Coherent sets from data: bifurcations, braiding, and predicting critical transitions

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with M. Stremler, A. BozorgMagham, S. Naik, P. Tallapragada, P. Grover, P. Kumar, S. Raben, B. Lin, P. Vlachos, A.J. Prussin, D. Schmale, F. Lekien

2013 SIAM Conference on Applications of Dynamical Systems (May 21, 2013)

Motivation: complex fluid motion, mixing, and control

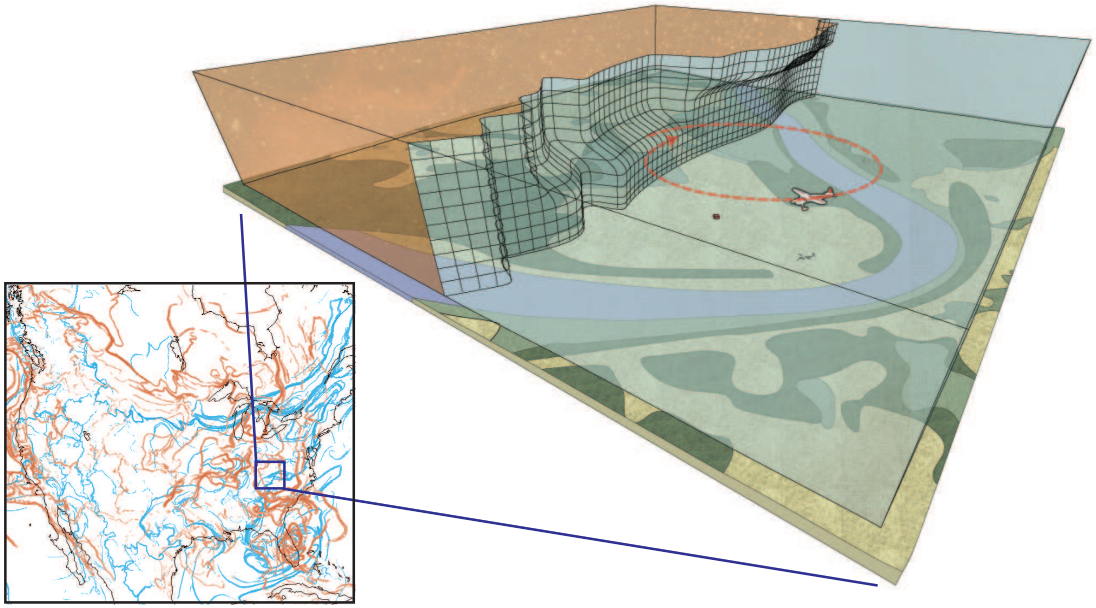
Oceans[1]

Atmosphere[2]

[1] LCS in an ocean model (Harrison, Siegel, Mitarai [2013] Marine Ecology Progress Series)

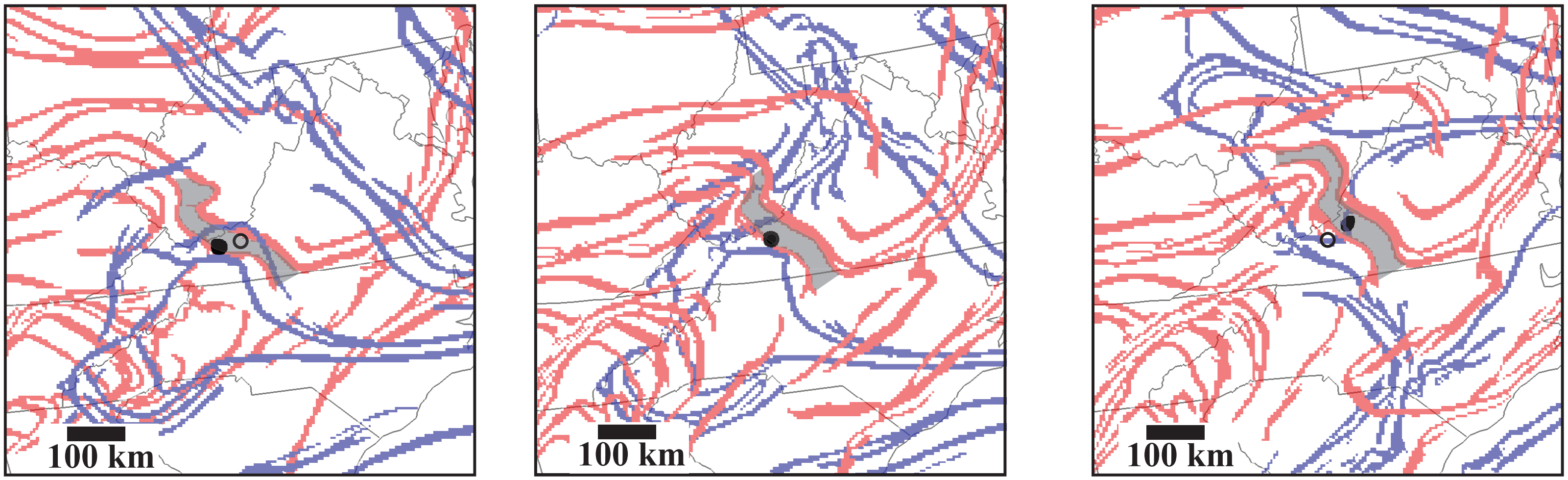
[2] LCS over North America: orange = repelling, blue = attracting

Aerial sampling of airborne diseases on either side of LCS

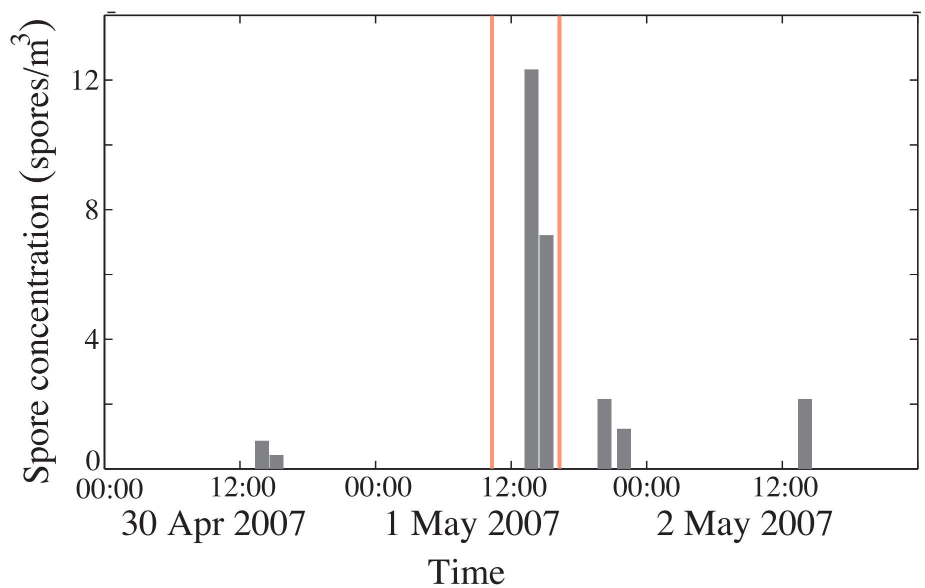


Filament with high pathogen values ‘sandwiched’ by LCS

12:00 UTC 1 May 2007 15:00 UTC 1 May 2007 18:00 UTC 1 May 2007



(a) (b) (c)



Tallapragada et al [2011] Chaos; Schmale et al [2012] Aerobiologia; BozorgMagham et al [2013] Physica D

