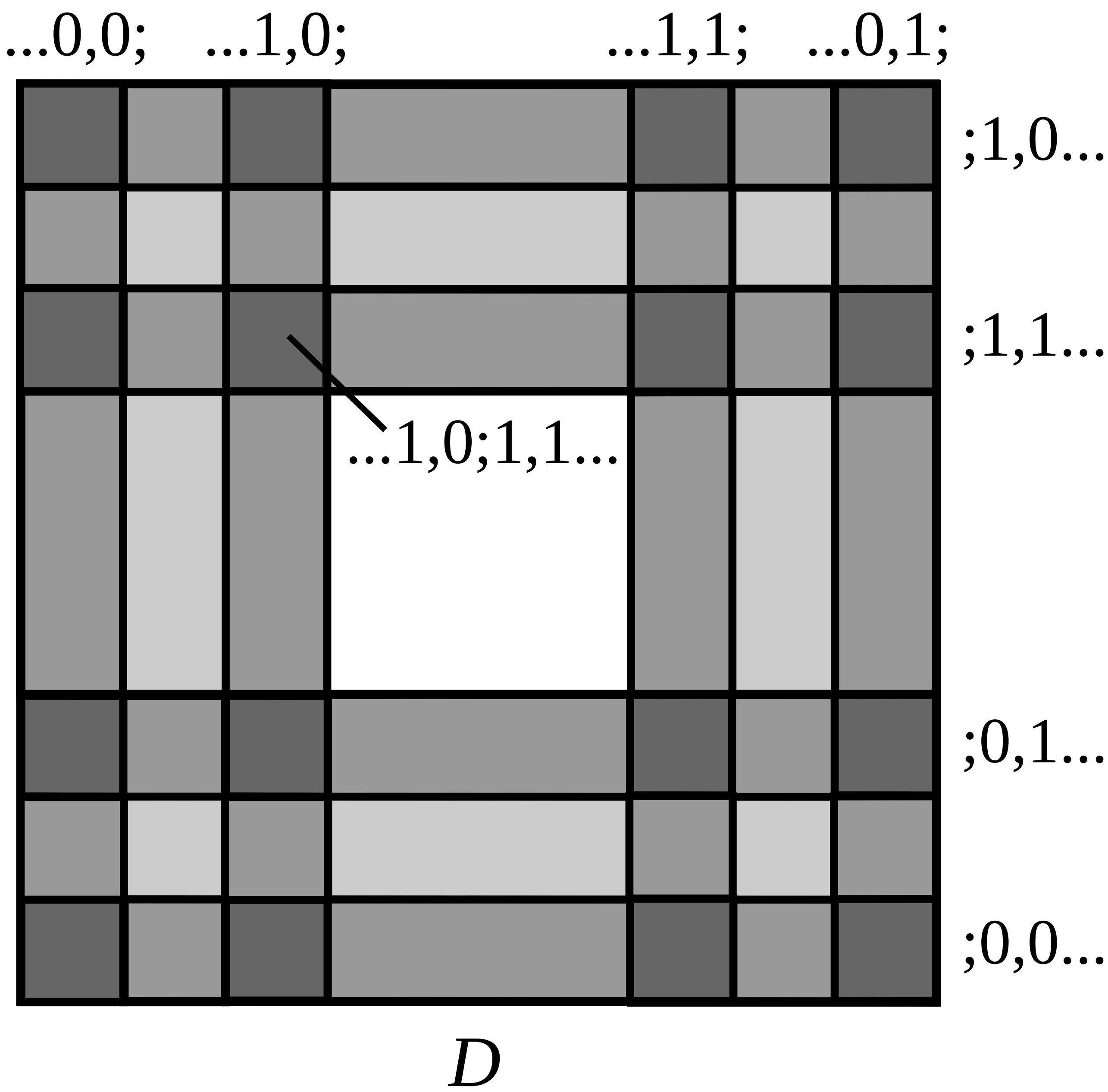




Review of Horseshoe Dynamics: Second Iteration

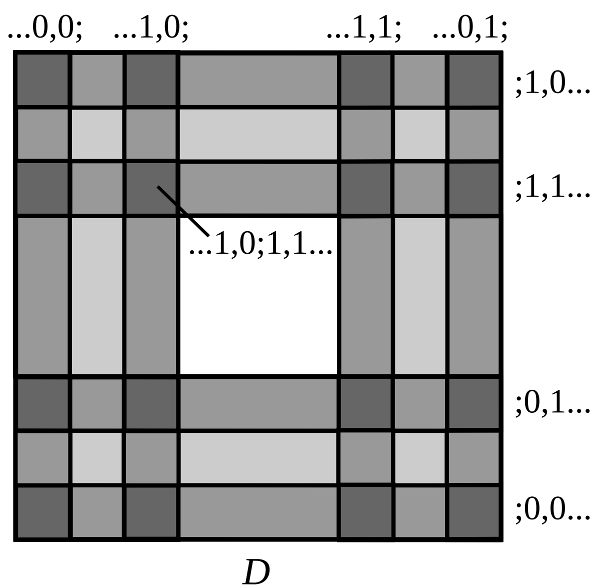


