Experimental verification of criteria for escape from a potential well in a multi-degree of freedom system

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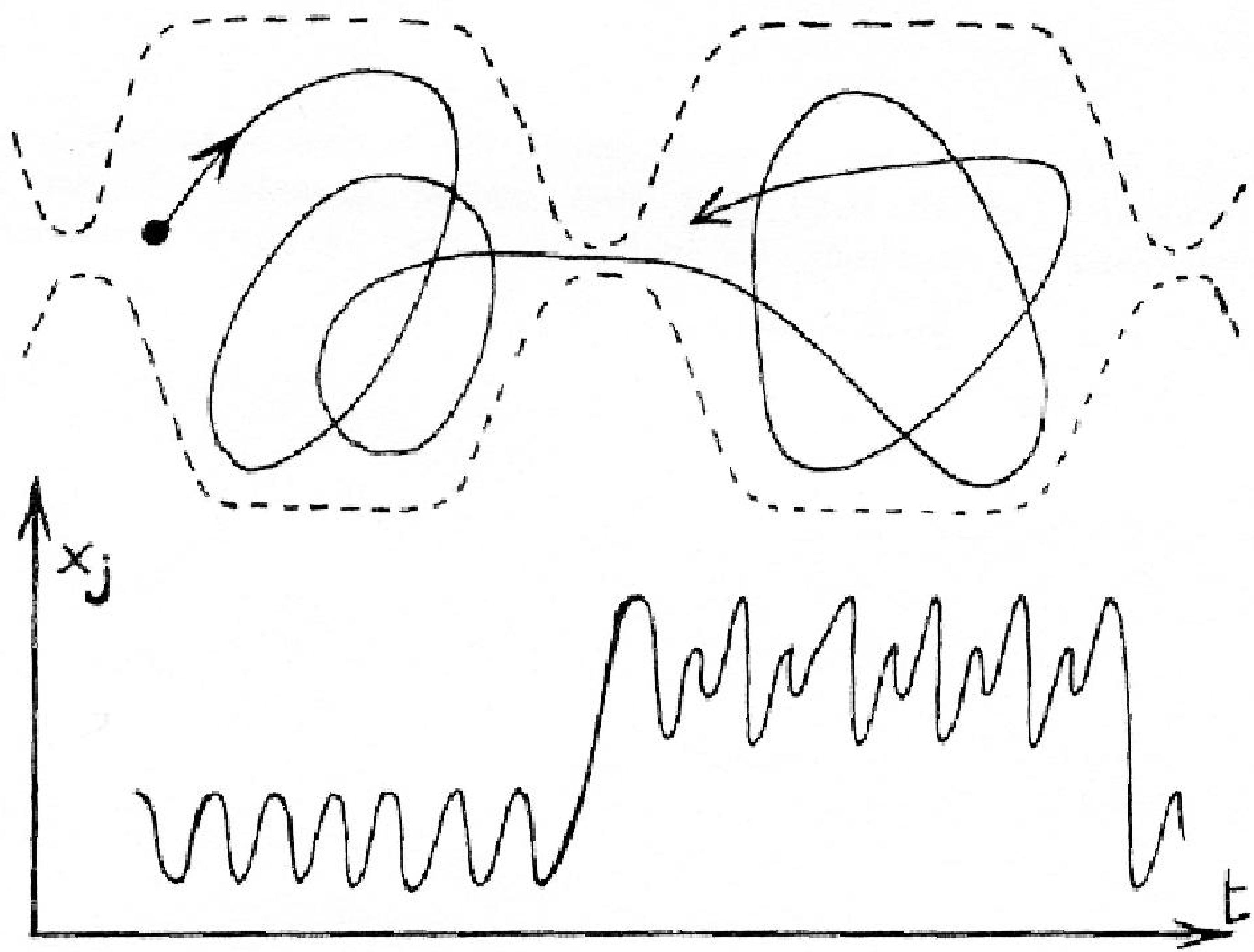
Intermittency and chaotic transitions

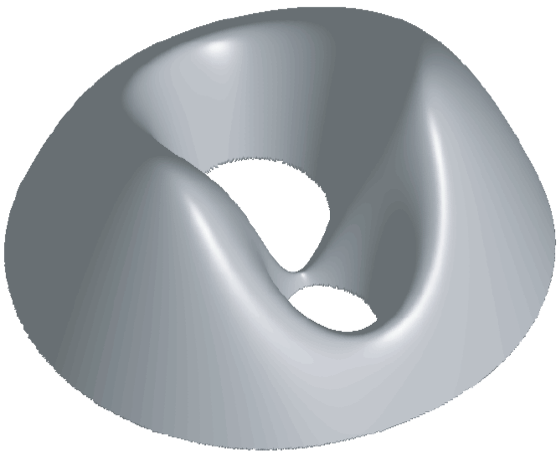
e.g., transitioning across "bottlenecks" in phase space

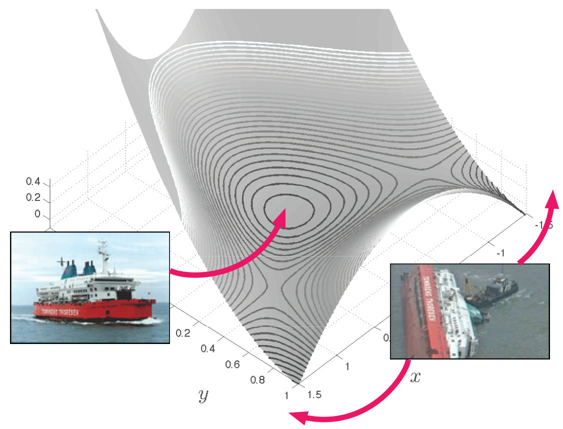


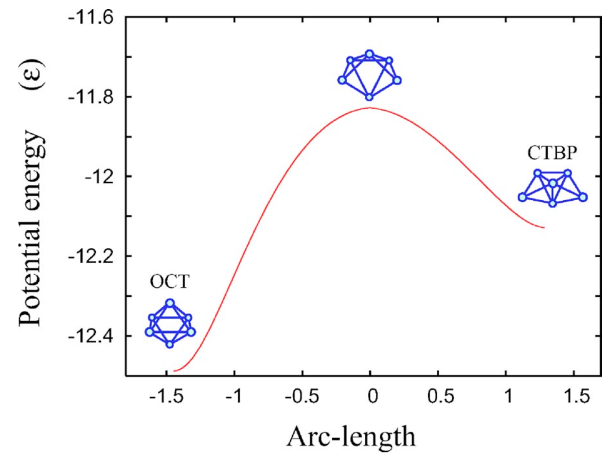
Marchal [1990]

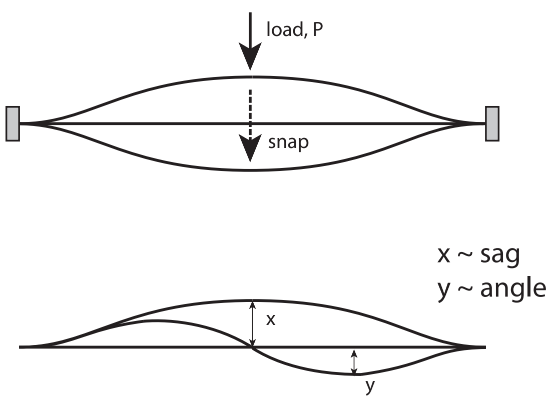
Multi-well multi-degree of freedom systems

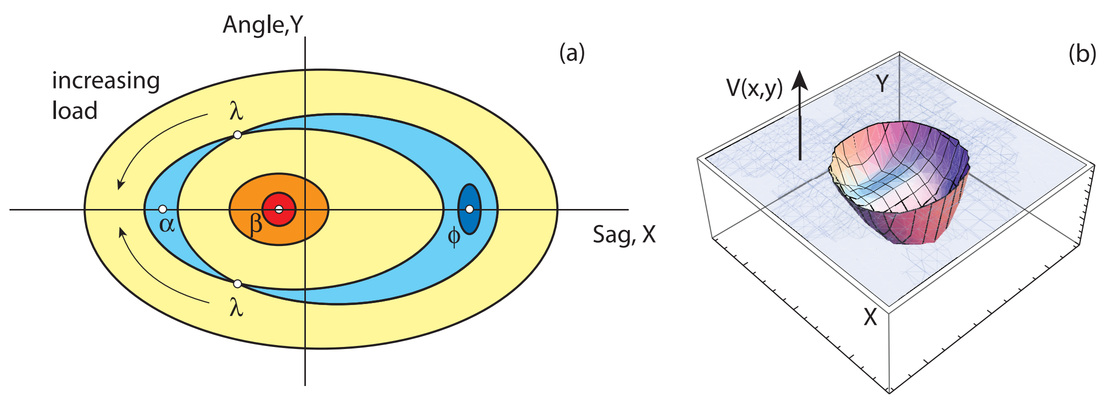
- Examples: chemistry, vehicle dynamics, structural mechanics



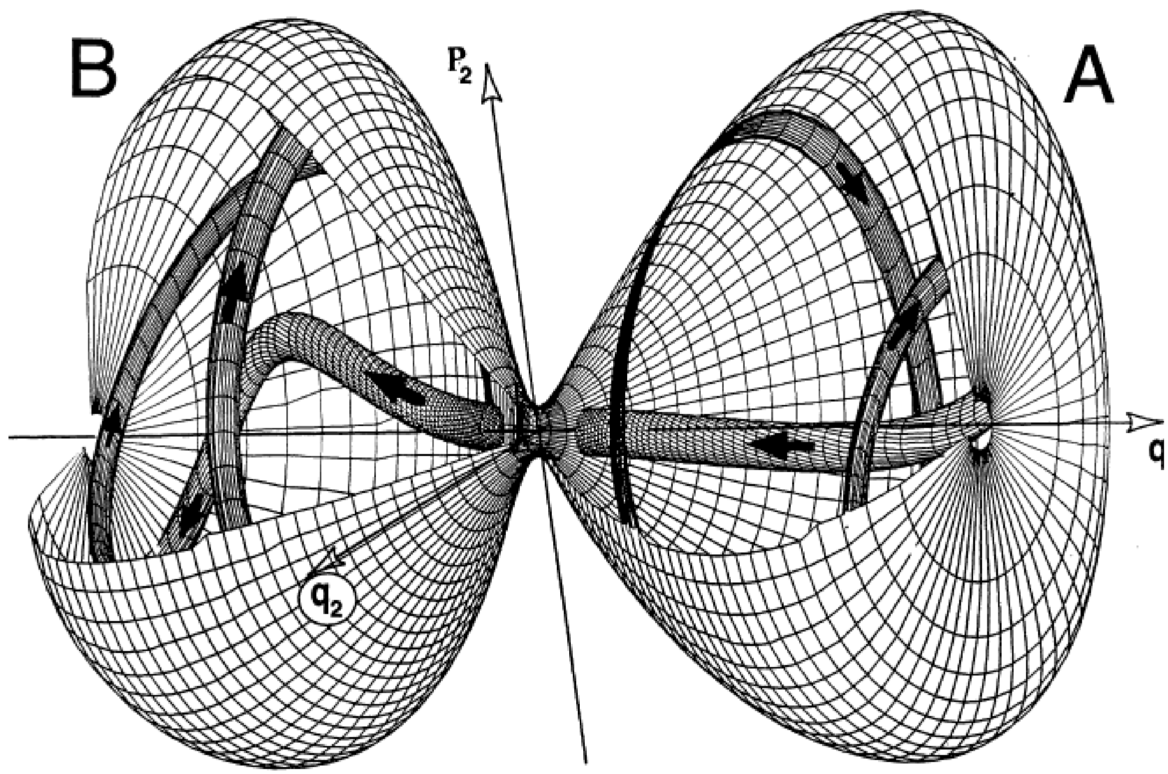








Transitions through bottlenecks via tubes

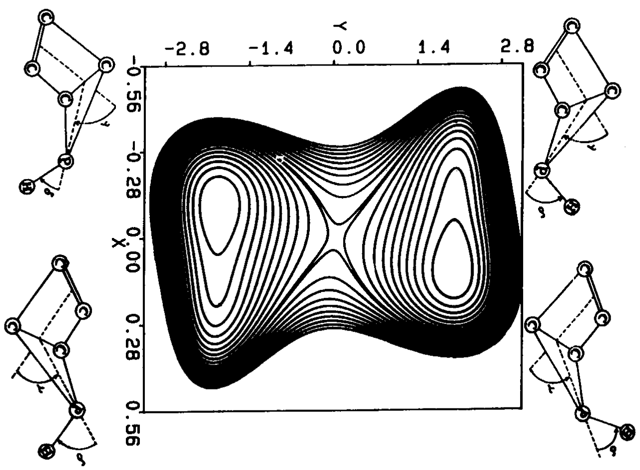


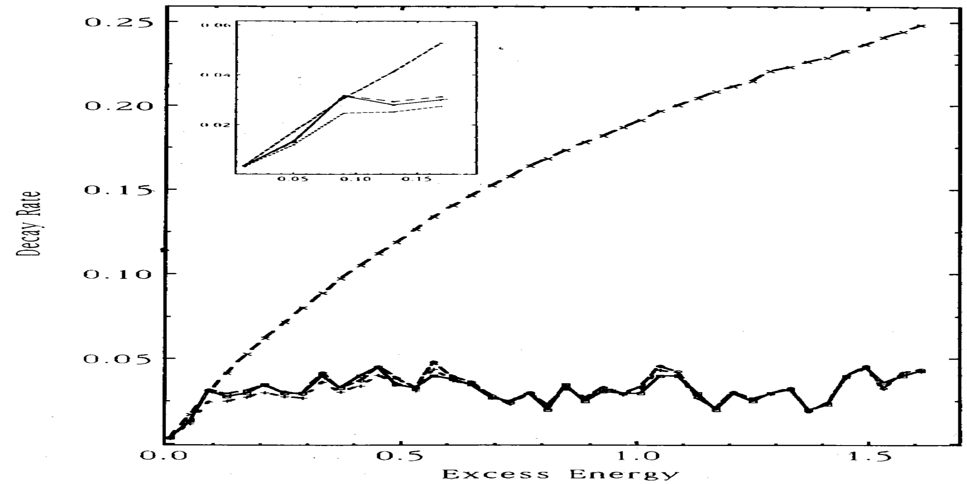
Topper [1997]

- Wells connected by phase space **transition tubes** $\simeq S^1 \times \mathbb{R}$ for 2 DOF
  — Conley, McGehee, 1960s
  — Llibre, Martínez, Simó, Pollack, Child, 1980s
  — De Leon, Mehta, Topper, Jaffé, Farrelly, Uzer, MacKay, 1990s
  — Gómez, Koon, Lo, Marsden, Masdemont, Ross, Yanao, 2000s

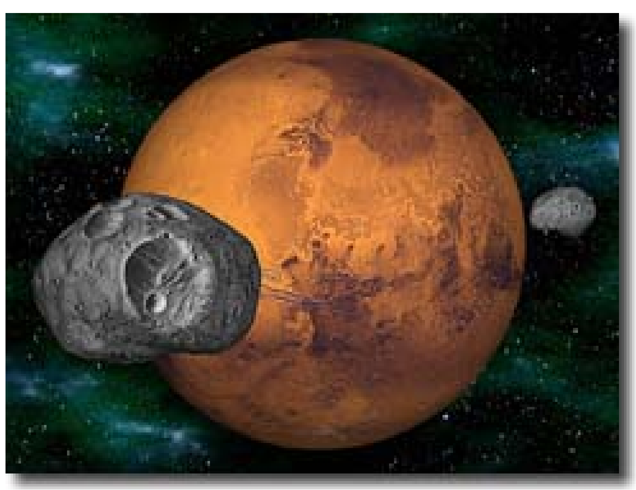
Is this geometric theory correct?