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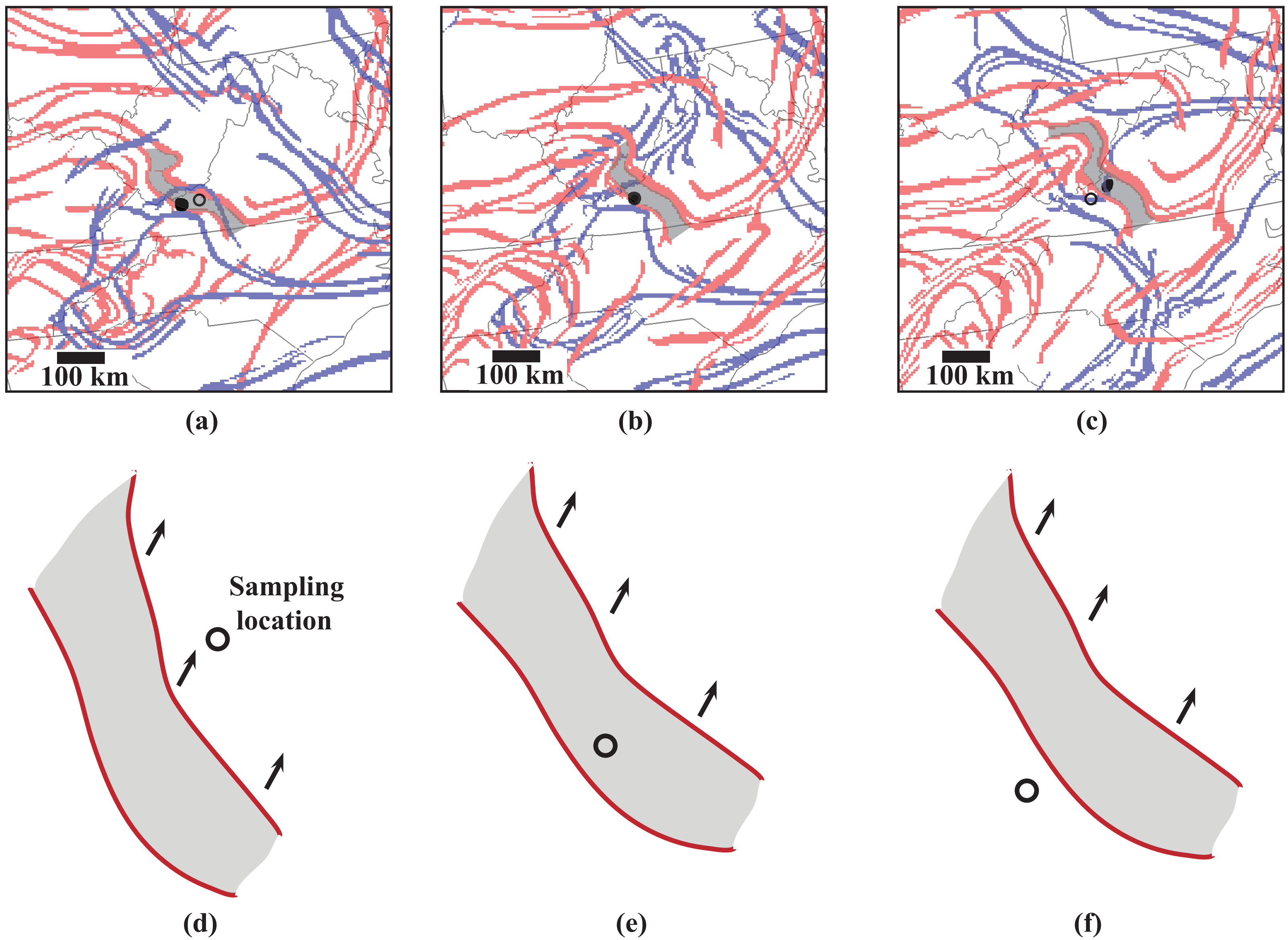

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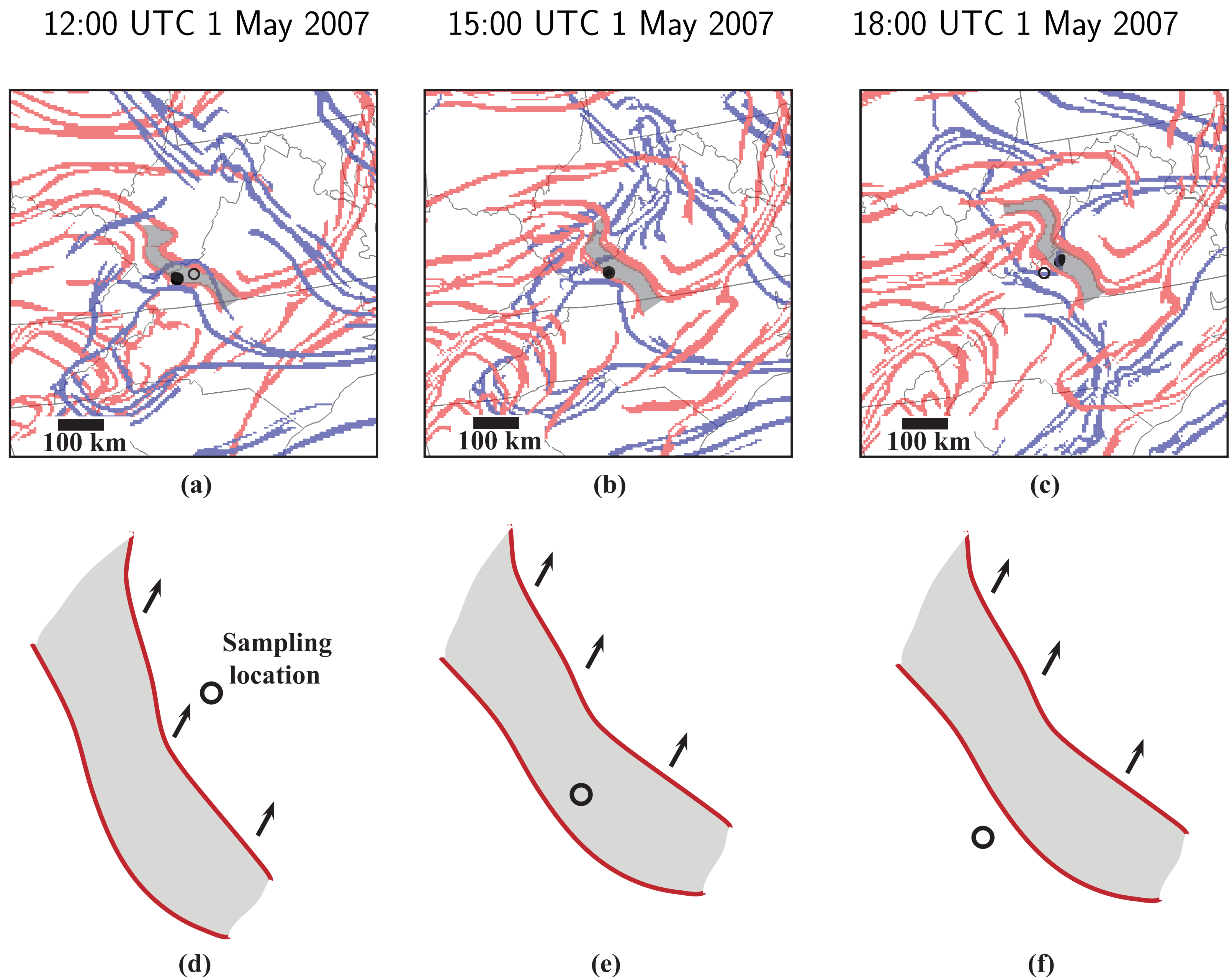
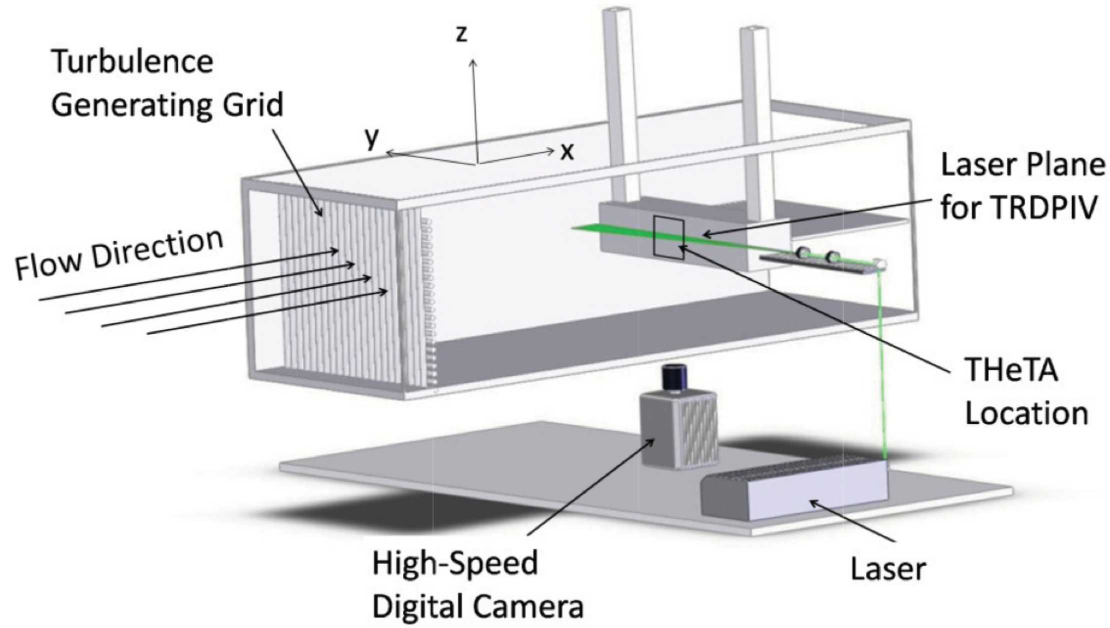
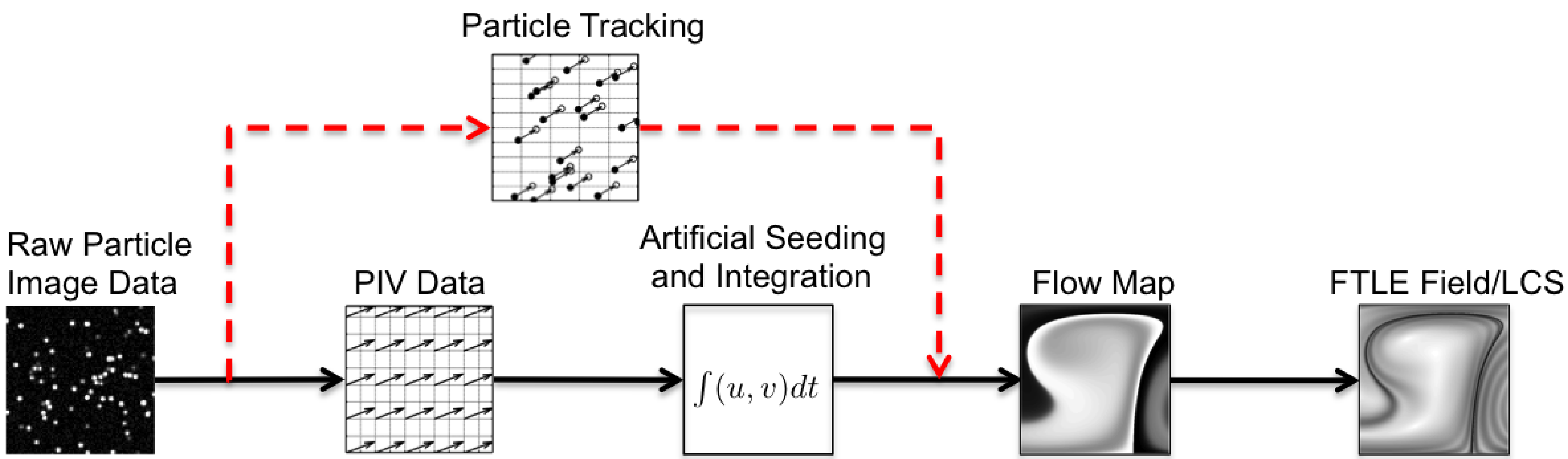