Review of Horseshoe Dynamics: Second Iteration



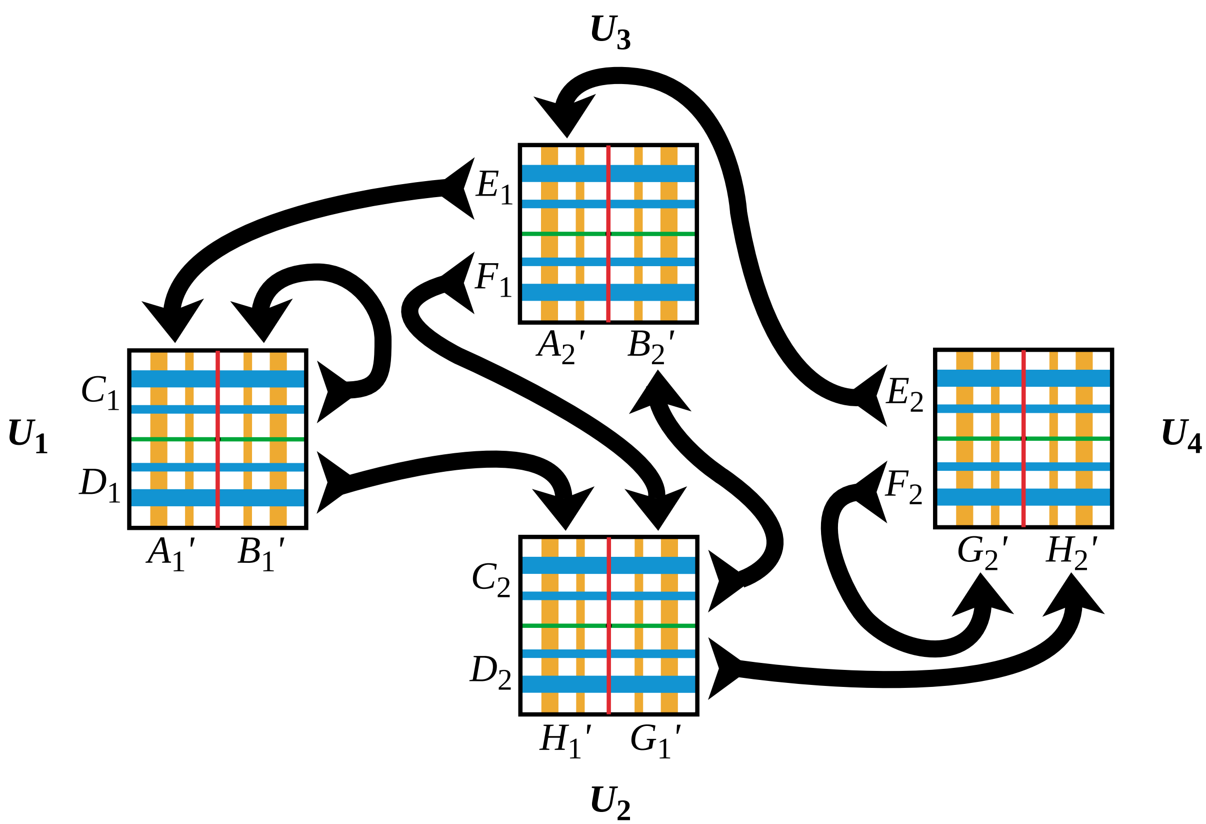
Conley-Moser Conditions: Horseshoe-type Map