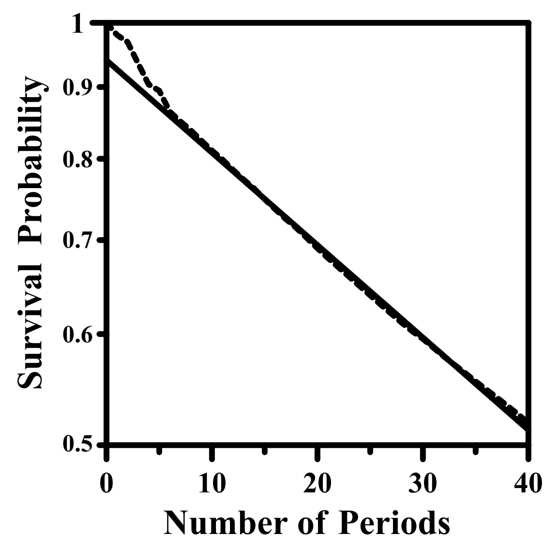
- Good agreement with **direct numerical simulation** — molecular reactions, 'reaction island theory' e.g., De Leon [1992]





— celestial mechanics, asteroid escape rates e.g., Jaffé, Ross, Lo, Marsden, Farrelly, Uzer [2002]





Is this geometric theory correct?

- but **experimental verification** has been lacking

- **Our goal:** We seek to perform experimental verification using a table top experiment with 2 degrees of freedom (DOF)

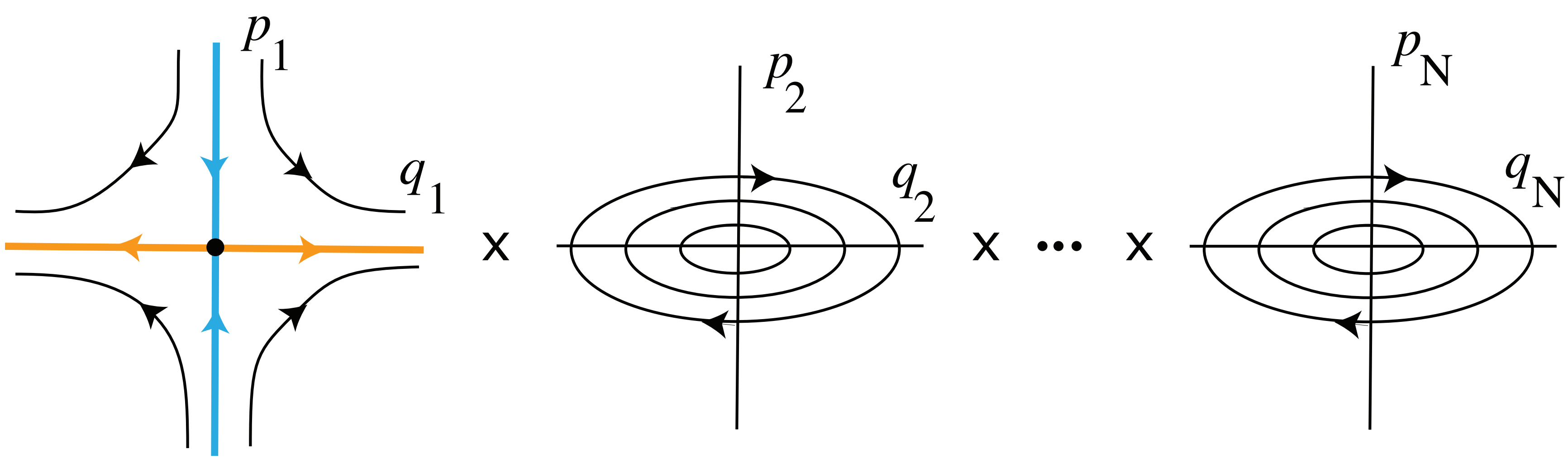
- If successful, apply theory to $\geq$2 DOF systems, combine with **control**:
- structural mechanics
  — re-configurable deformation of flexible objects
  — adaptive structures that can bend, fold, and twist to provide advanced engineering opportunities for deployable structures, mechanical sensors
- vehicle stability
  — capsize problem, etc.

Motion near saddles

- Near **rank 1 saddles** in N DOF, linearized vector field eigenvalues are

$$\pm\lambda \text{ and } \pm i\omega_j,\ j = 2, \ldots, N$$

- Equilibrium point is of type saddle $\times$ center $\times \cdots \times$ center ($N-1$ centers).



the saddle-space projection and $N-1$ center projections — the N canonical planes

Motion near saddles

- For **excess energy** $\Delta E > 0$ above the saddle, there's a normally hyperbolic invariant manifold (NHIM) of bound orbits

$$\mathcal{M}_{\Delta E} = \left\{ \sum_{i=2}^{N} \frac{\omega_i}{2} \left(p_i^2 + q_i^2\right) = \Delta E \right\}$$

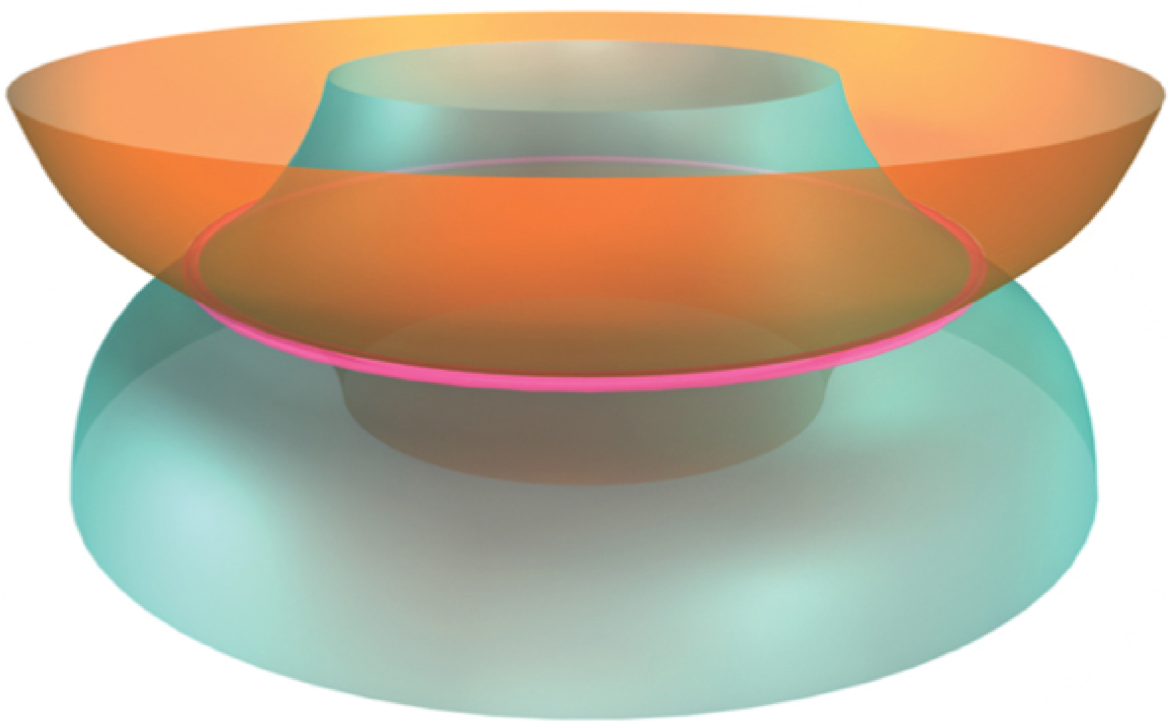
- So, $\mathcal{M}_{\Delta E} \simeq S^{2N-3}$, topologically, a $(2N-3)$-sphere
- $N = 2$,

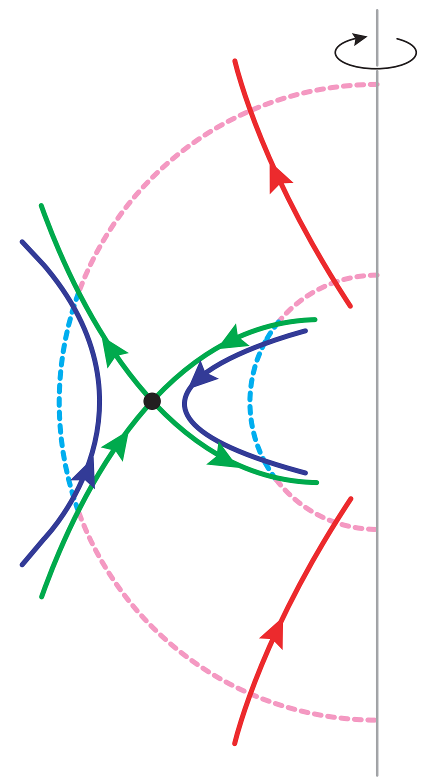
$$\mathcal{M}_{\Delta E} = \left\{ \tfrac{\omega}{2} \left(p_2^2 + q_2^2\right) = \Delta E \right\}$$

$\mathcal{M}_{\Delta E} \simeq S^1$, a periodic orbit of period $T_{\rm po} = \frac{2\pi}{\omega}$

Motion near saddles: 2 DOF

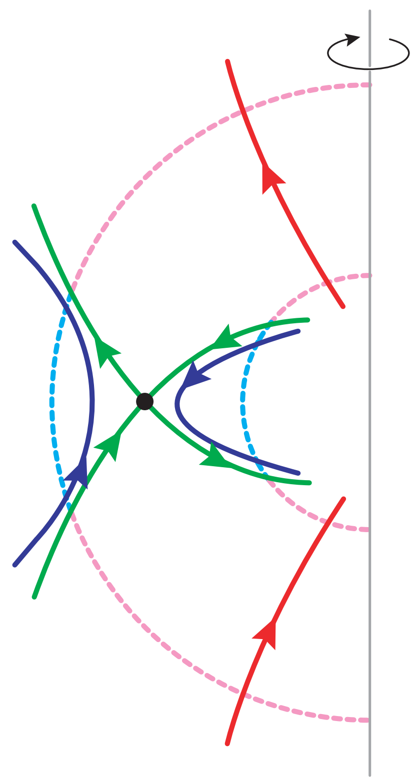
- Cylindrical **tubes** of orbits asymptotic to $\mathcal{M}_{\Delta E}$: stable and unstable invariant manifolds, $W^s_\pm(\mathcal{M}_{\Delta E}), W^u_\pm(\mathcal{M}_{\Delta E})$, $\simeq S^1 \times \mathbb{R}$
- Enclose transitioning trajectories

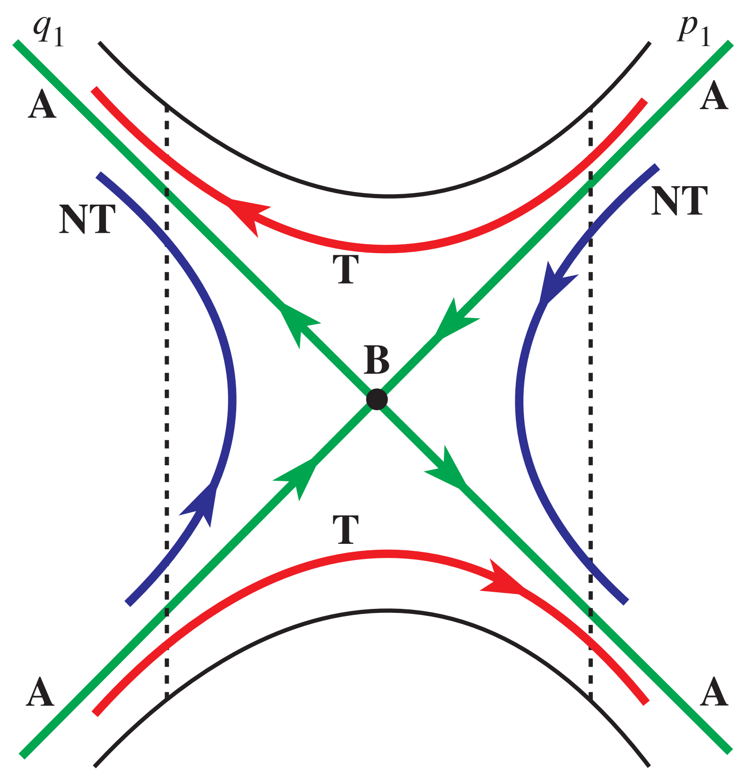




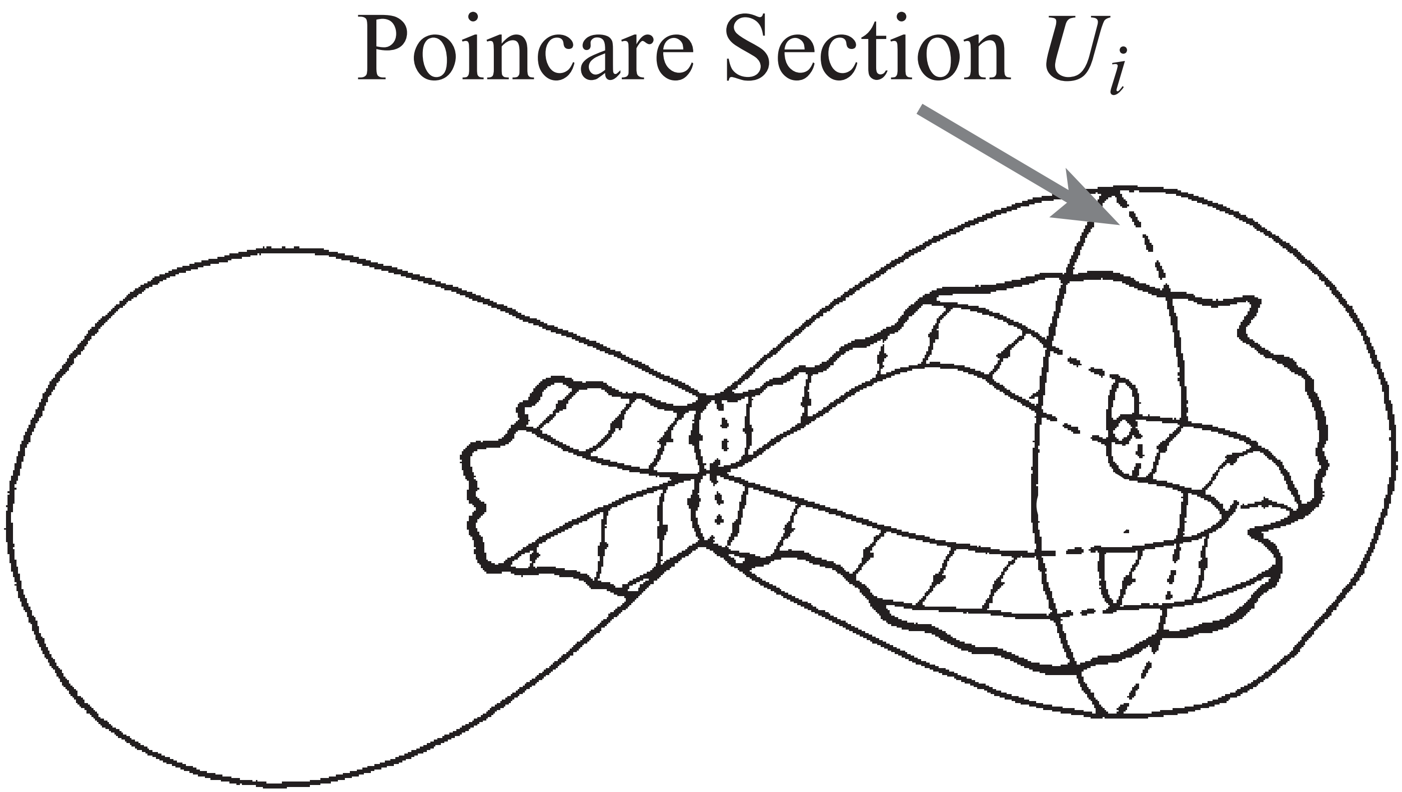
Motion near saddles: 2 DOF

- **B** : **bounded orbits** (periodic): S^1
- **A** : **asymptotic orbits** to 1-sphere: $S^1 \times \mathbb{R}$ **(tubes)**
- **T** : **transitioning** and **NT** : **non-transitioning** orbits.