See Amir BozorgMagham’s talk on Thursday, MS126, 3D Geophysical Fluid Flows

Laboratory fluid experiments

A typical PIV experiment (Hubble [2011])

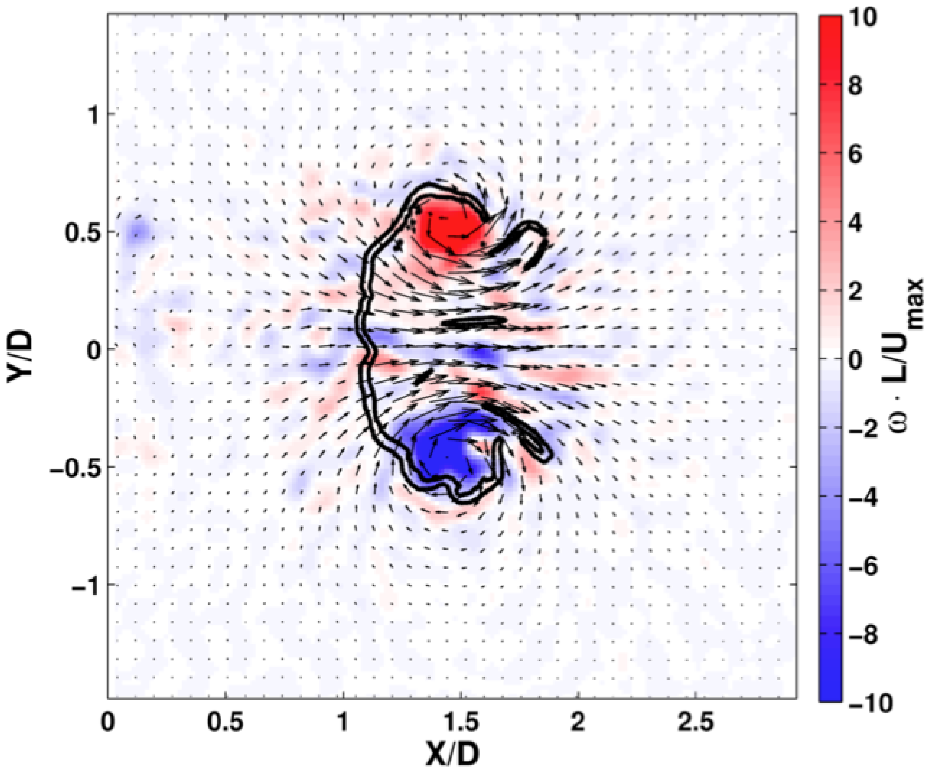




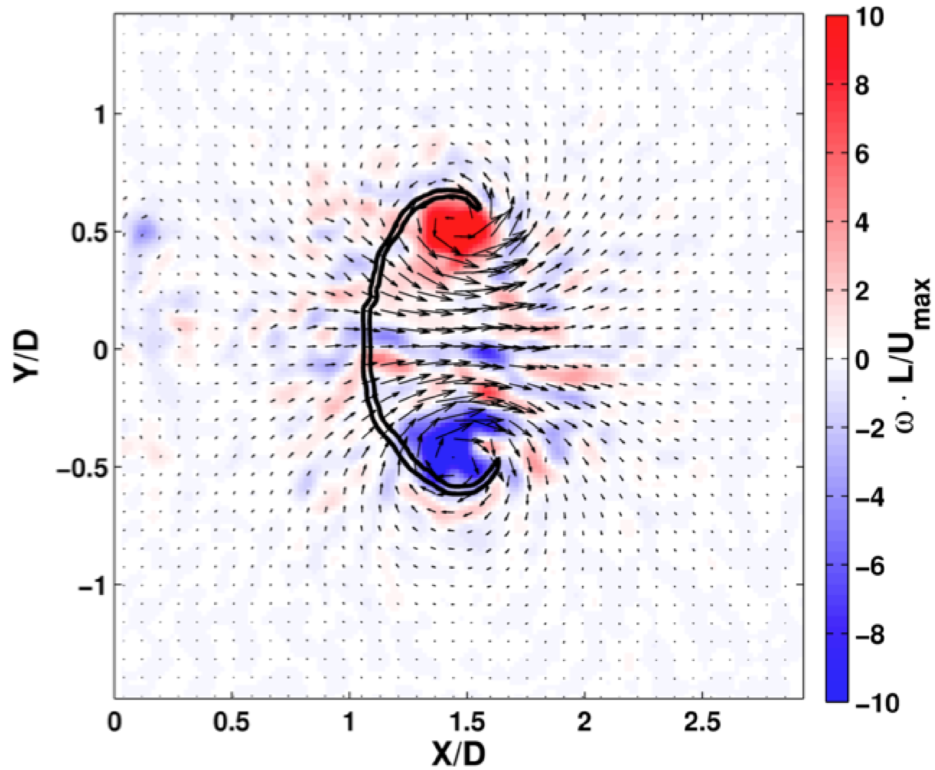
Using sequences of particle images to directly calculate flow structures (Raben, Ross, Vlachos [2013])

Improved resolution of Lagrangian structure

Numerical Integration



Interpolated Tracking



Based on a vortex ring experiment (Raben, Ross, Vlachos [2013])

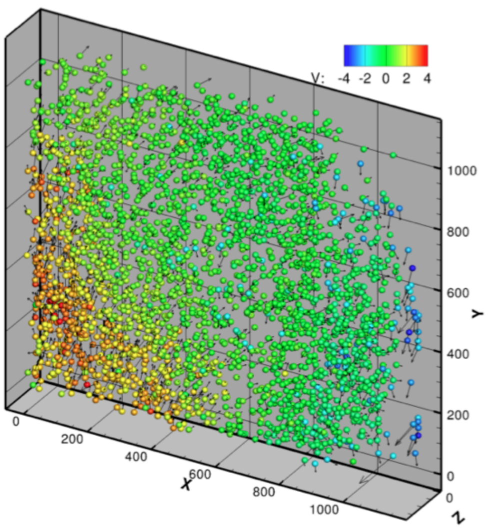
Ideal for low particle seeding situations, e.g., natural tracers in the ocean

- Builds off flow map composition method of Brunton & Rowley [2010]

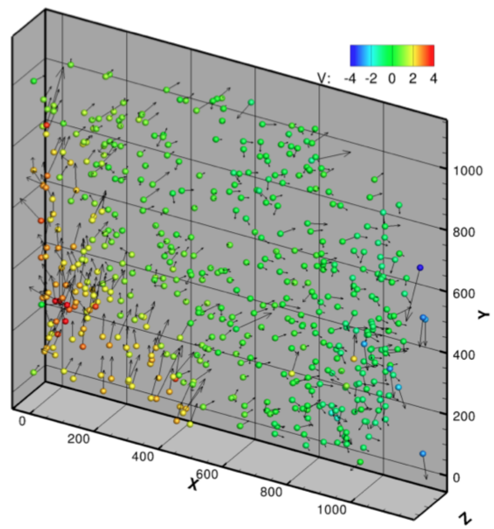
3D Lagrangian structure for non-tracer particles