- For horseshoe-type map h satisfying **Conley-Moser conditions**, the **invariant set** of all iterations, $\Lambda_h = \cap_{n=-\infty}^{\infty} h^n(Q)$, can be constructed and visualized in a standard way.
  - **Strip condition**: h maps "horizontal strips" H_0, H_1 to "vertical strips" V_0, V_1, (with horizontal boundaries to horizontal boundaries and vertical boundaries to vertical boundaries).
  - **Hyperbolicity condition**: h has uniform contraction in horizontal direction and expansion in vertical direction.



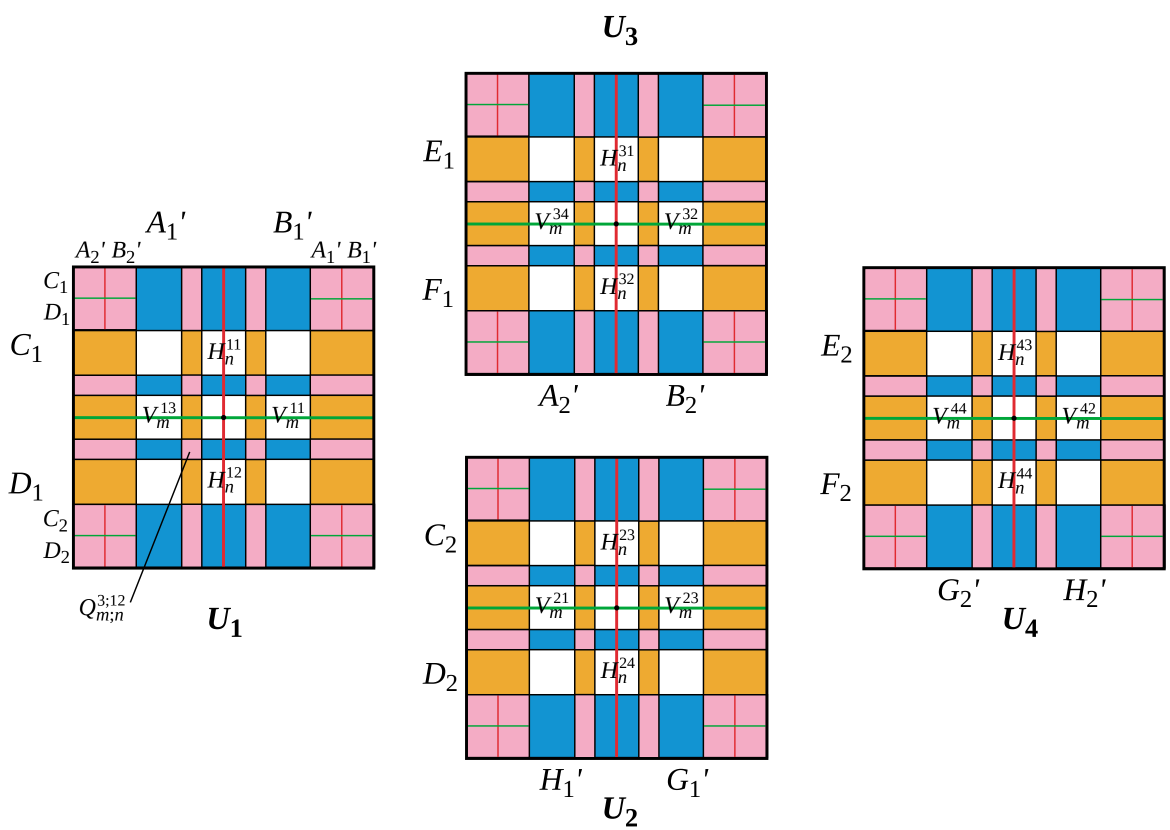
Generalized Conley-Moser Conditions

- Proved P satisfies **Generalized** Conley-Moser conditions:
  - **Strip** condition: it maps "horizontal strips" H_n^{ij} to "vertical strips" V_n^{ji}.
  - **Hyperbolicity** condition: it has uniform contraction in horizontal direction and expansion in vertical direction.