Tube dynamics

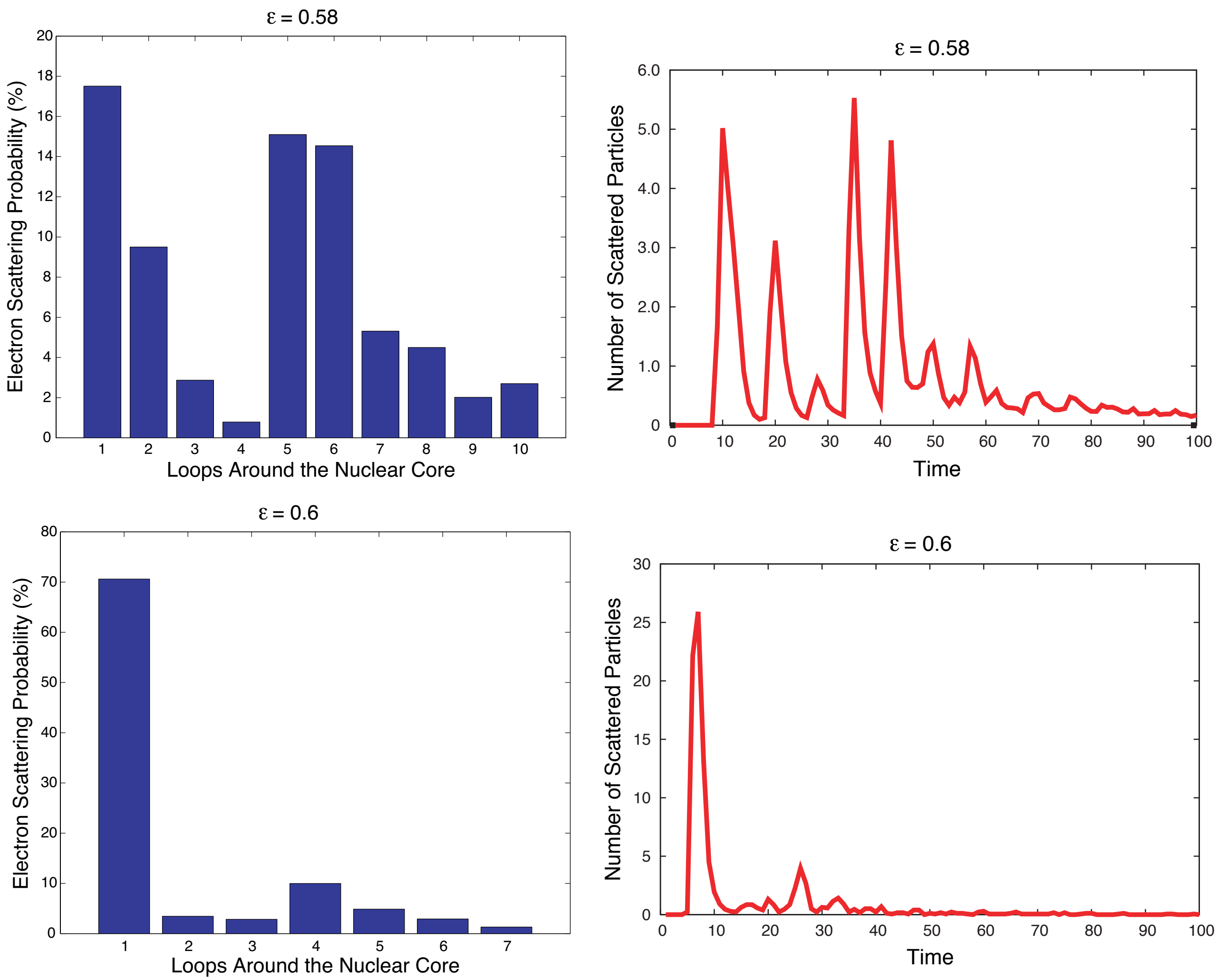


De Leon [1992]

- **Tube dynamics:** All transitioning motion between wells connected by bottlenecks must occur through tube
  - Imminent transition regions, transitioning fractions
  - Consider k Poincaré sections U_i, various excess energies ΔE

Verification by simulation

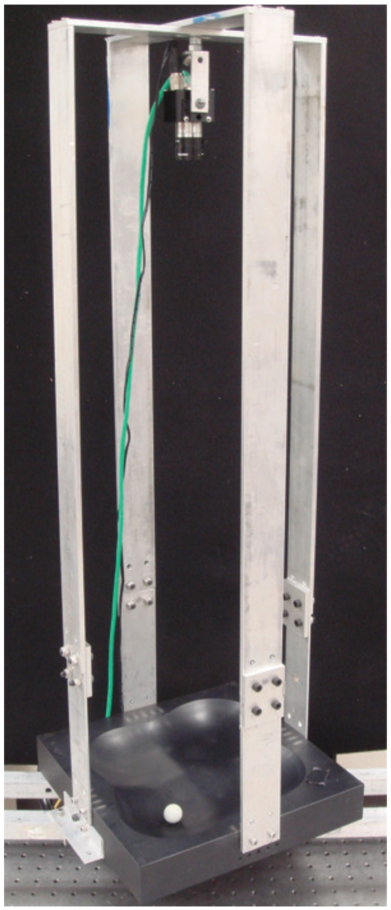
- Structured transition statistics in chemistry, etc 3+ DOF

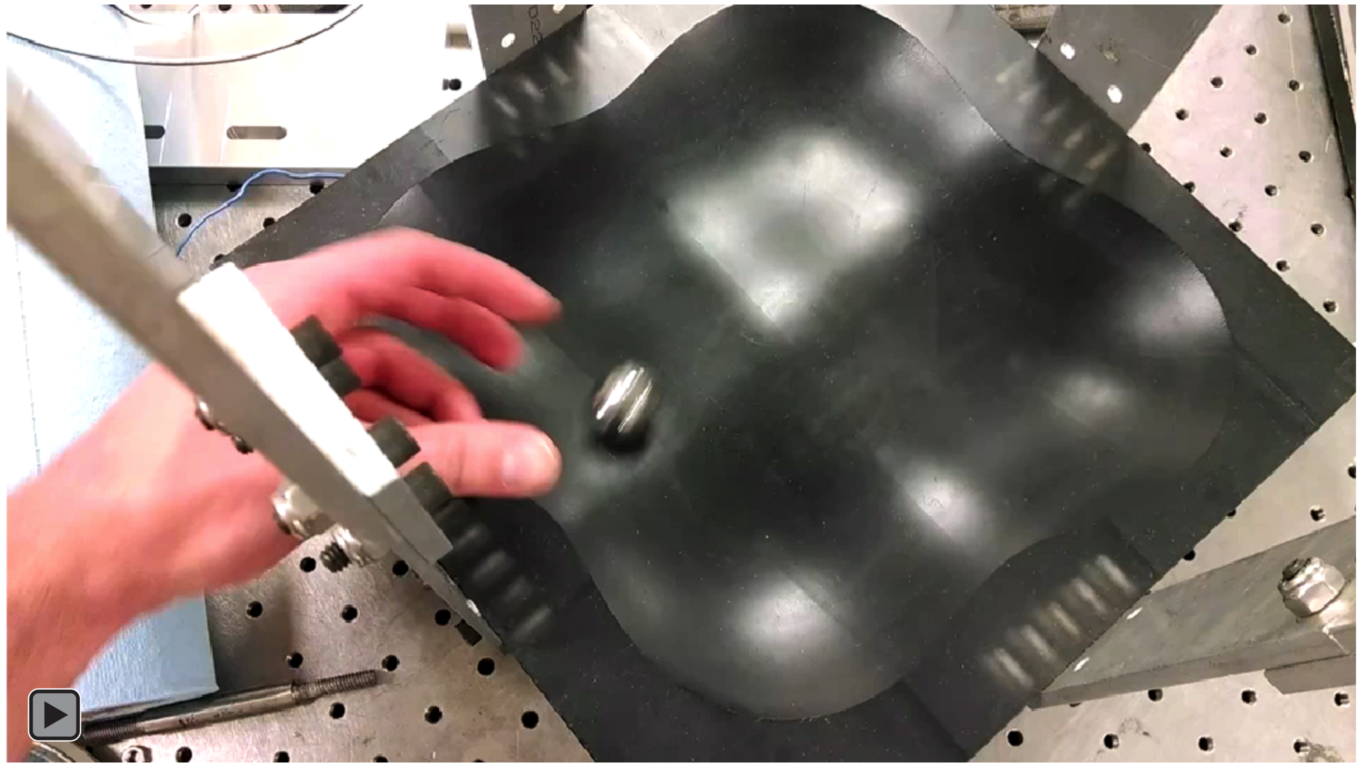


Gabern, Koon, Marsden, Ross [2005]

Verification by experiment

- Simple table top experiments; e.g., ball rolling on a 3D-printed surface



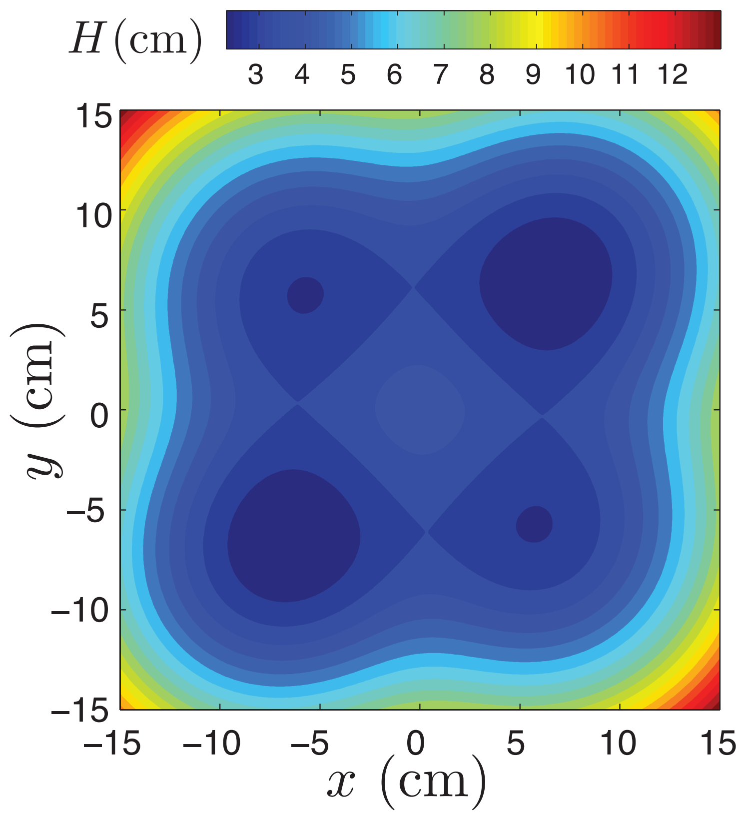


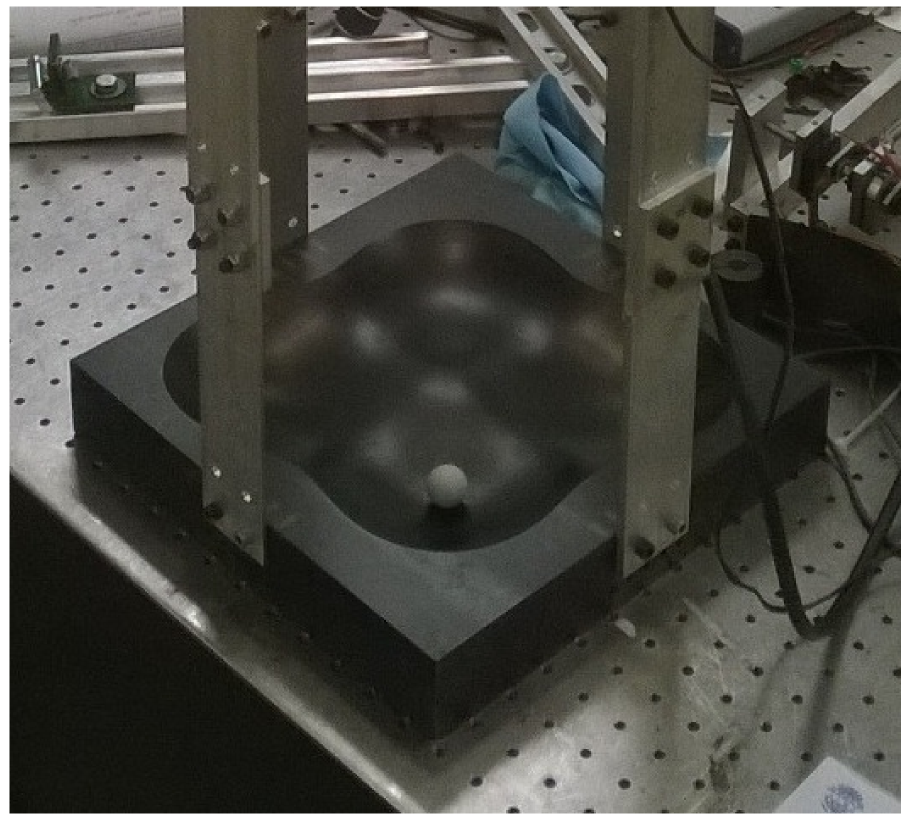
Virgin, Lyman, Davis [2010] Am. J. Phys.

Ball rolling on a surface — 2 DOF

- The potential energy is $V(x,y) = gH(x,y) - V_0$, where the surface is arbitrary, e.g., we chose

$$H(x,y) = \alpha(x^2 + y^2) - \beta(\sqrt{x^2+\gamma} + \sqrt{y^2+\gamma}) - \xi xy + H_0.$$