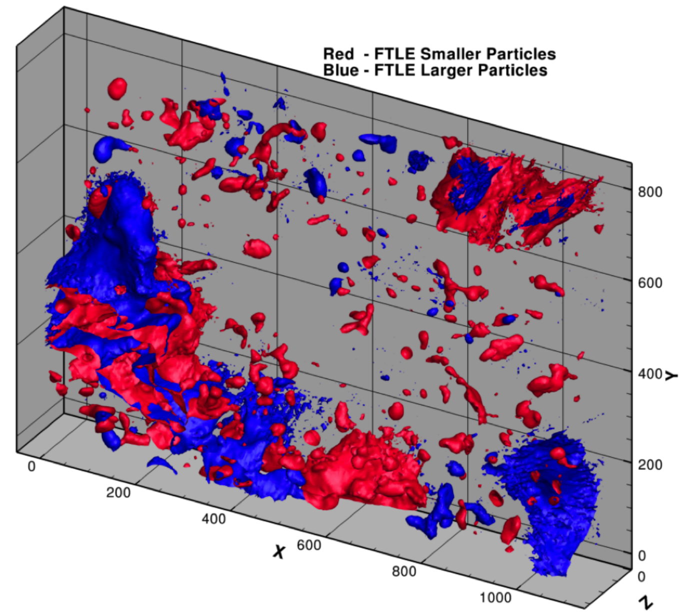
Also ideal for inertial particles which do not follow fluid velocity

Above 75 voxels



Above 175 voxels

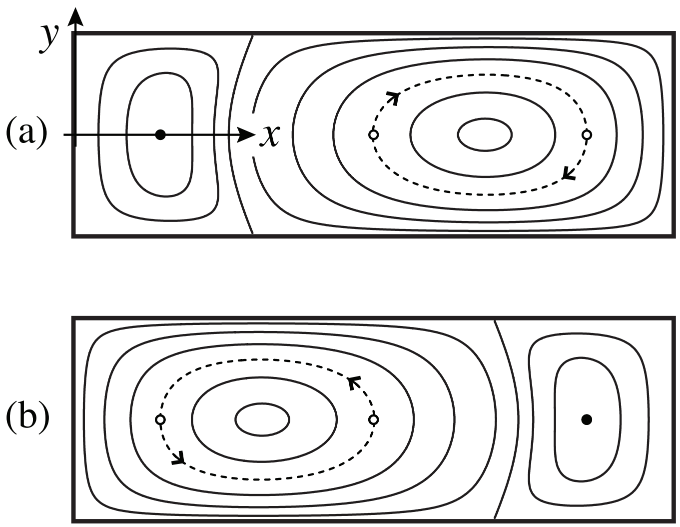




e.g., allows further exploration of physics of multi-phase flows

Ghost rods in microfluidic mixer

- Viscous flow in a 2D box (described by Mark Stremler on Monday)



streamlines for $\tau_f = 1$

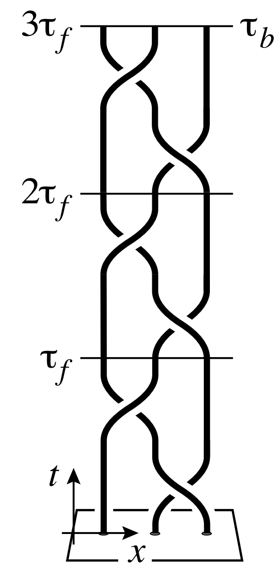
tracer blob ($\tau_f > 1$)

- $t \in [n\tau_f, (n+1)\tau_f/2)$, top streamline pattern
- $t \in [(n+1)\tau_f/2, (n+1)\tau_f)$, bottom streamline pattern
- System has parameter τ_f, which we treat as a bifurcation parameter — critical point $\tau_f^* = 1$

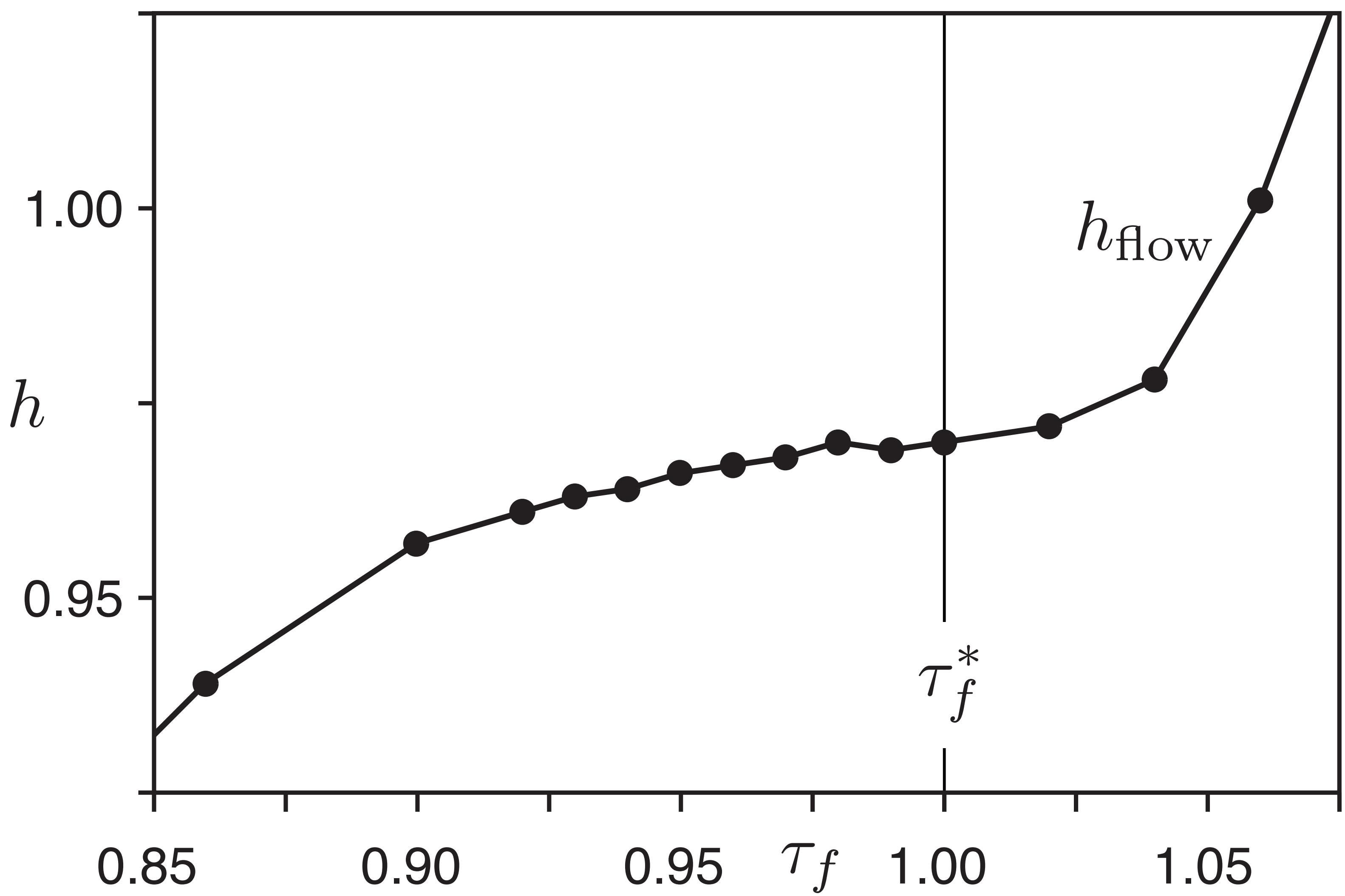
Ghost rods in microfluidic mixer

Recall Jean-Luc's talk

- For $\tau_f = 1$, braid on 3 strands act as 'ghost rods' stirring the fluid
- Their braid has $h_{\text{TN}} = 0.962$ from Thurston-Nielsen Classification Theorem
- Actual for flow $h_{\text{flow}} = 0.964$
- $\Rightarrow h_{\text{TN}}$ is an excellent lower bound