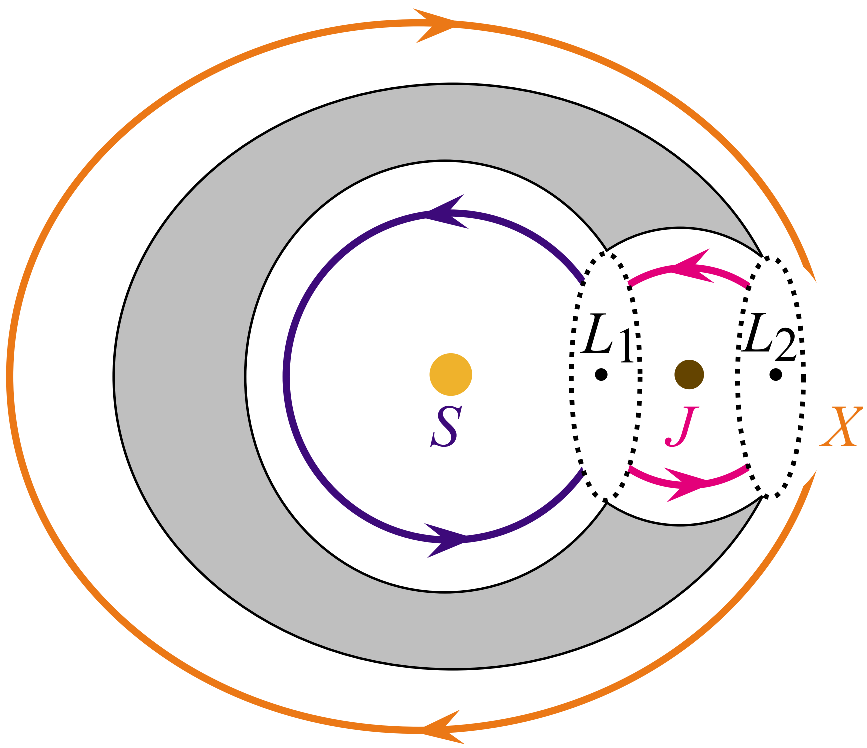
Generalized Conley-Moser Conditions

- Shown are invariant set Λ^1 under **first iteration**.
- Since P satisfies Generalized Conley-Moser Conditions, this process can be repeated **ad infinitum**.
- What remains is **invariant set** of points Λ which are in 1-to-1 corr. with set of bi-infinite **sequences** $(\ldots, u_i, m; u_j, n, u_k, \ldots)$.