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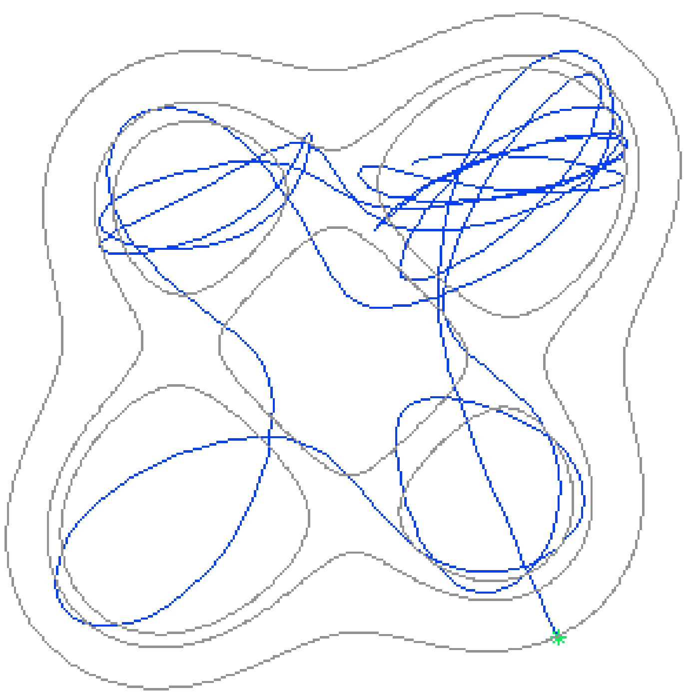
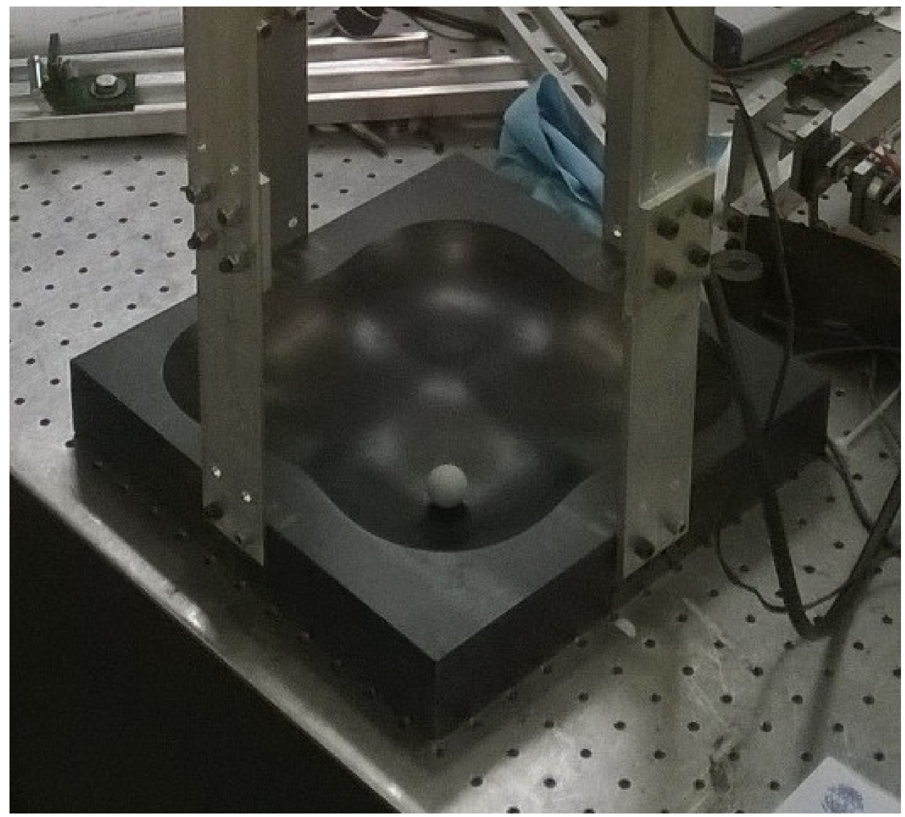
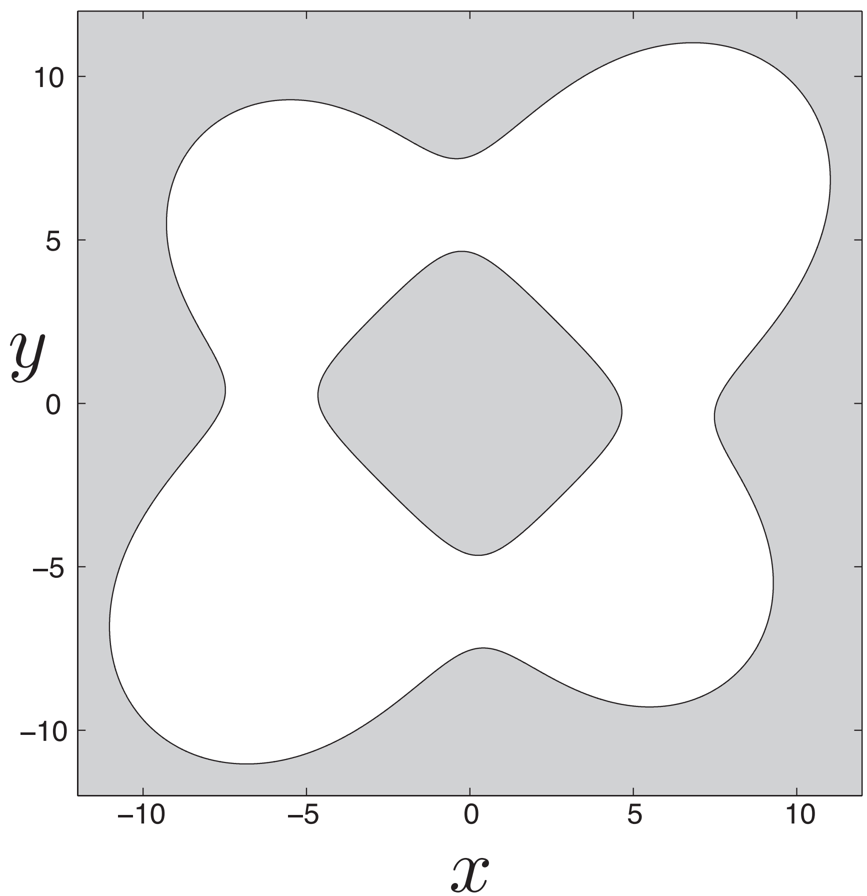


typical experimental trial



Transition tubes in the rolling ball system



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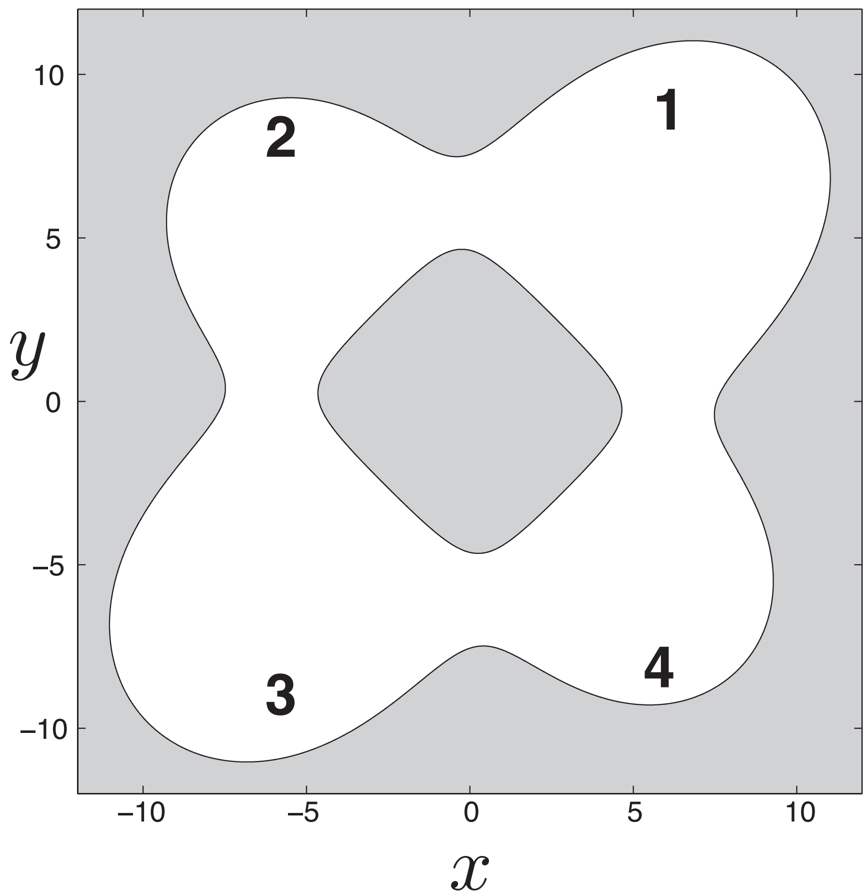

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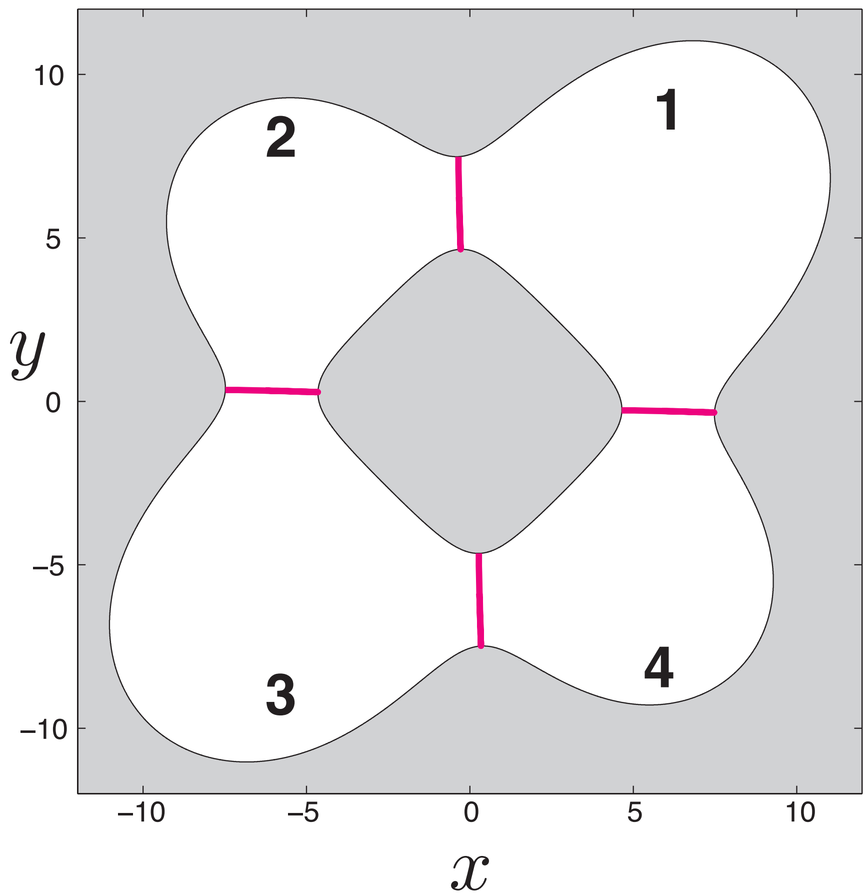

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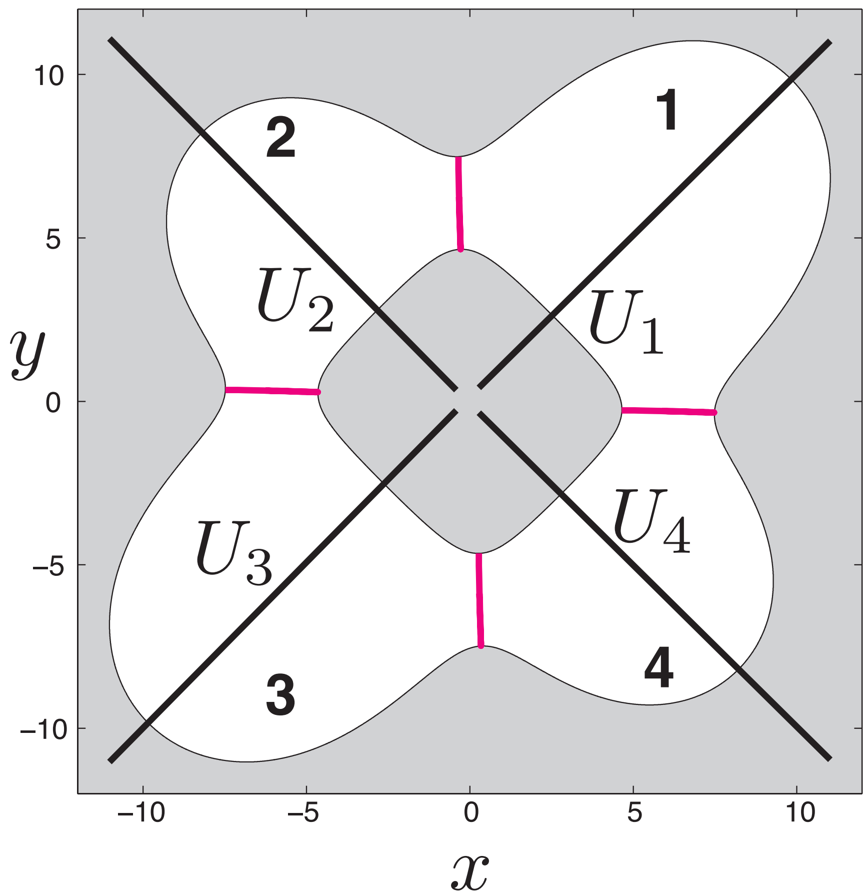

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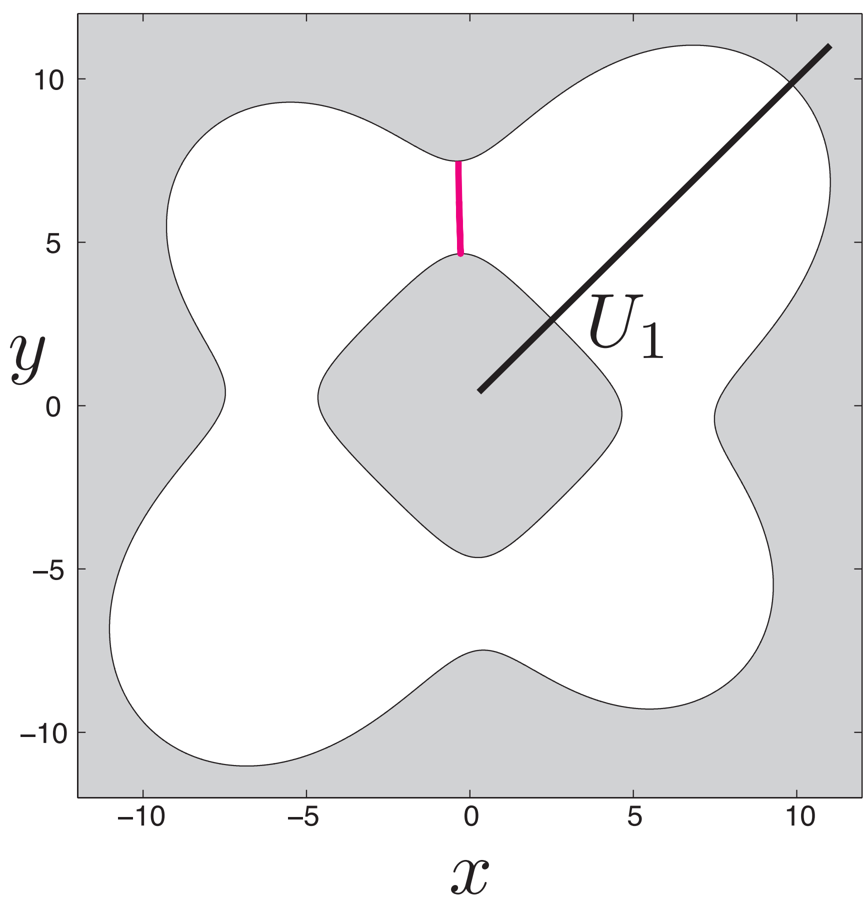

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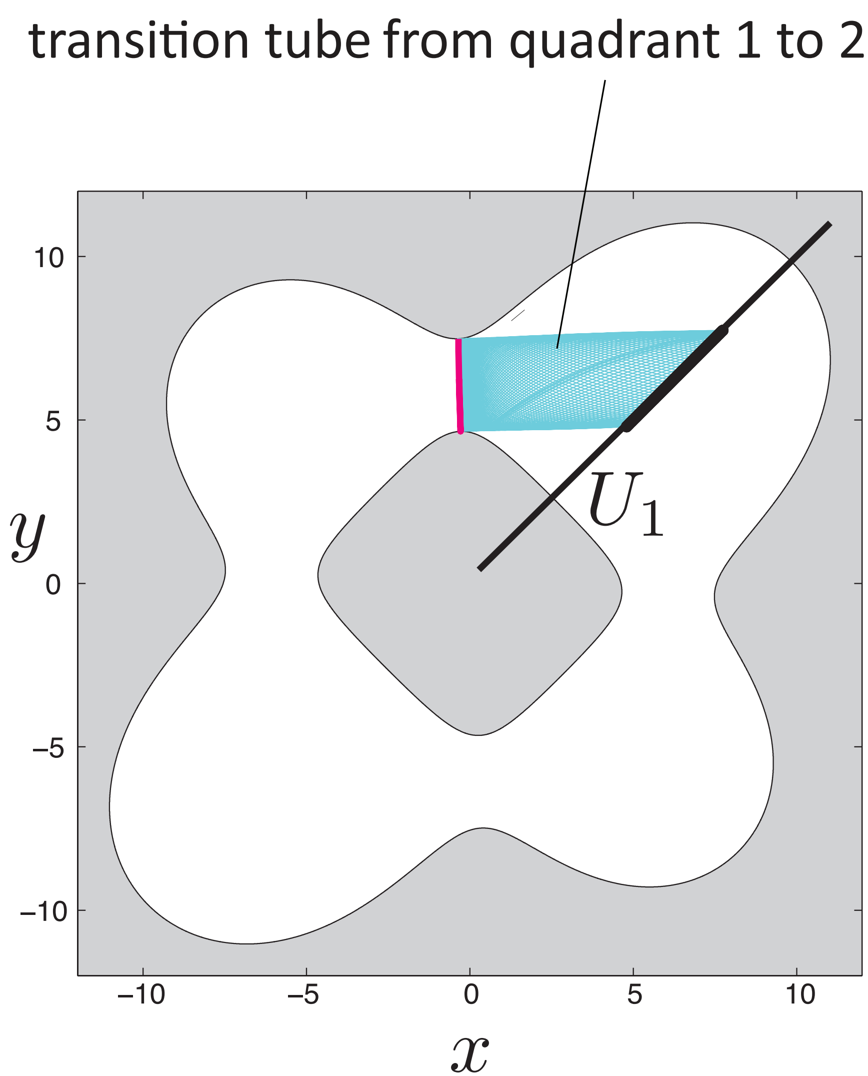