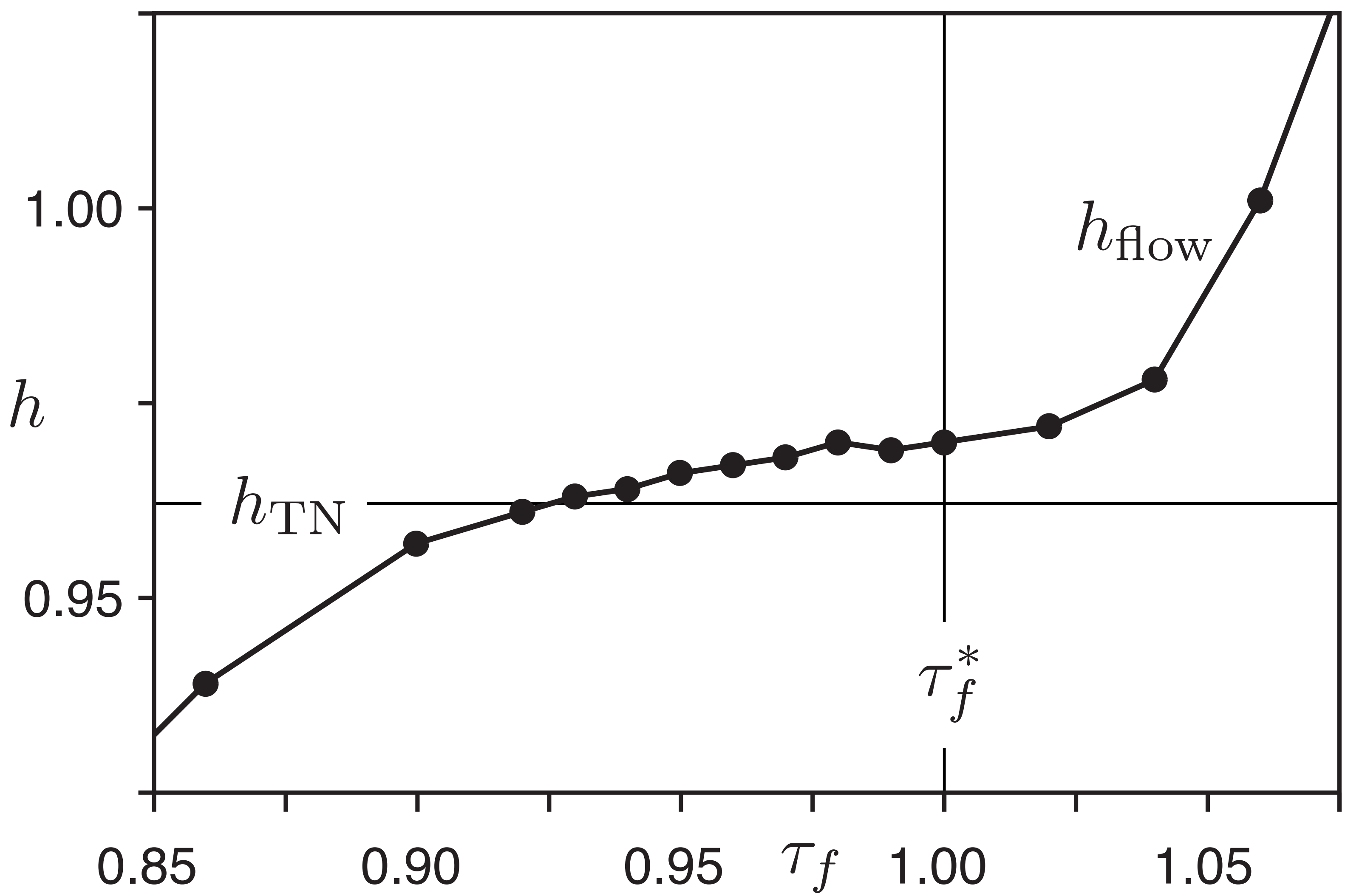


Topological entropy continuity across critical point



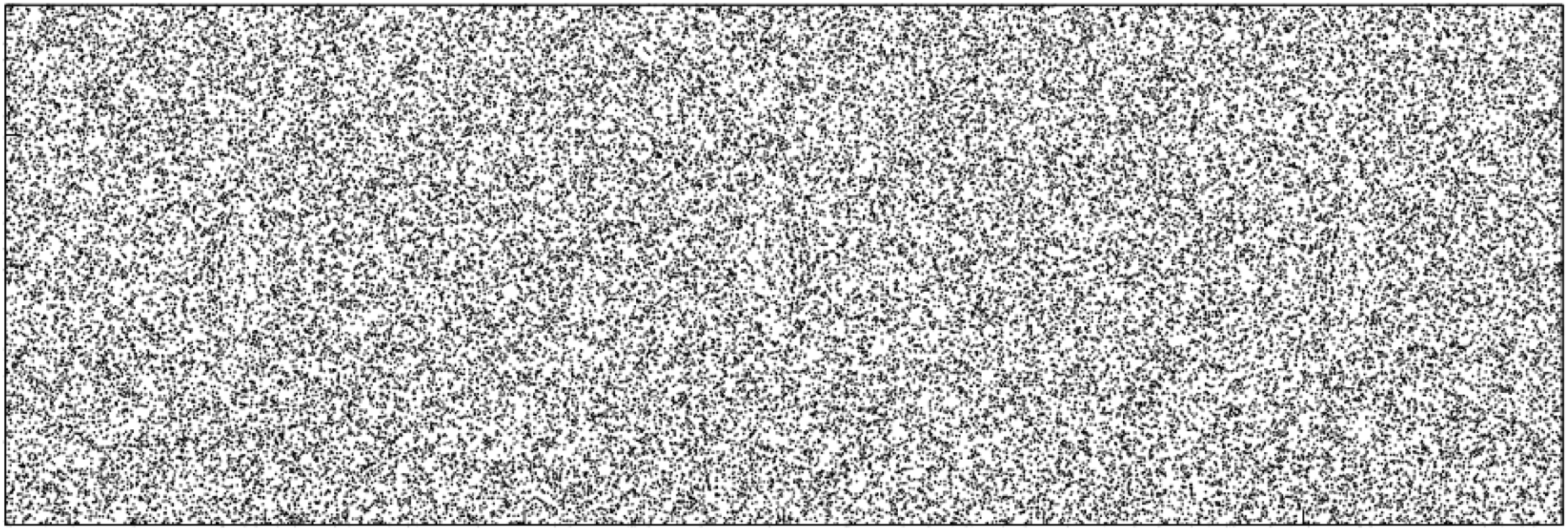
topological entropy as a function of τ_f

Topological entropy continuity across critical point



topological entropy as a function of τ_f

Identifying 'ghost rods'?



Poincaré section for $\tau_f < 1 \Rightarrow$ no obvious structure!

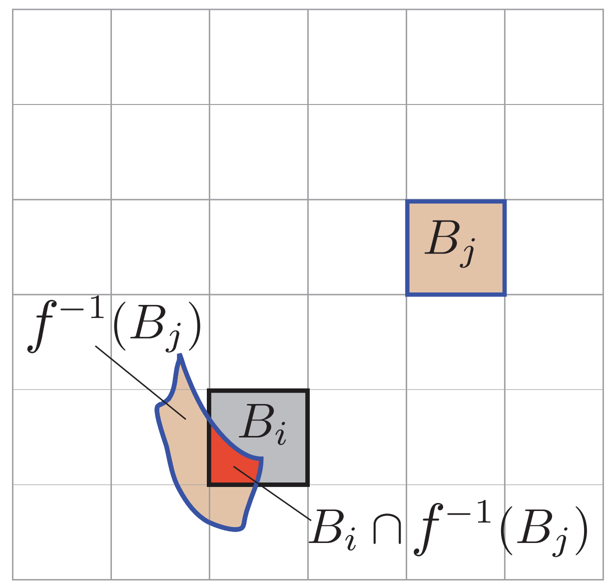
- Note the absence of any elliptical islands
- No periodic orbits of low period were found
- Is the phase space featureless?

Almost-invariant set approach

- Identify **almost-invariant sets** (AISs) using probabilistic point of view
- Relatedly, **almost-cyclic sets** (ACSs)[1]
- Create box partition of phase space $\mathcal{B} = \{B_1, \ldots B_q\}$, with q large
- Consider a q-by-q **transition (Ulam) matrix**, P, where

$$P_{ij} = \frac{m(B_i \cap f^{-1}(B_j))}{m(B_i)},$$

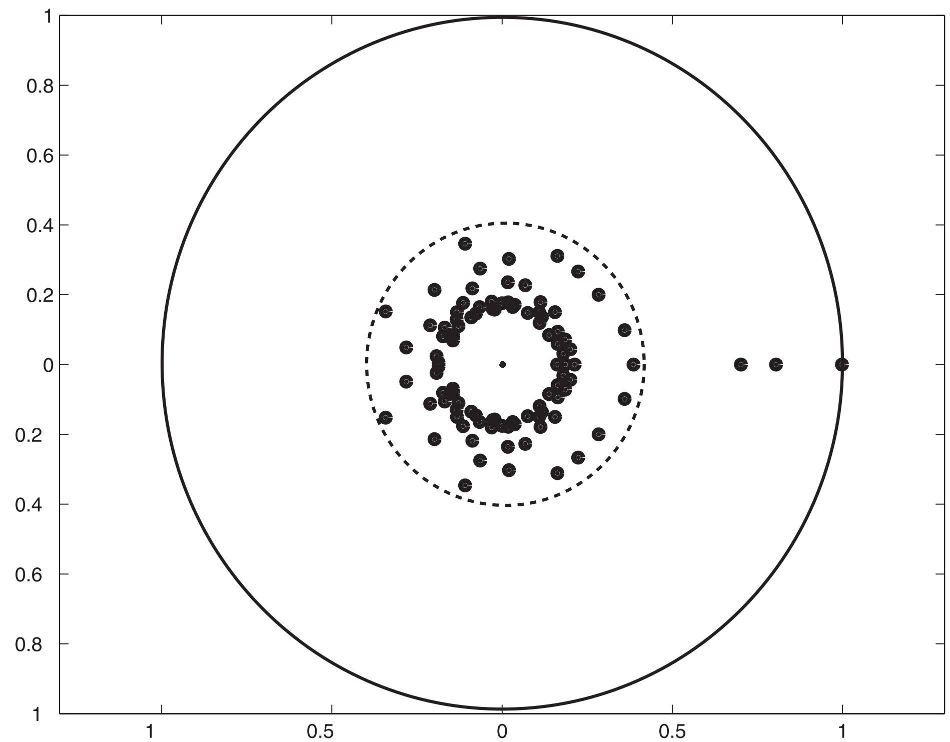
the *transition probability* from B_i to B_j using, e.g., $f = \phi_t^{t+T}$, computed numerically