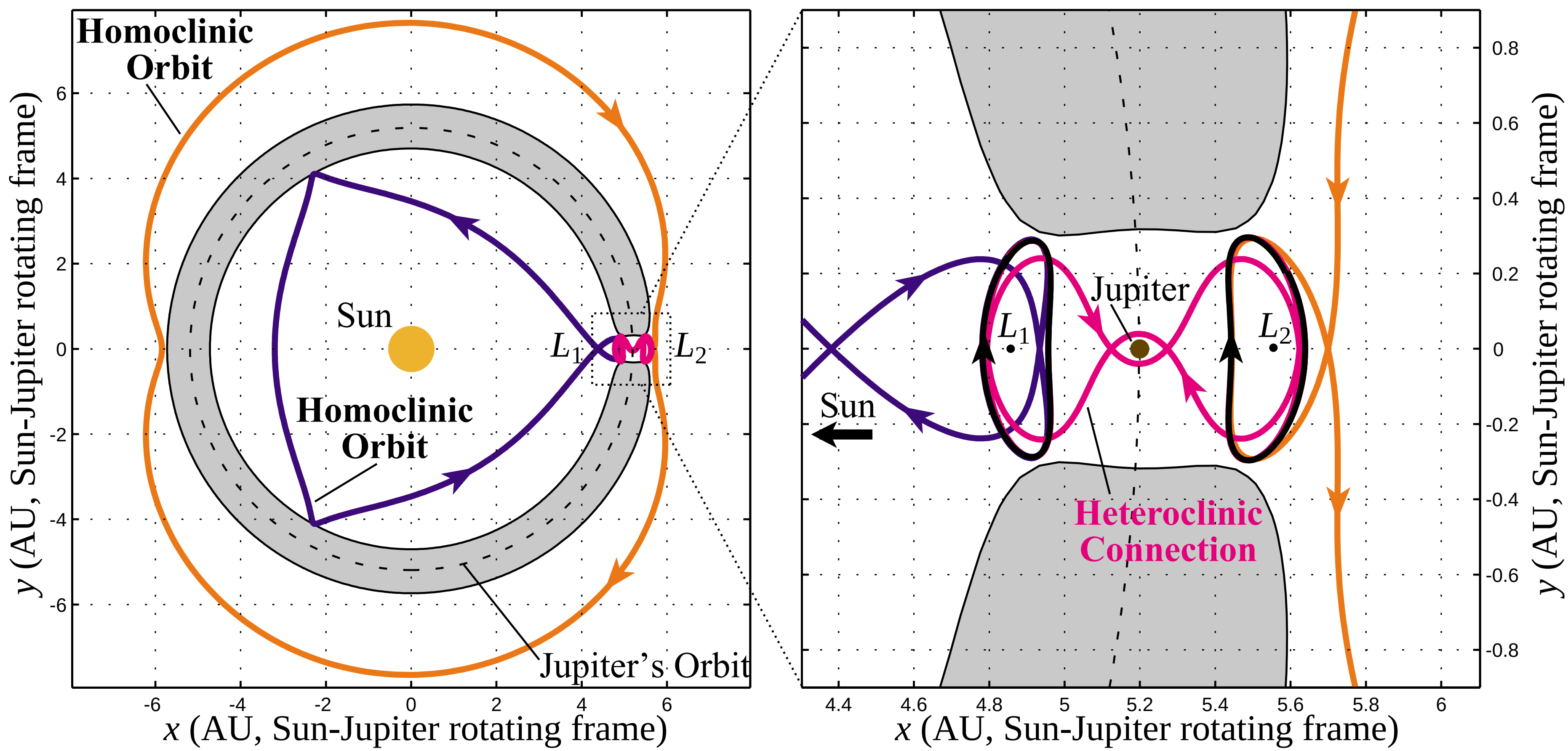
Global Orbit Structure: Main Theorem

- ***Main Theorem***: For any admissible **itinerary**, e.g., $(\ldots, \mathbf{X}, \mathbf{1}, \mathbf{J}, \mathbf{0}; \mathbf{S}, \mathbf{1}, \mathbf{J}, \mathbf{2}, \mathbf{X}, \ldots)$, there exists an orbit whose **whereabouts** matches this **itinerary**.
- Can even specify **number of revolutions** the comet makes around Sun & Jupiter.