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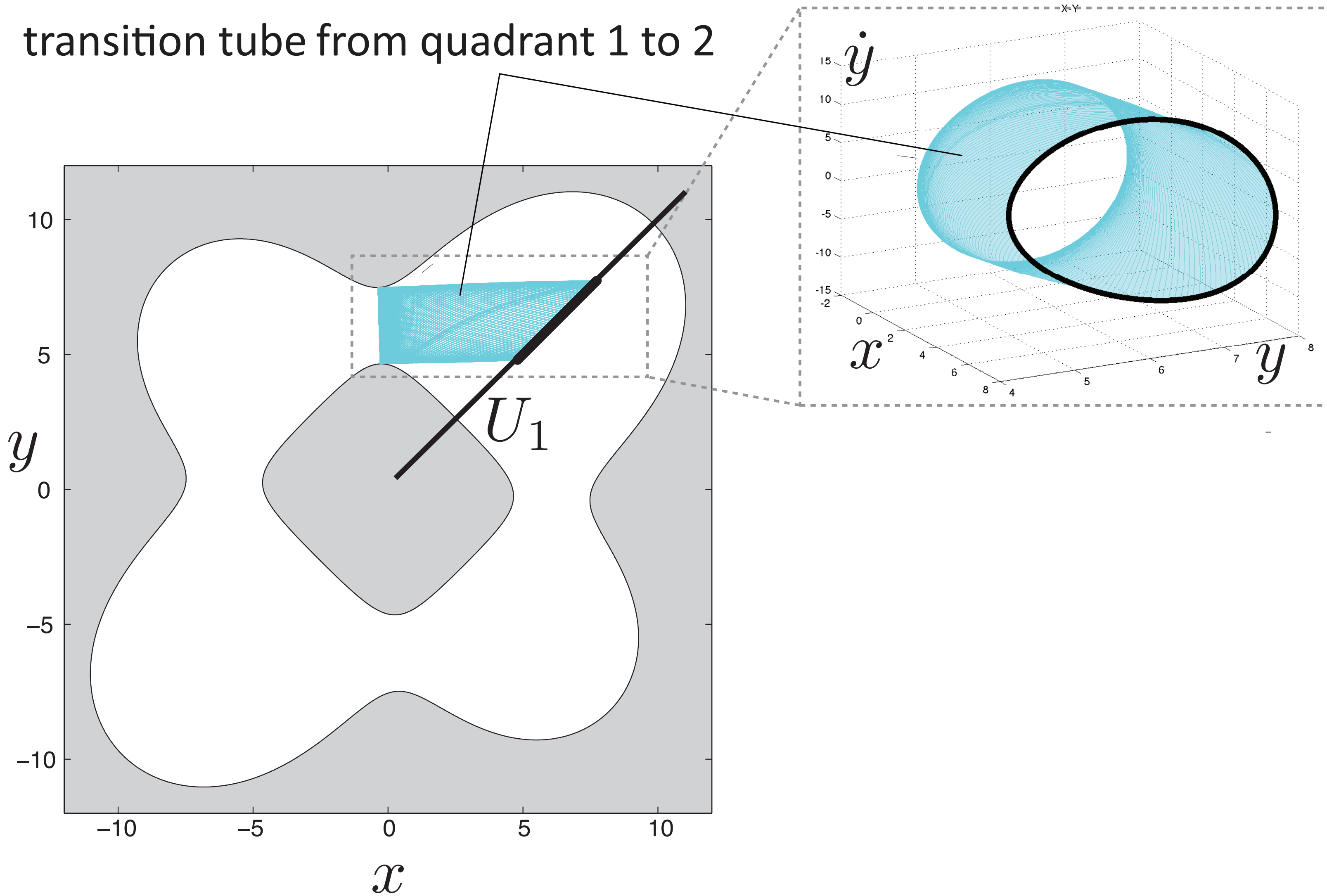

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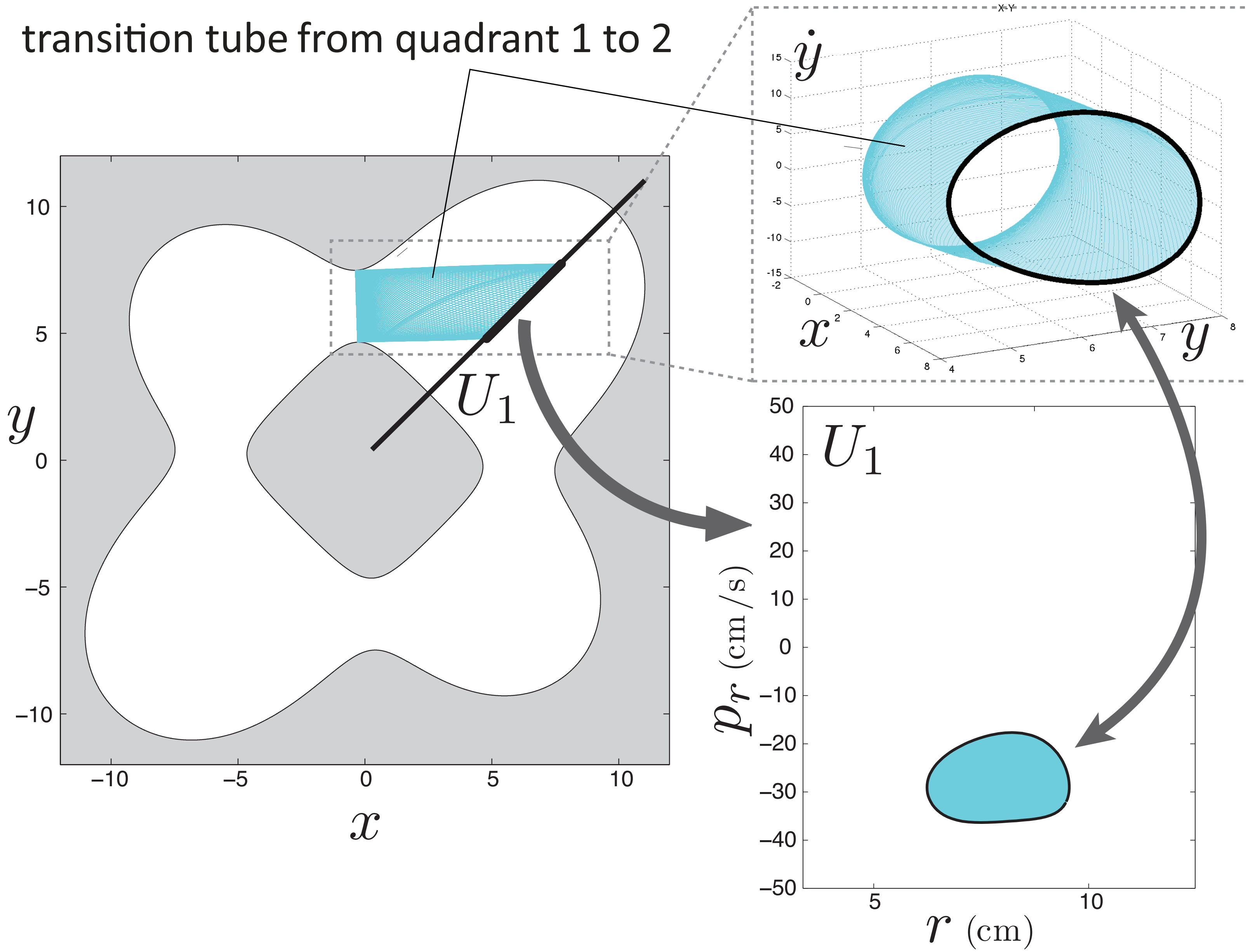

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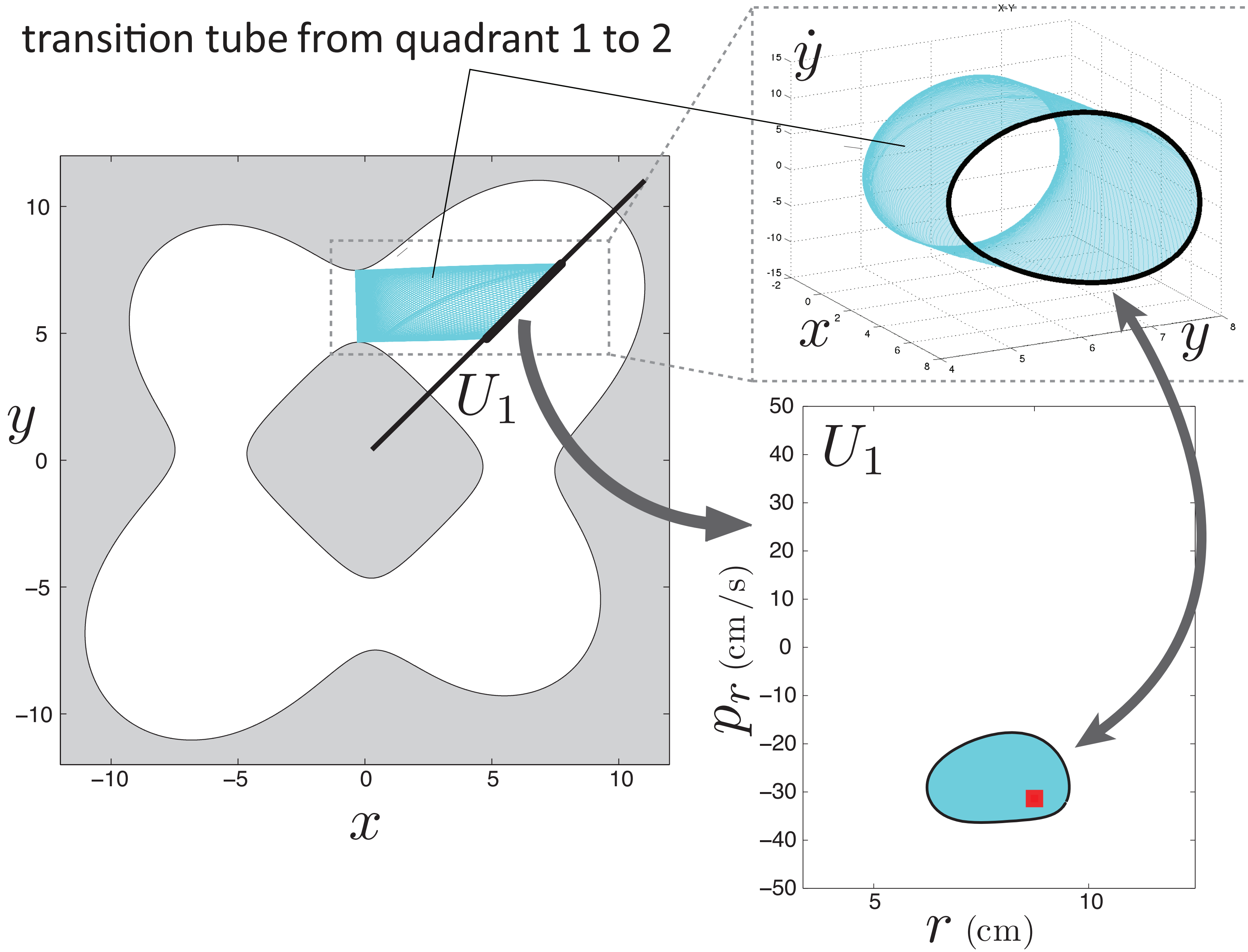

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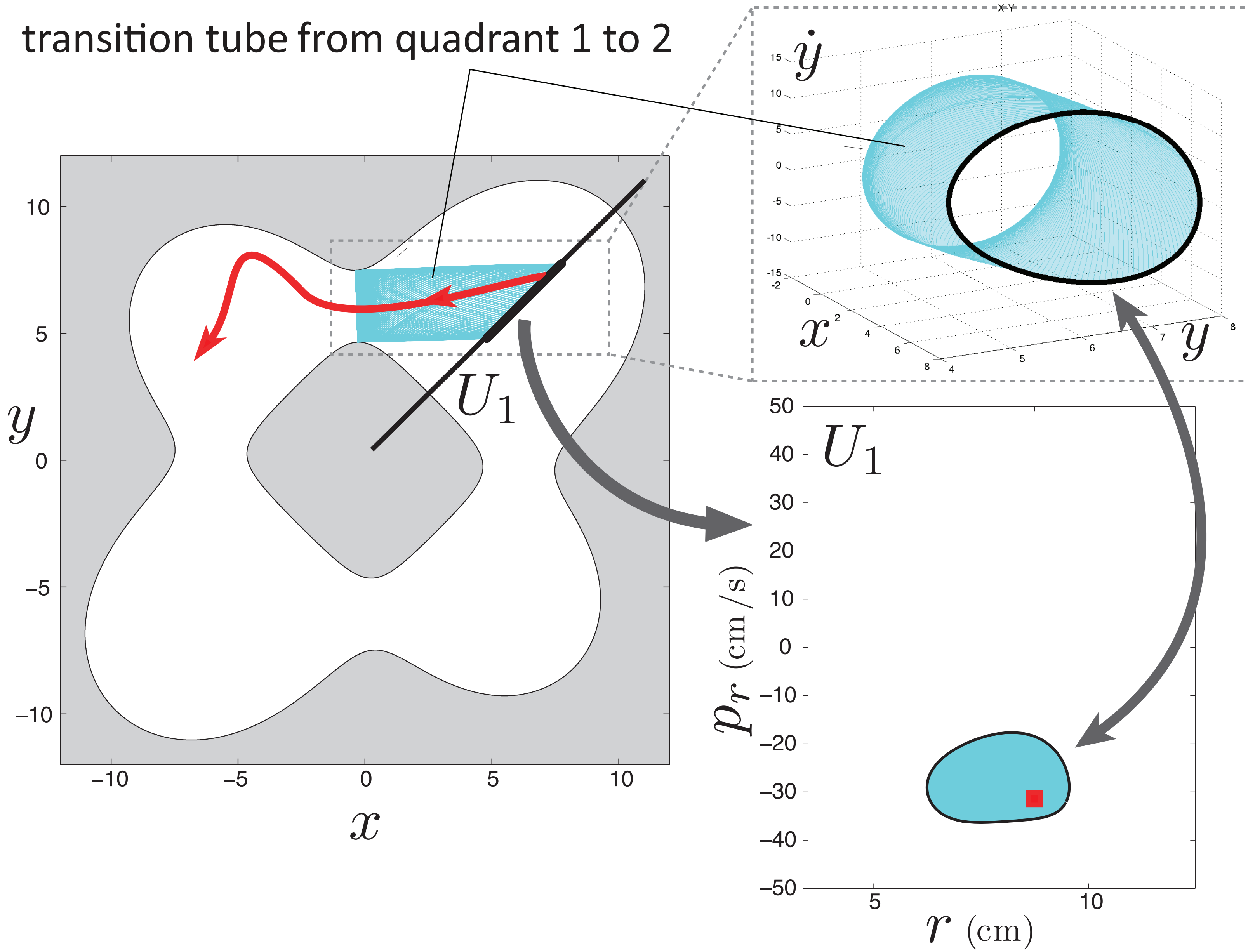