- P approximates $\mathcal{P}$, Perron-Frobenius operator — which evolves densities, ν, over one iterate of f, as $\mathcal{P}\nu$
- Typically, we use a reversibilized operator R, obtained from P

[1] Dellnitz & Junge [1999], Froyland & Dellnitz [2003]

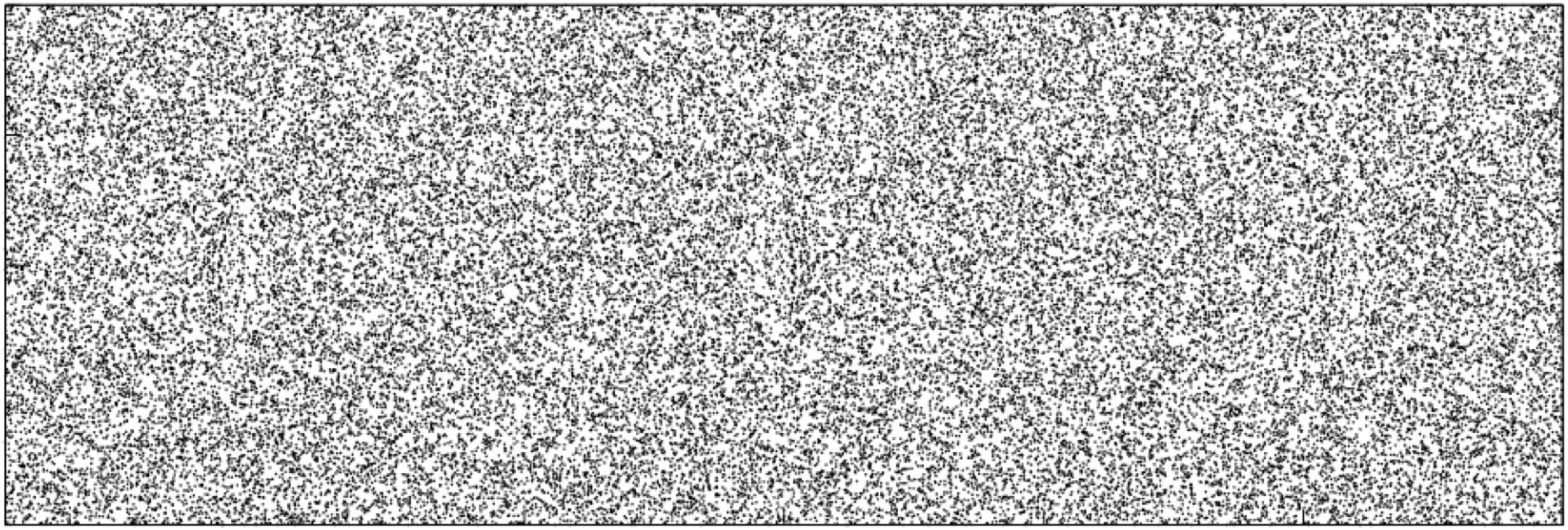
Identifying AISs by spectrum-partitioning



- **Invariant** densities are those fixed under P, $P\nu = \nu$, i.e., eigenvalue 1
- Essential spectrum lies within a disk of radius $r < 1$ which depends on the weakest expansion rate of the underlying system.
- The other real eigenvalues identify **almost-invariant** sets

Dellnitz, Froyland, Sertl [2000] Nonlinearity

Identifying 'ghost rods': almost-cyclic sets

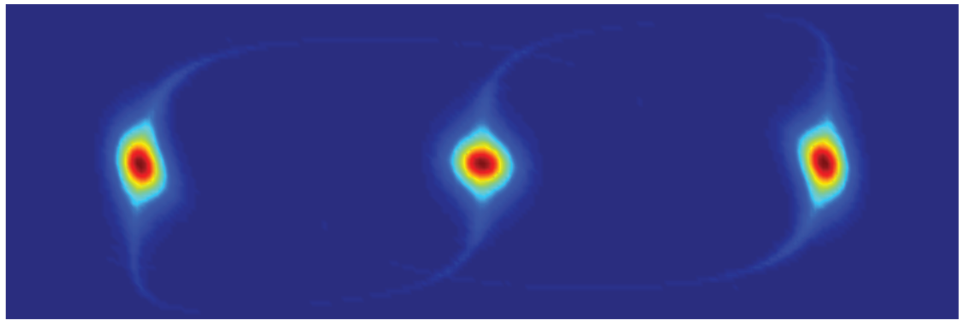


Poincaré section with no obvious structure

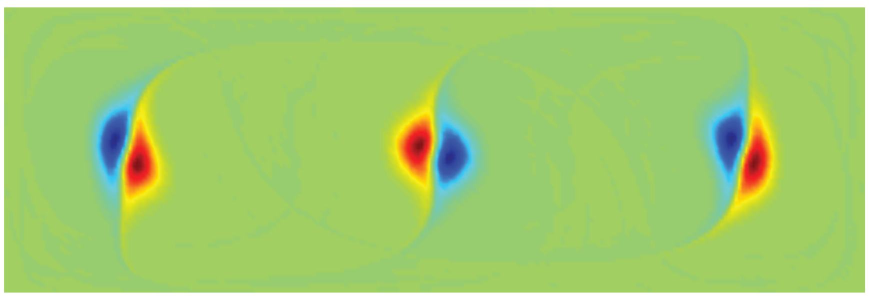
- Return to $\tau_f < 1$ case, where no periodic orbits of low period known
- What are the AISs and ACSs here?

Identifying 'ghost rods': almost-cyclic sets

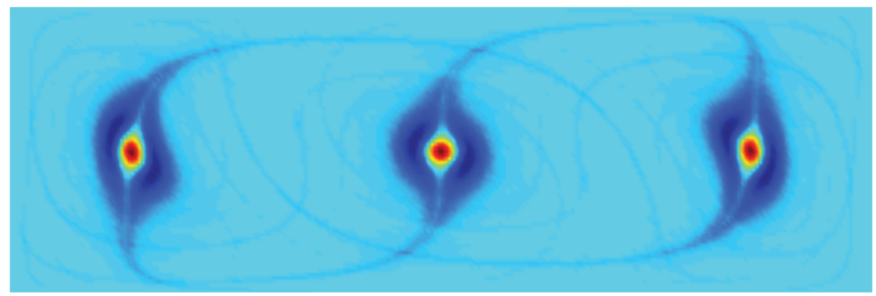
Top eigenvectors for $\tau_f = 0.99$ reveal hierarchy of phase space structures