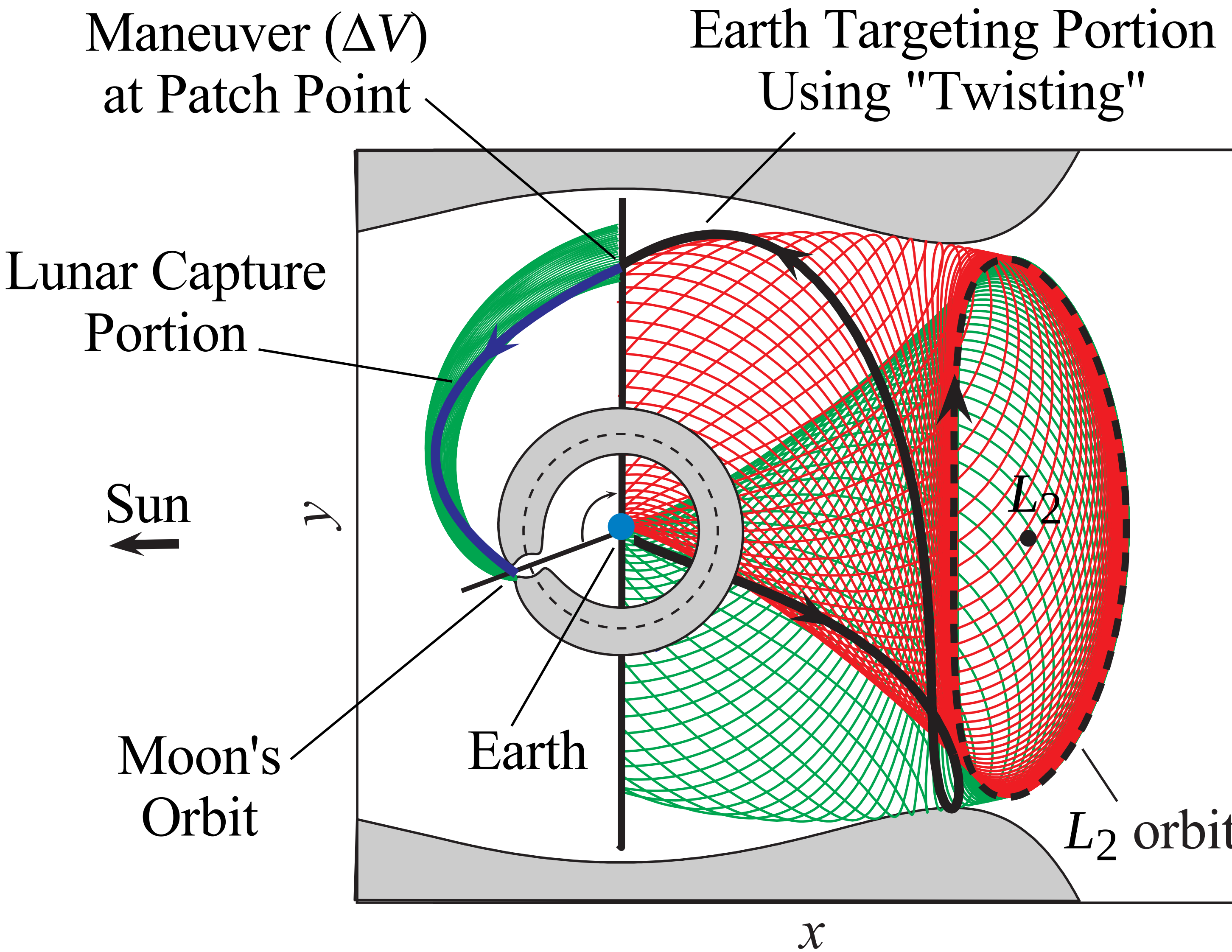
Global Orbit Structure: Dynamical Channels

- Found a large class of **orbits** near homo/heteroclinic ***chain***.
- Comet can follow these ***channels*** in rapid transition.