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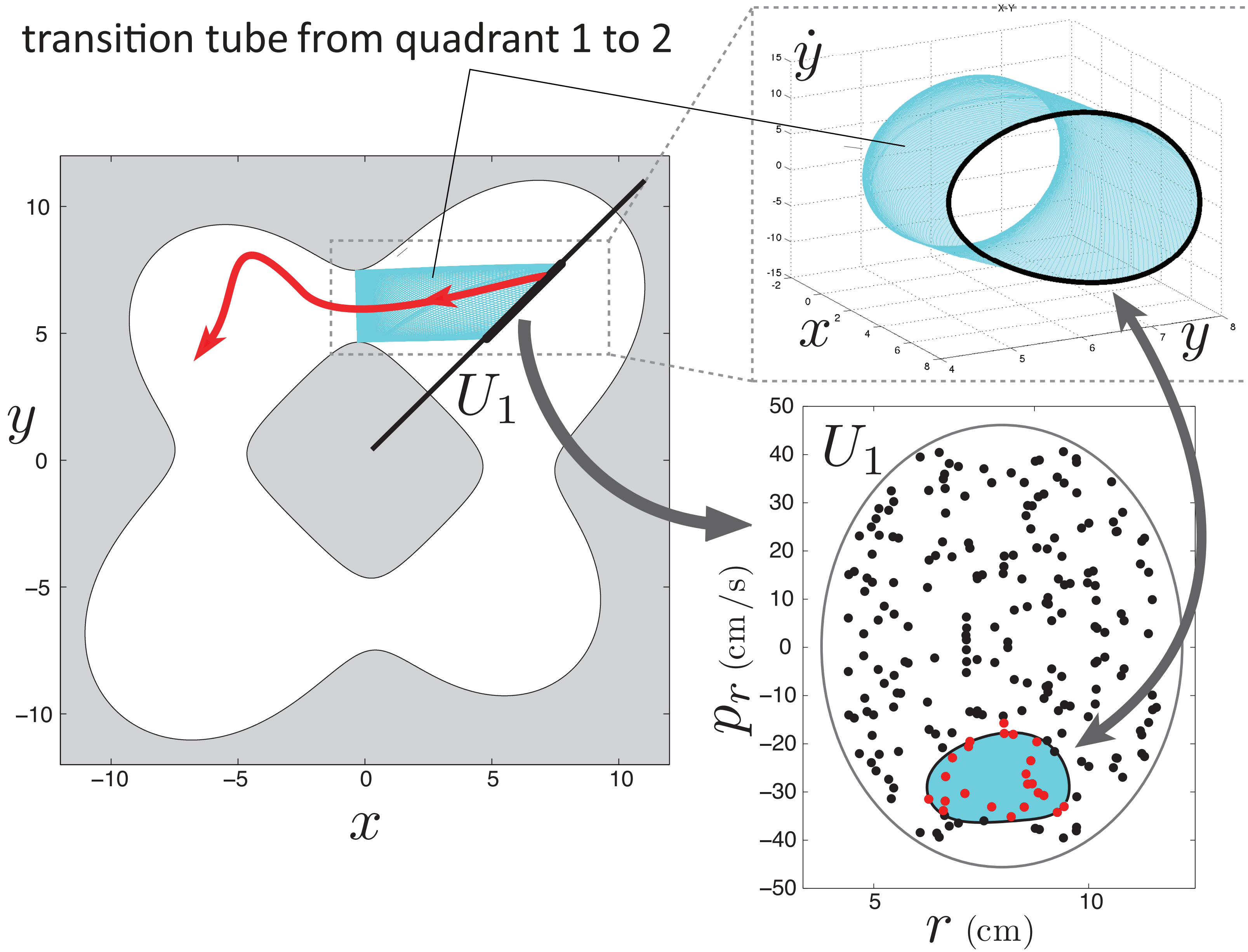
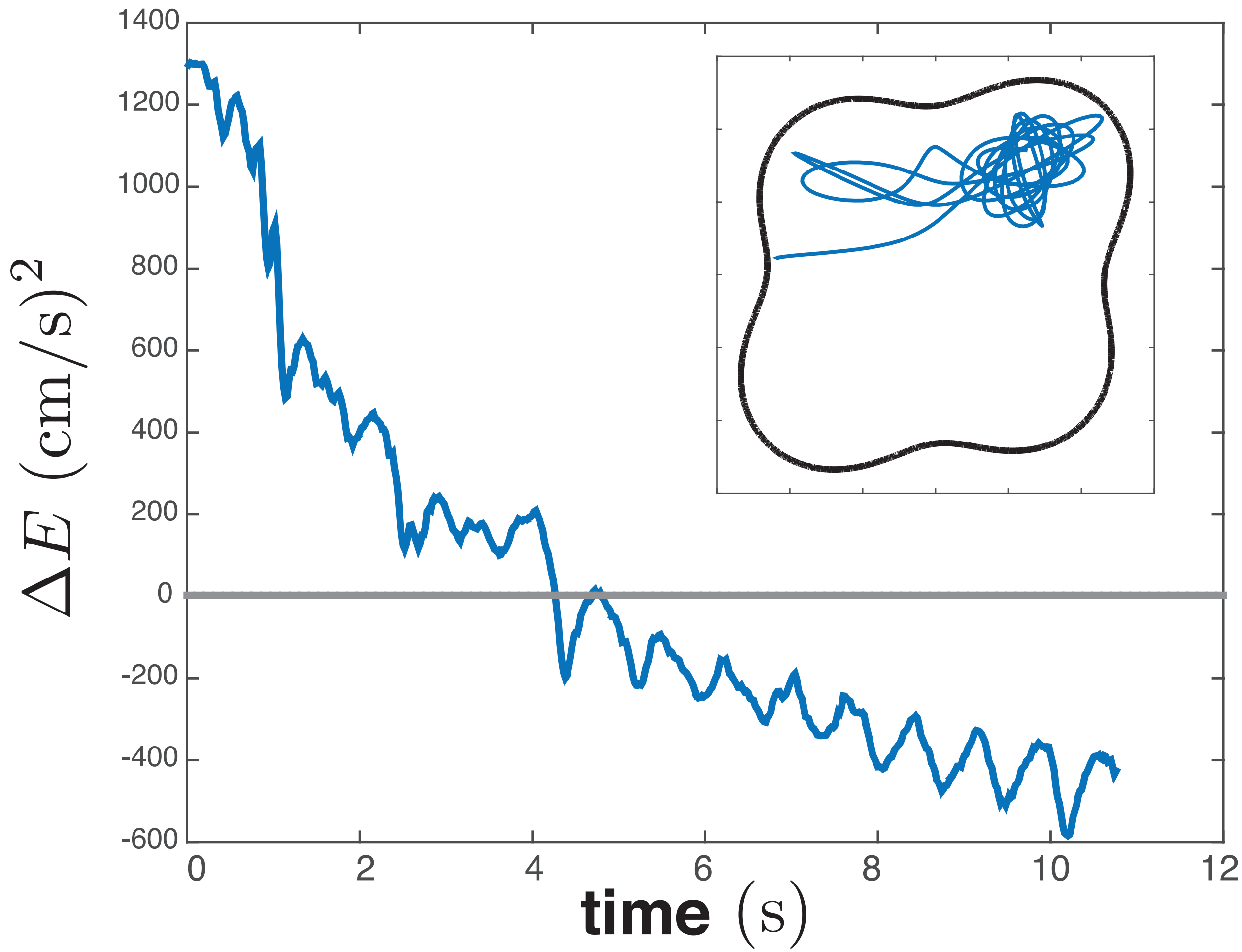
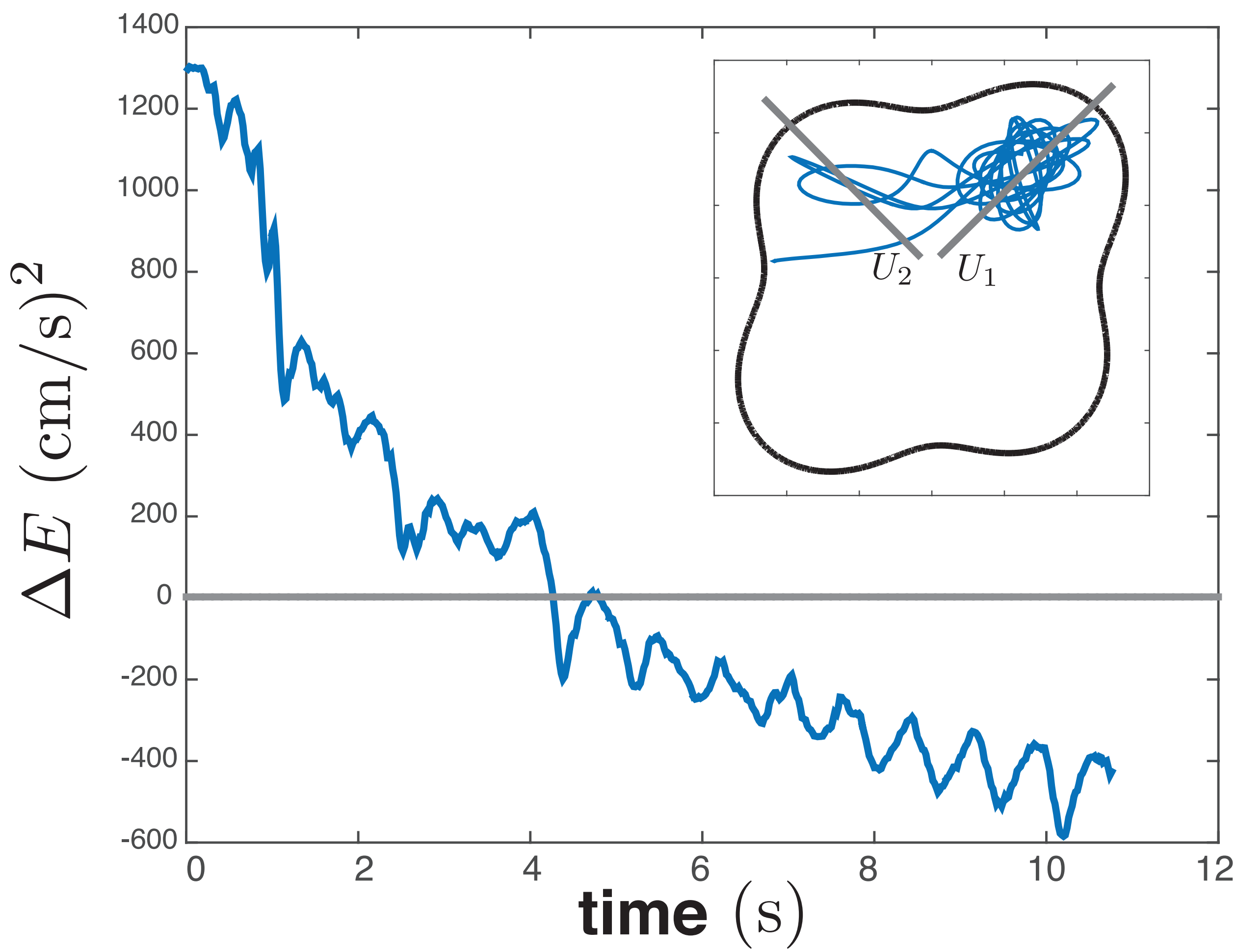


Analysis of experimental data



- 120 experimental trials of about 10 seconds each, recorded at 50 Hz

Analysis of experimental data



- 120 experimental trials of about 10 seconds each, recorded at 50 Hz
- 3500 intersections of Poincaré sections, sorted by energy

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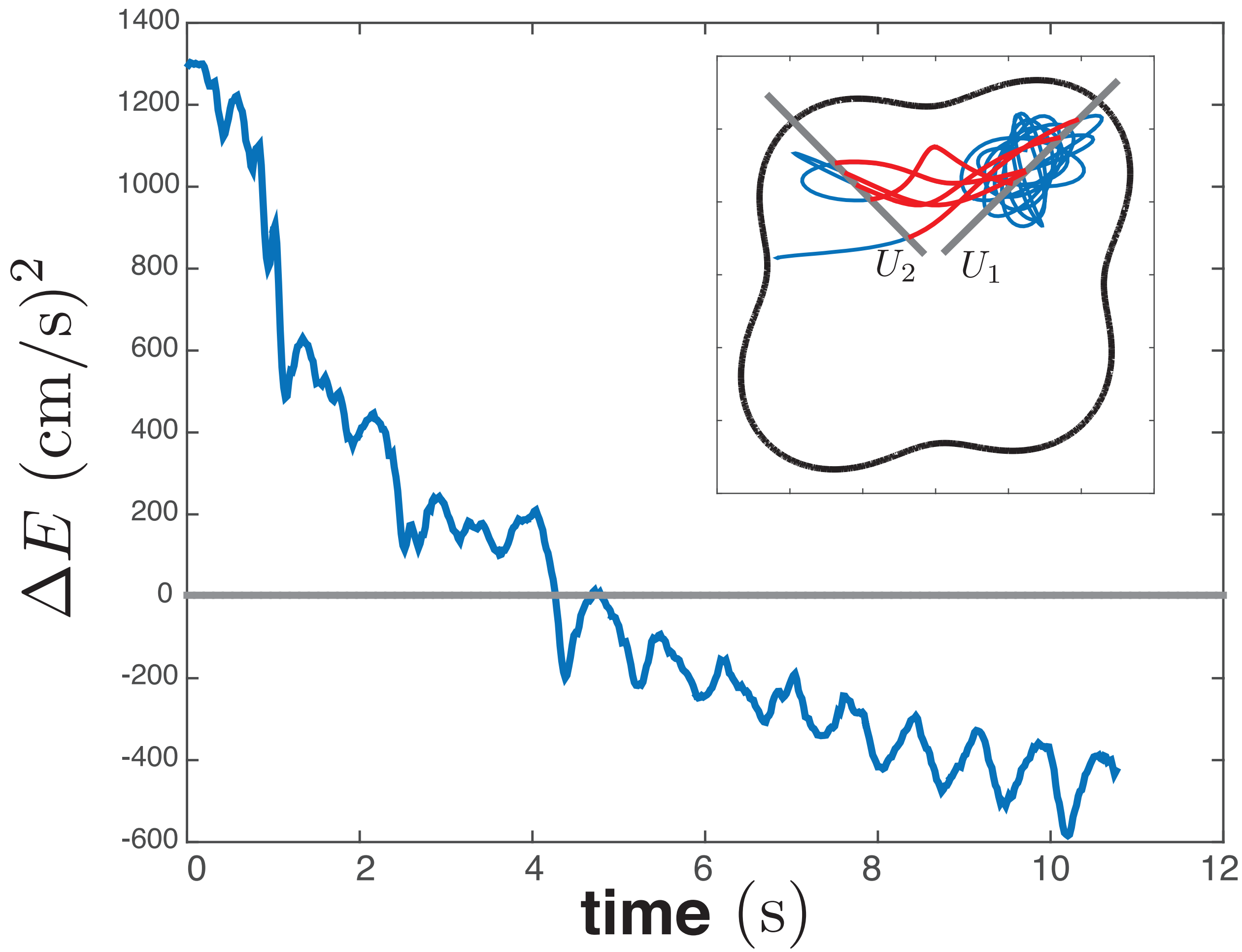

- 120 experimental trials of about 10 seconds each, recorded at 50 Hz
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- 400 transitions detected