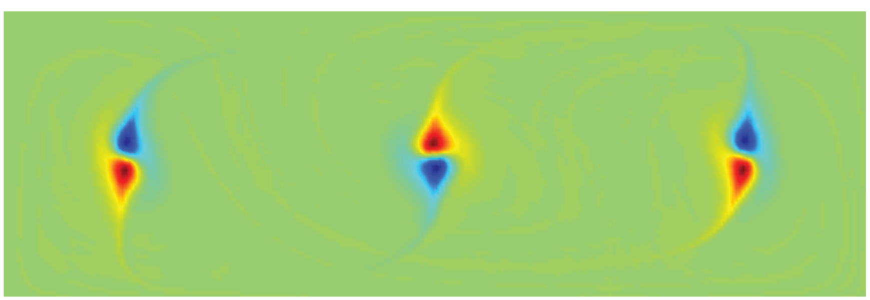
ν_2



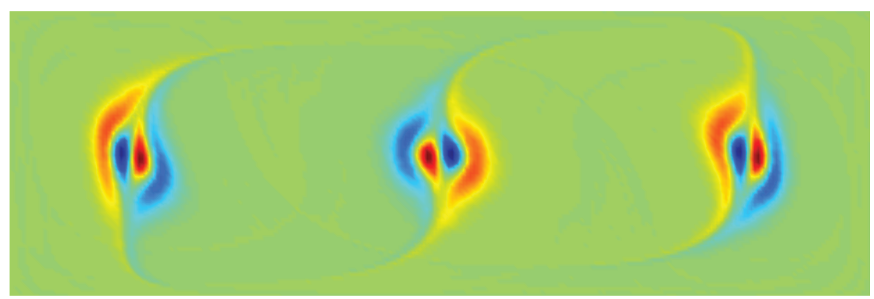
ν_3



ν_4

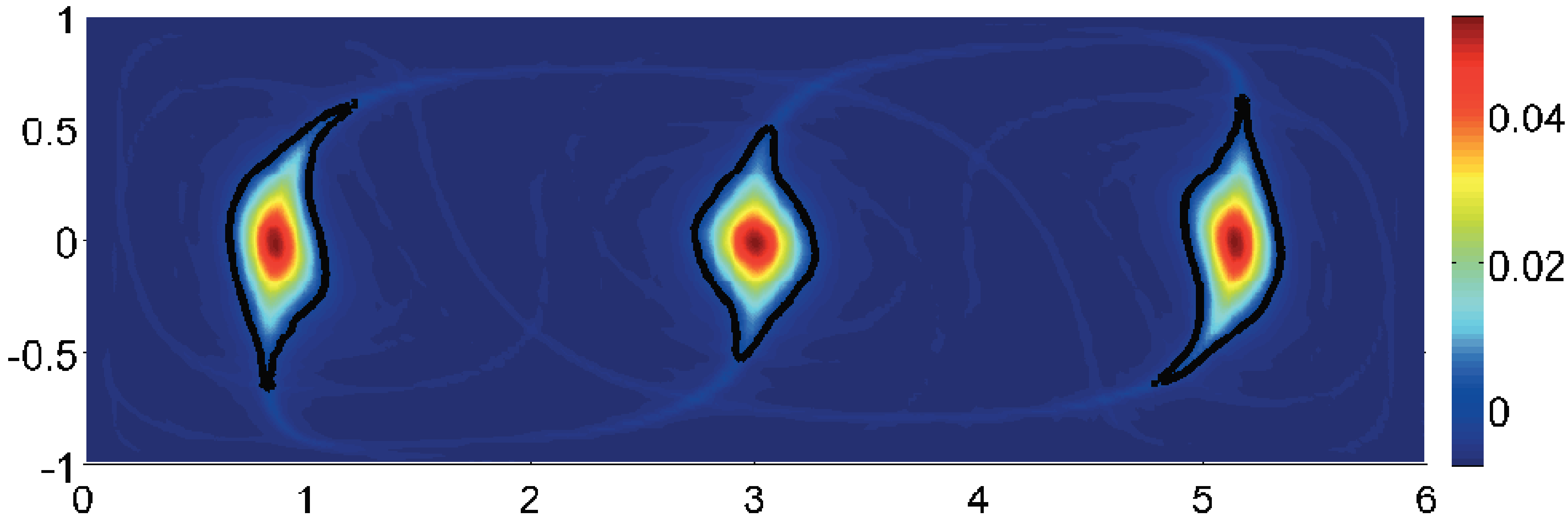


ν_5



ν_6

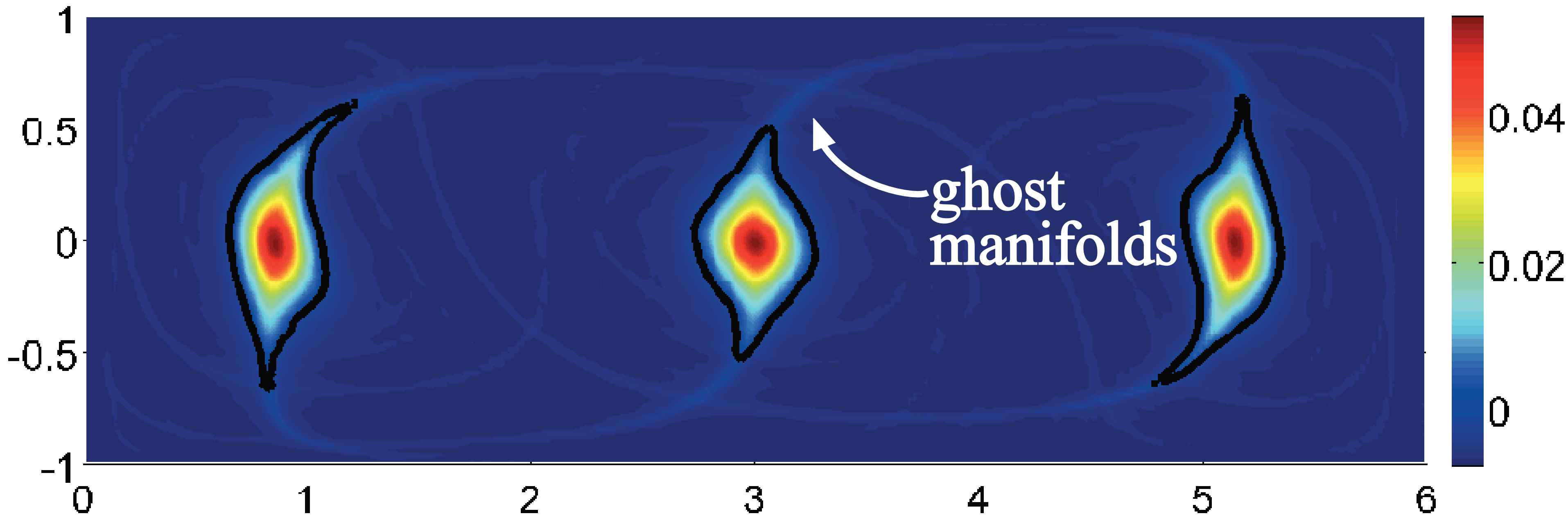
Identifying ‘ghost rods’: almost-cyclic sets



The zero contour (black) is the boundary between the two almost-invariant sets.

- Three-component AIS made of 3 ACSs of period 3
- ACSs, in effect, replace periodic orbits for TNCT

Identifying 'ghost rods': almost-cyclic sets



The zero contour (black) is the boundary between the two almost-invariant sets.

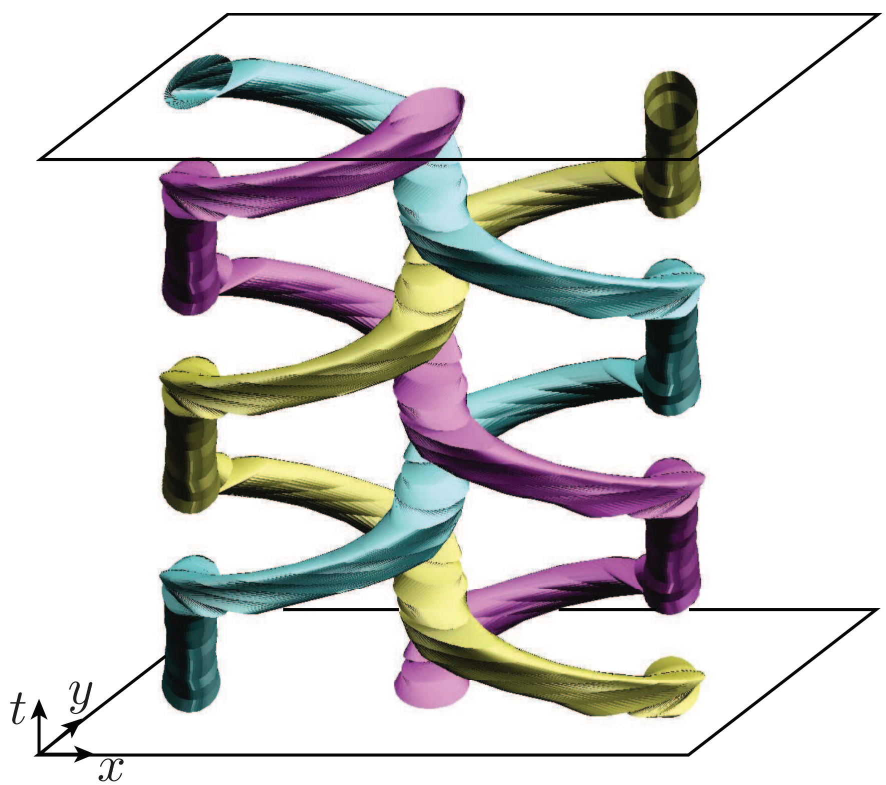
- Three-component AIS made of 3 ACSs of period 3
- ACSs, in effect, replace periodic orbits for TNCT
- Also: we see a **remnant of the 'stable and unstable manifolds' of the saddle points**, despite no saddle points – 'ghost manifolds'?