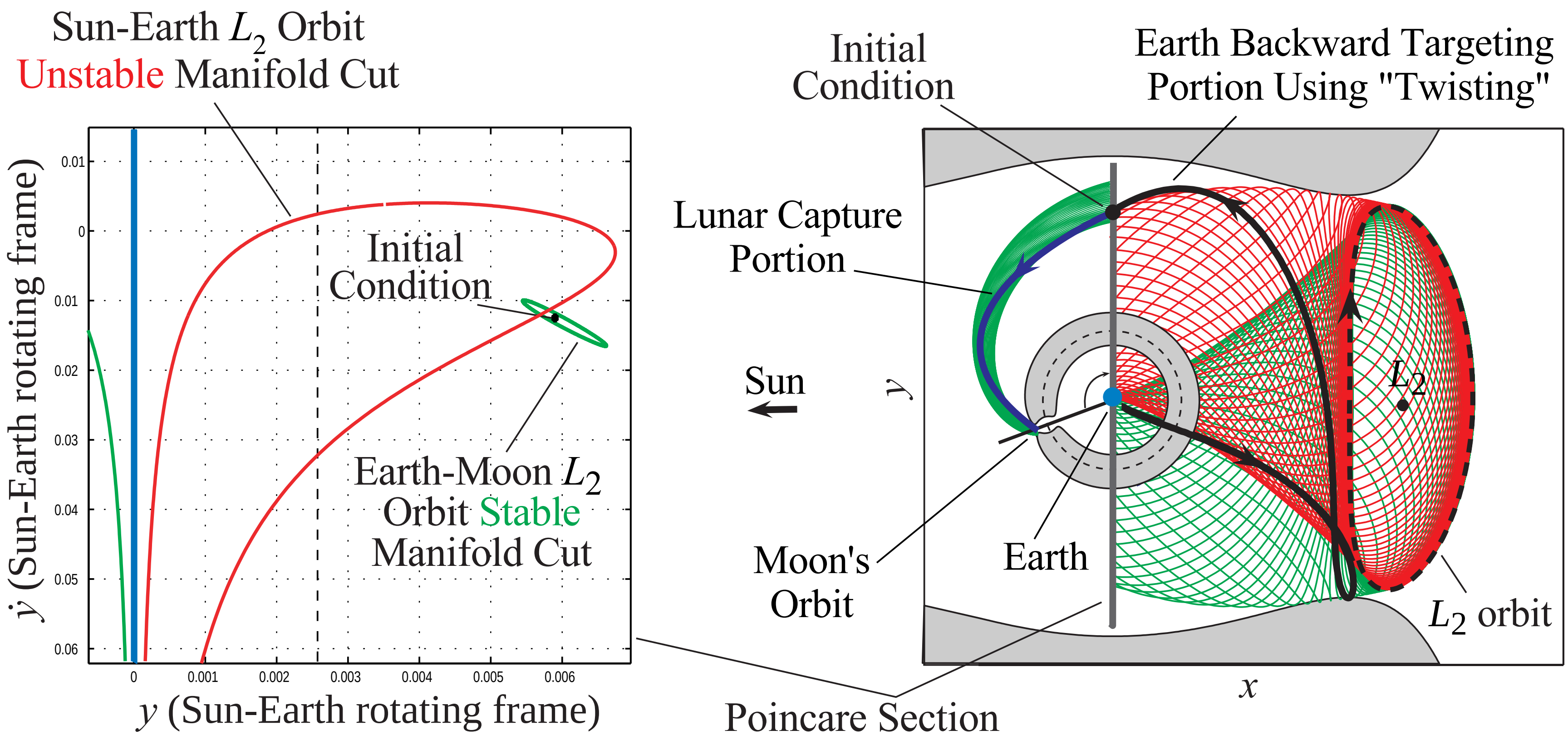


Lunar Capture: How to get to the Moon Cheaply

- Using the invariant manifold tubes as the building blocks, we can construct interesting, fuel saving space mission trajectories.
  - For instance, an Earth-to-Moon ballistic capture orbit.
  - Uses Sun's perturbation.
  - Jump from Sun-Earth-S/C system to Earth-Moon-S/C system.
  - Saves about 20% of onboard fuel compared to Apollo-like transfer.


Maneuver (ΔV)
at Patch Point
Earth Targeting Portion
Using "Twisting"
Lunar Capture
Portion
Sun
y
L_2
Moon's
Orbit
Earth
L_2 orbit
x

- Intersection found between Earth-Moon **stable manifold** and Sun-Earth **unstable manifold**, which targets trajectory back to Earth.



Movie: Shoot the Moon in rotating frame

Future Research Directions

- For a single 3-body system:
  - When is 3-body effect more important than 2-body?
  - Find "sweet spot" within tubes where transport is most efficient/fastest?
  - Consider continuous low-thrust control, optimal control.
- For coupling multiple 3-body systems:
  - Where to jump from one 3-body system to another?
  - Optimal control: trade off between travel time and fuel.
  - Efficient use of resonances
- Planetary science/astronomy applications:
  - Statistics: transport rates, capture probabilities, etc.
- Chemical/atomic physics applications?

- **Final Thought**: For a class of Hamiltonian systems which have phase space bottlenecks containing unstable periodic orbits, the unstable and stable manifolds of those periodic orbits partition the part of the energy surface where transport is possible. The manifolds not only provide a picture of the global behavior of the system, but are the starting point for obtaining the statistical properties of the system.

Further Information

- **Koon, W.S., M.W. Lo, J.E. Marsden and S.D. Ross [2000]**, Heteroclinic connections between periodic orbits and resonance transitions in celestial mechanics, **Chaos**, vol. **10(2)**, pp. 427-469.
- **Koon, W.S., M.W. Lo, J.E. Marsden and S.D. Ross [2000]**, Low energy transfer to the Moon.
- **Jaffé, C., D. Farrelly and T. Uzer [1999]**, Transition state in atomic physics, **Phys. Rev. A**, vol. **60(5)**, pp. 3833-3850.
- **http://www.cds.caltech.edu/~shane/**