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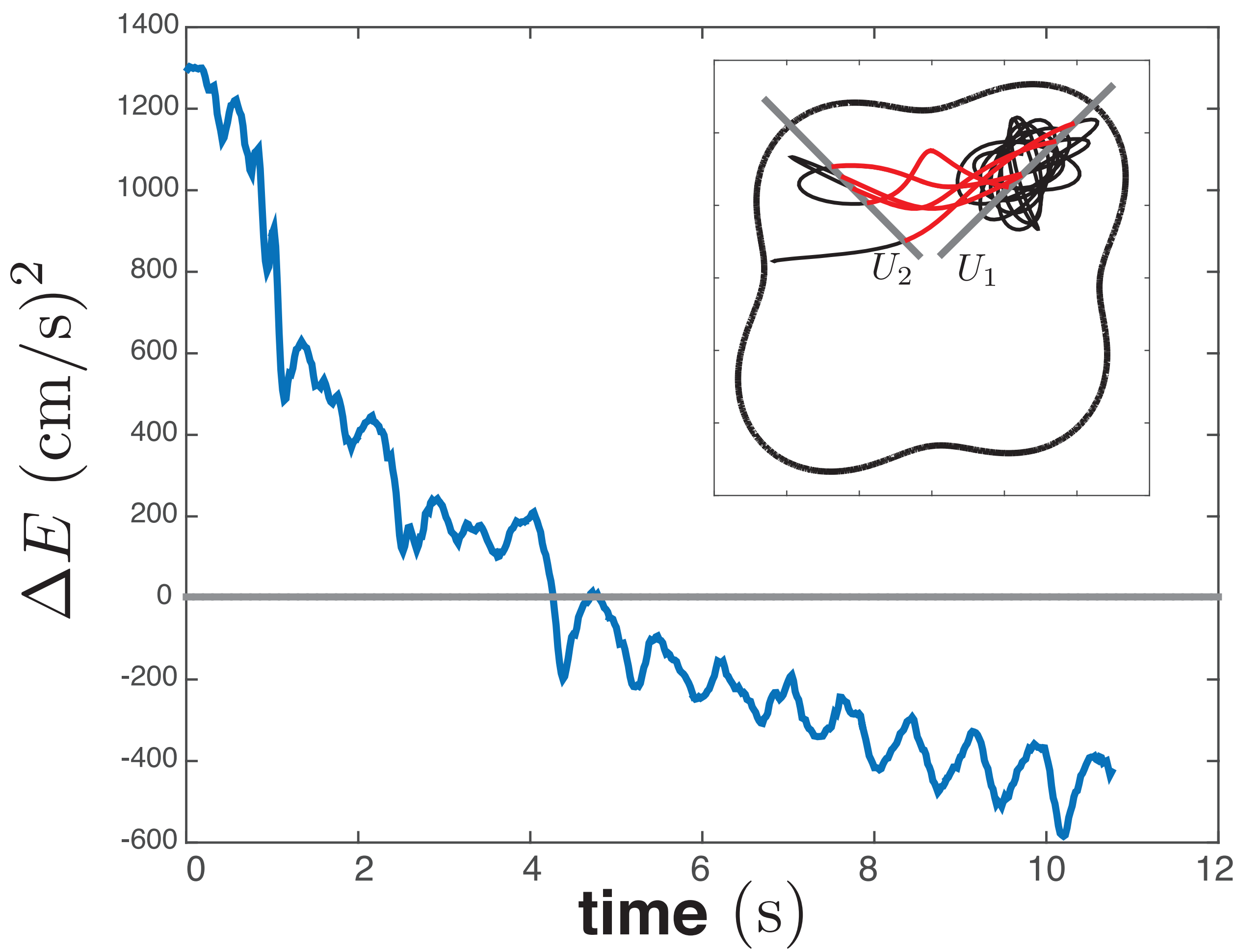

- 120 experimental trials of about 10 seconds each, recorded at 50 Hz
- 3500 intersections of Poincaré sections, sorted by energy
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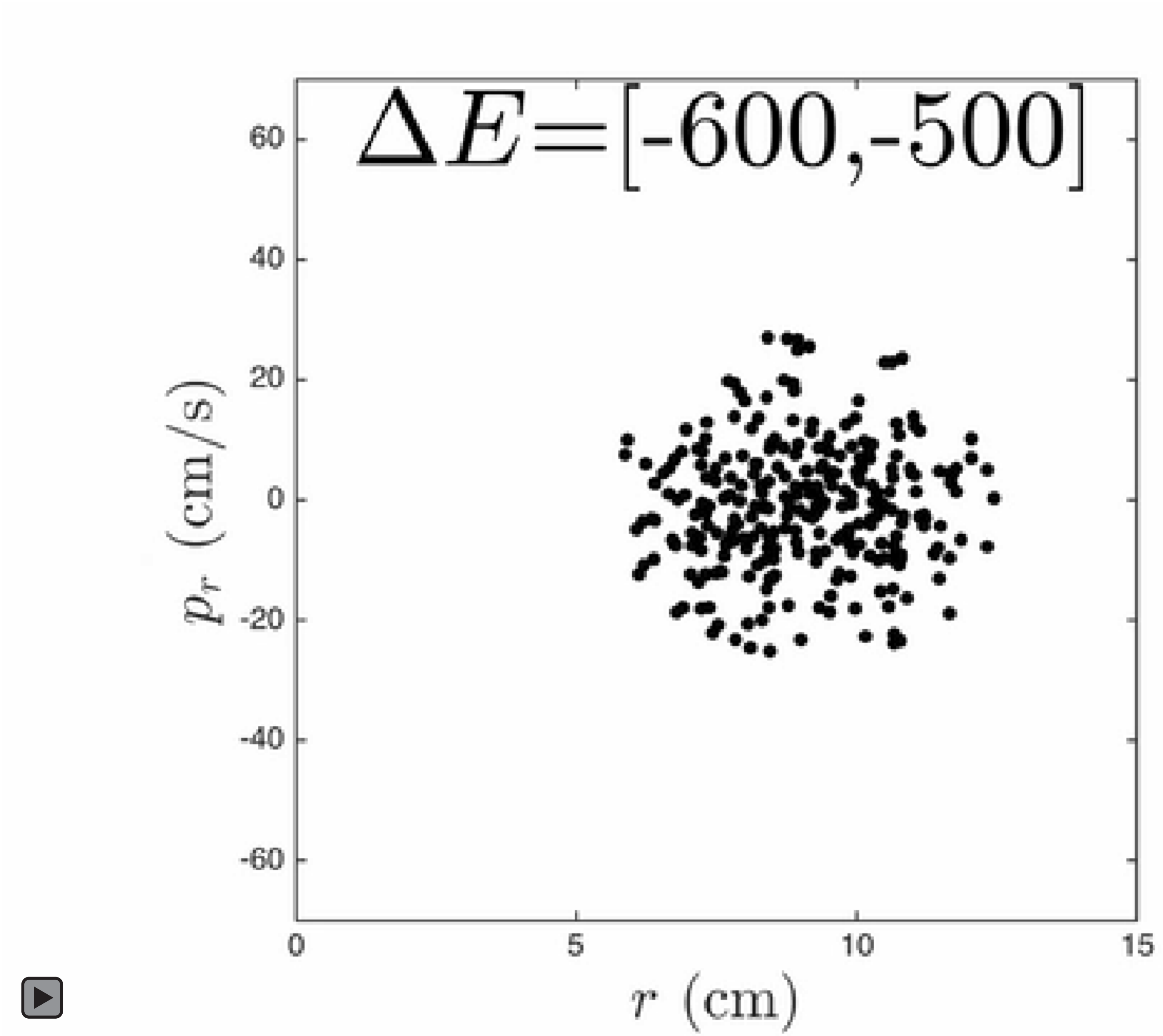
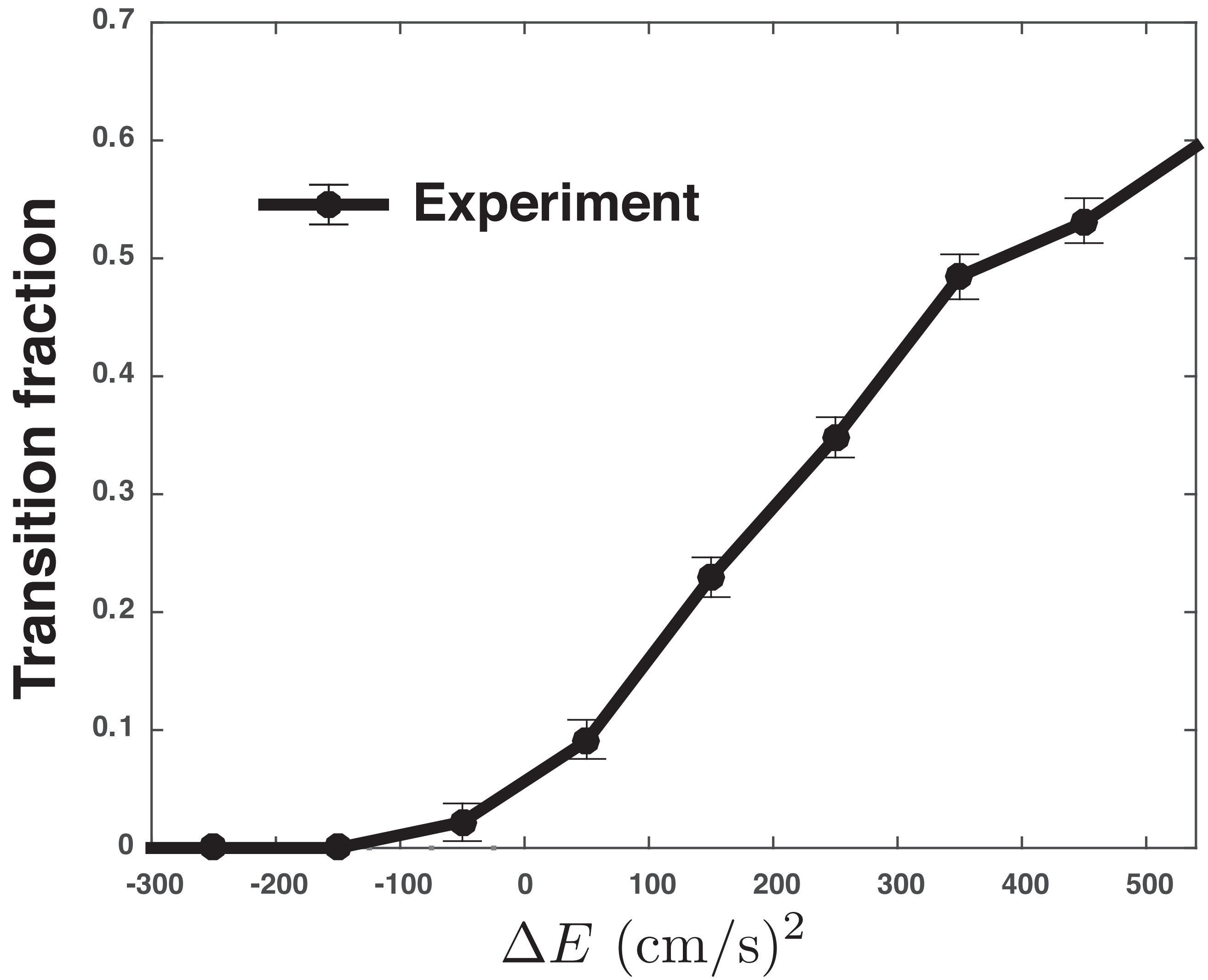
Poincaré sections at various energy ranges



Experimental confirmation of transition tubes

- Theory predicts $> 95\%$ of transitions
- Consider overall trend in transition fraction as excess energy grows



Theory for small excess energy, ΔE

- Area of the transitioning region, the tube cross-section (MacKay [1990])

$$A_{\text{trans}} = T_{\text{po}} \Delta E$$

where $T_{\text{po}} = 2\pi/\omega$ period of unstable periodic orbit in bottleneck

- Area of energy surface

$$A_{\Delta E} = A_0 + \tau \Delta E$$

where

$$A_0 = 2 \int_{r_{\min}}^{r_{\max}} \sqrt{-\frac{14}{5} g H(r) (1 + \frac{\partial H}{\partial r}^2 (r))} dr$$

and

$$\tau = \int_{r_{\min}}^{r_{\max}} \sqrt{\frac{\frac{14}{5}(1 + \frac{\partial H}{\partial r}^2 (r))}{-gH(r)}} dr$$

Theory for small excess energy, ΔE

- The transitioning fraction, under well-mixed assumption,

$$p_{\text{trans}} = \frac{A_{\text{trans}}}{A_{\Delta E}} = \frac{T_{\text{po}}}{A_0}\Delta E\left(1 - \frac{\tau}{A_0}\Delta E + \mathcal{O}(\Delta E^2)\right)$$

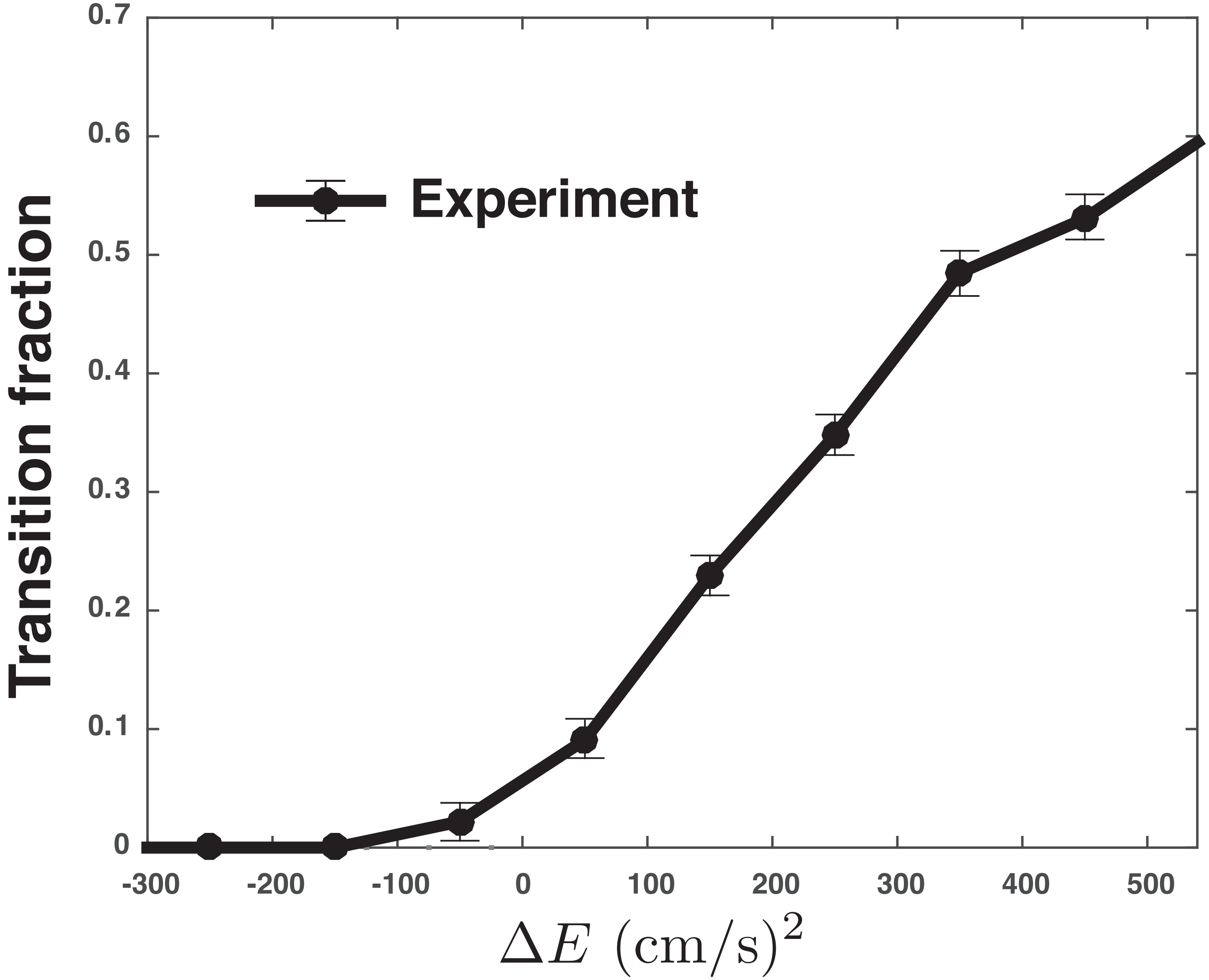
- For small ΔE, growth in p_{trans} with ΔE is linear, with slope

$$\frac{\partial p_{\text{trans}}}{\partial \Delta E} = \frac{T_{\text{po}}}{A_0}$$

- For slightly larger values of ΔE, there will be a correction term leading to a decreasing slope,

$$\frac{\partial p_{\text{trans}}}{\partial \Delta E} = \frac{T_{\text{po}}}{A_0}\left(1 - 2\frac{\tau}{A_0}\Delta E\right)$$

Theory for small excess energy, ΔE



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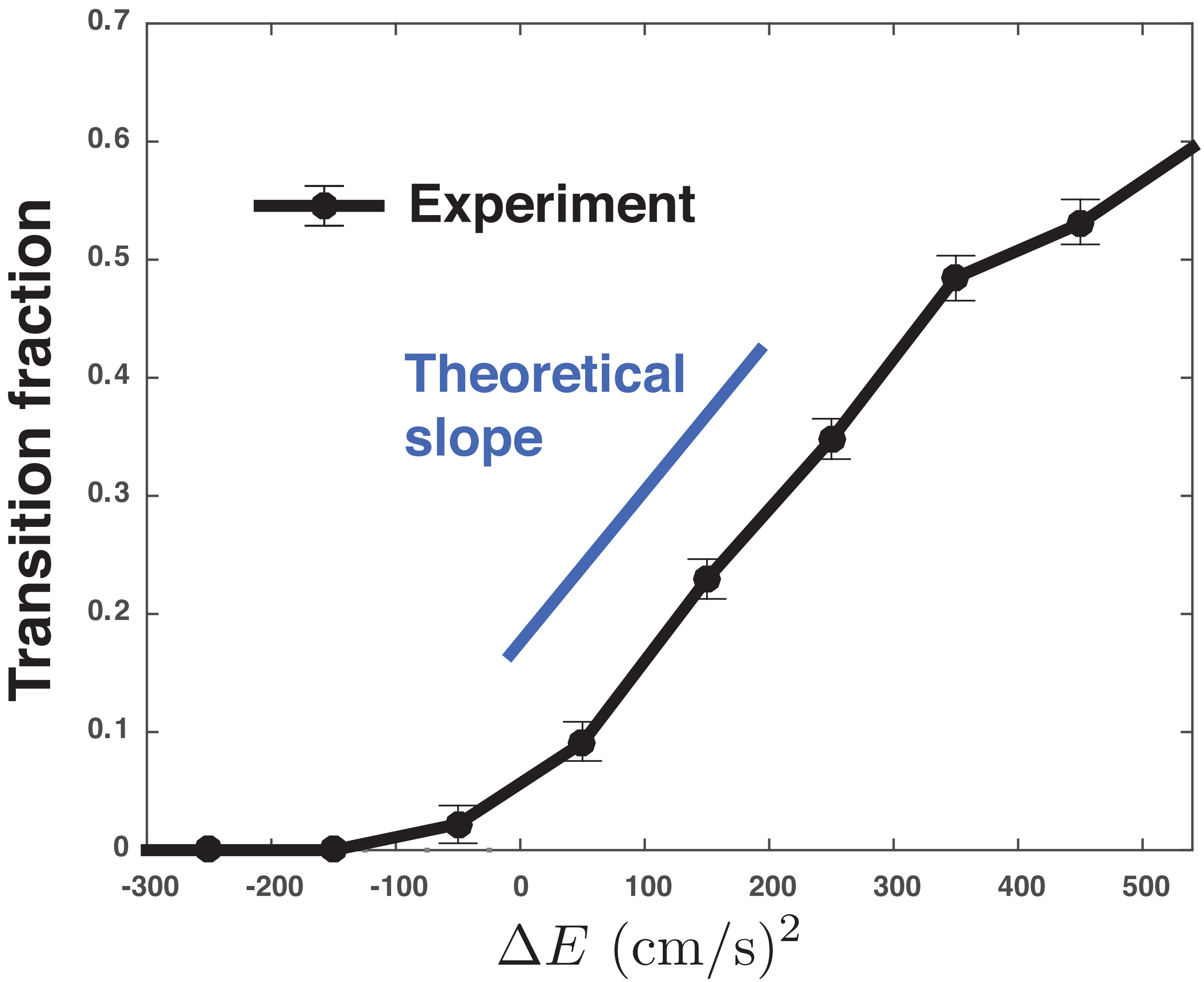

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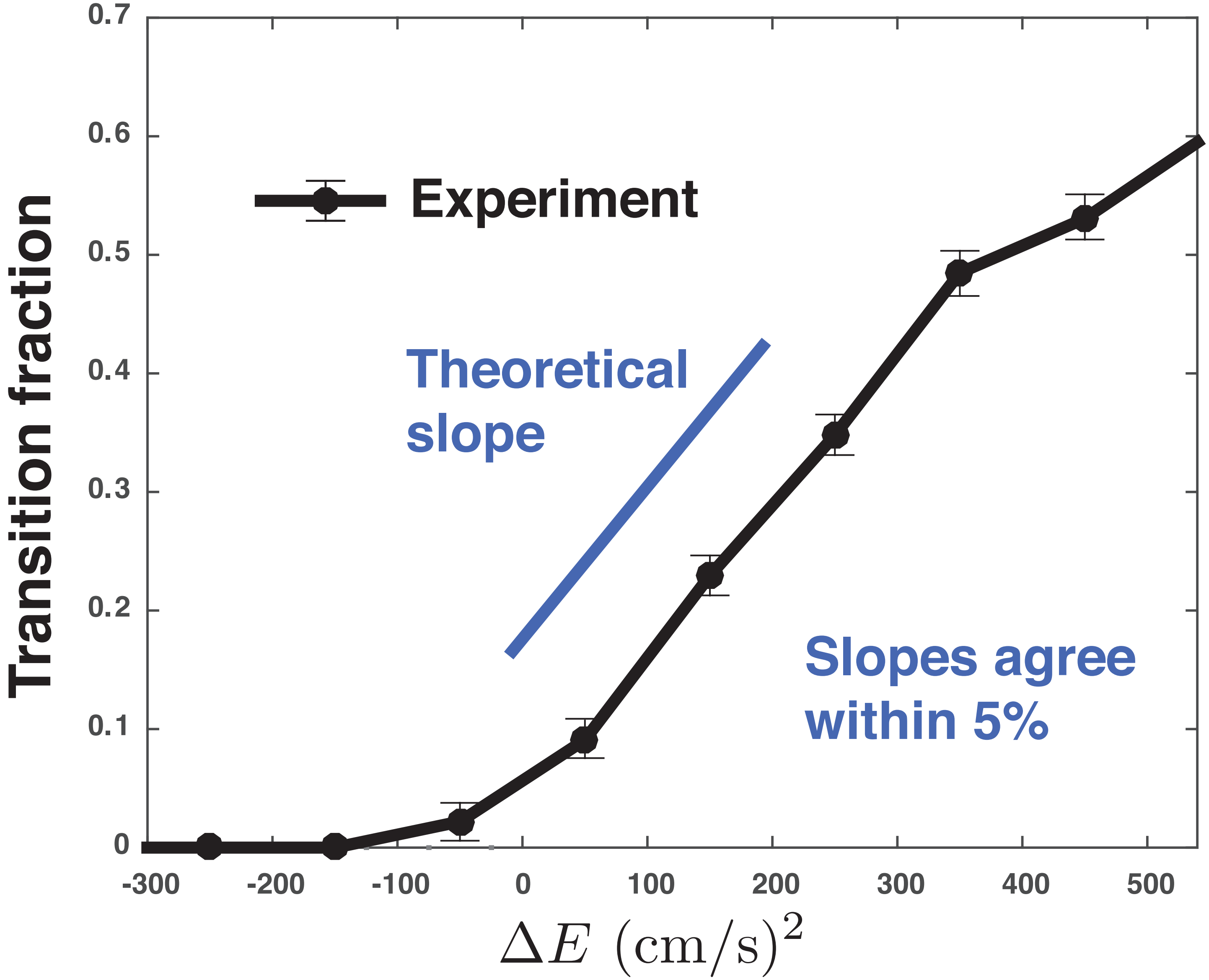
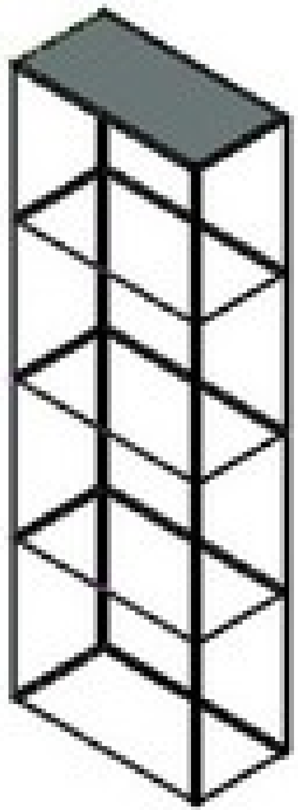
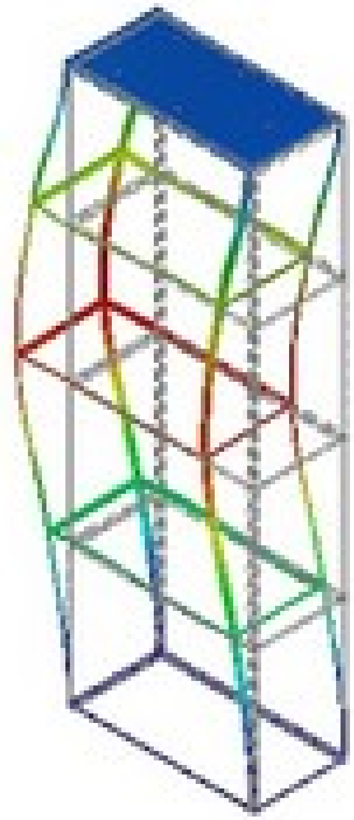
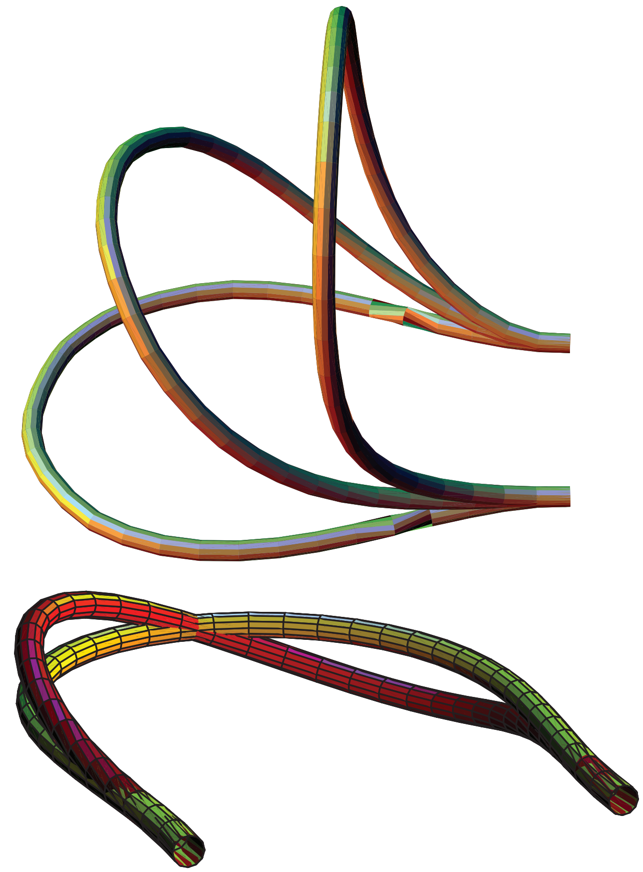


Next steps — structural mechanics

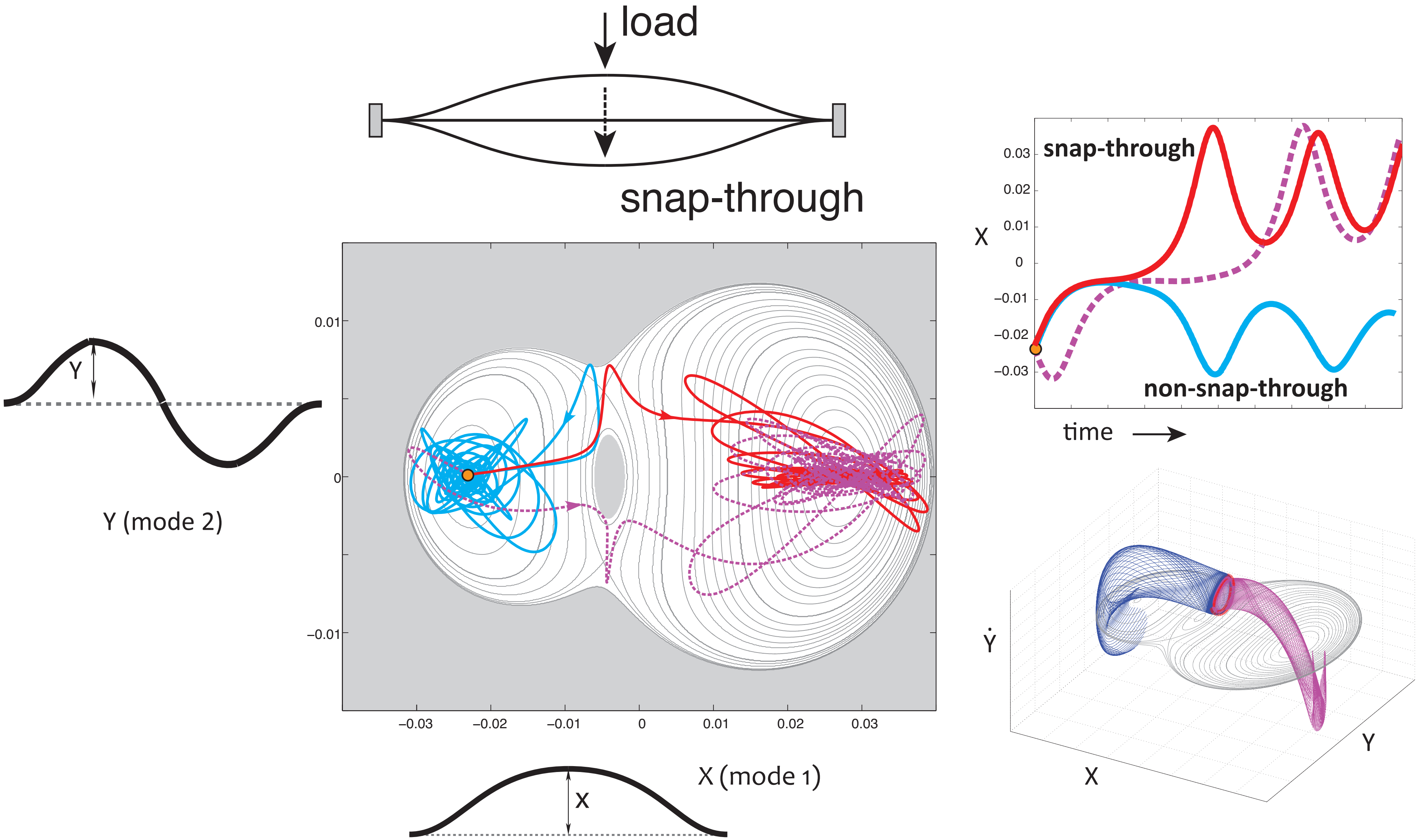






Buckling, bending, twisting, and crumpling of flexible bodies

Next steps — structural mechanics



Final words

- 2 DOF experiment for understanding geometry of transitions — verified geometric theory of tube dynamics
- Unobserved unstable periodic orbits have observable consequences
- Future work: control of transitions in multi-DOF systems
  e.g., triggering and avoidance of buckling in flexible structures, capsize avoidance for ships in rough seas and floating structures
- **For more, see Lawrie Virgin's talk tomorrow, 3:45pm, in 'CP25 Topics in Classical and Fluid Dynamical Systems'**
- **also Isaac Yeaton's talk tomorrow, 4:45pm (CP25)**
  Snakes on An Invariant Plane: Dynamics of Flying Snakes

Paper in preparation; check status at:
shaneross.com