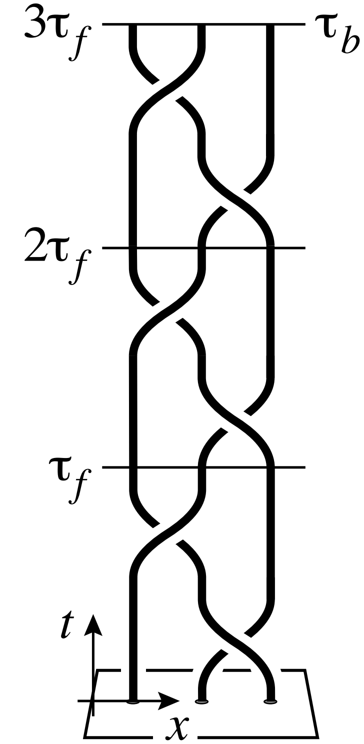
Identifying 'ghost rods': almost-cyclic sets

Almost-cyclic sets stirring the surrounding fluid like 'ghost rods'
— **works even when periodic orbits are absent!**

Movie shown is second eigenvector for $R_t^{t+\tau_f}$ for $t \in [0, \tau_f)$

Identifying 'ghost rods': almost-cyclic sets





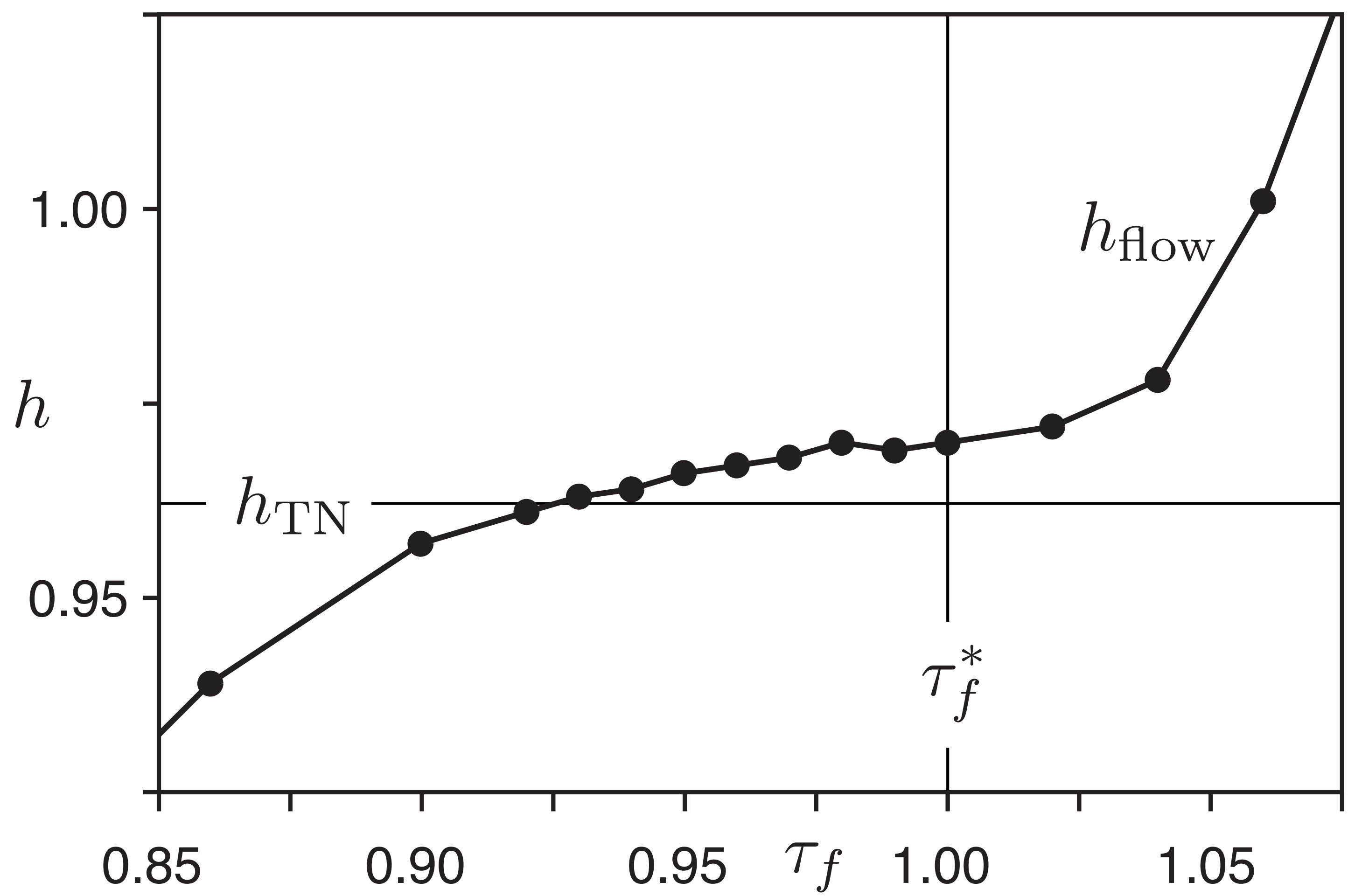
Braid of ACSs gives lower bound of entropy via Thurston-Nielsen

— One only needs approximately cyclic blobs of fluid

— But, theorems apply only to periodic points!

— Stremler, Ross, Grover, Kumar [2011] Phys. Rev. Lett.

Topological entropy vs. bifurcation parameter



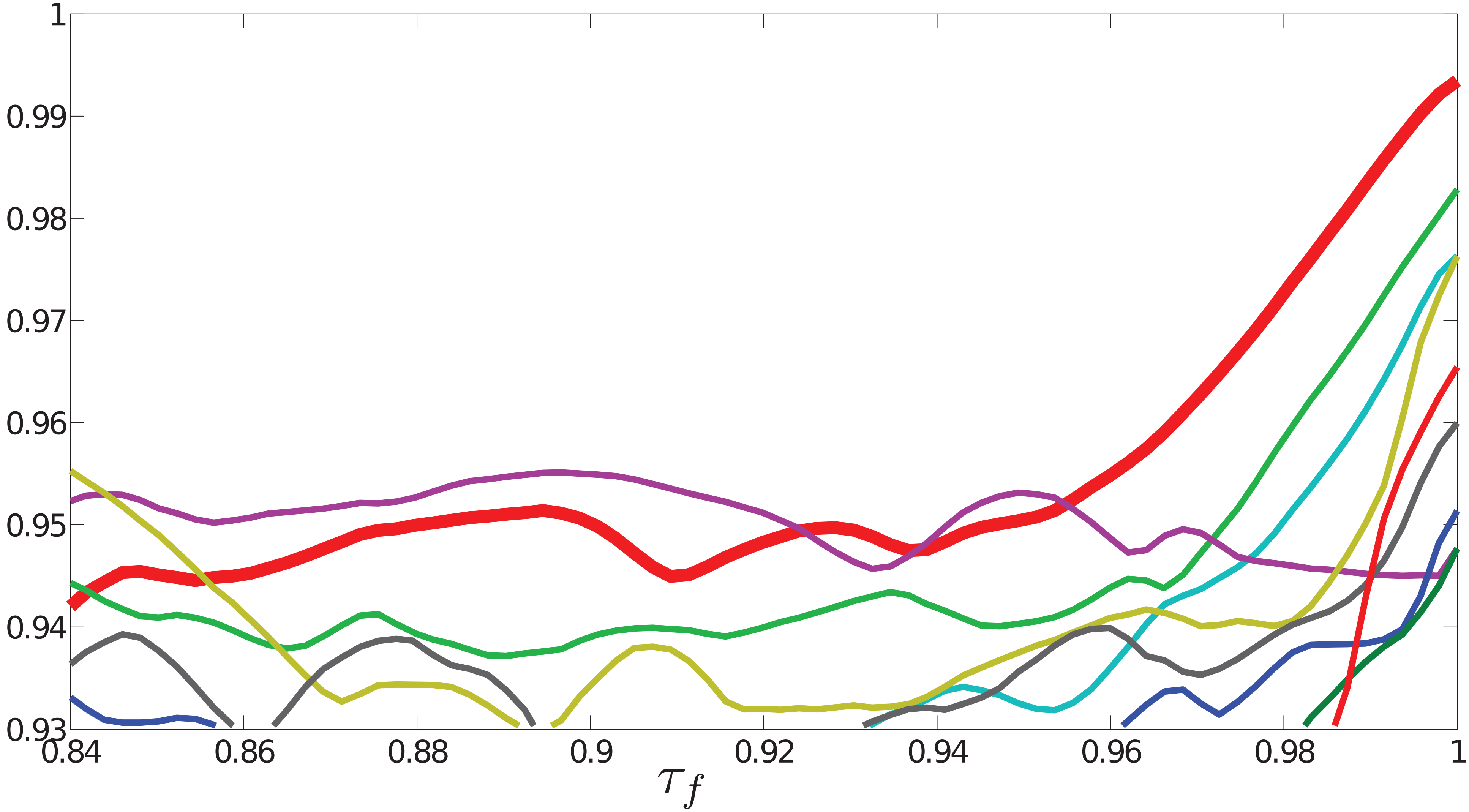
topological entropy as a function of τ_f

- h_{TN} shown for ACS braid on 3 strands

Eigenvalues/eigenvectors vs. bifurcation parameter

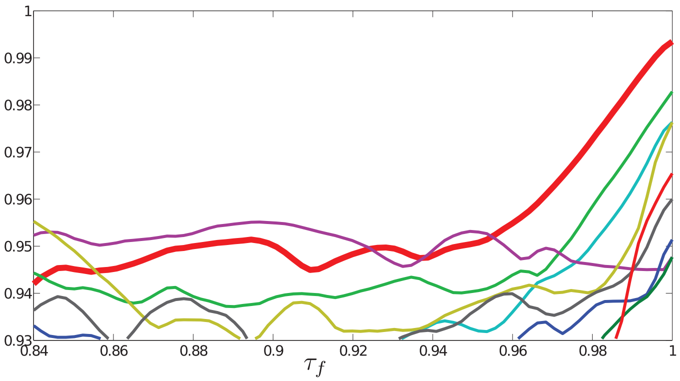
Eigenspectrum of P changes with the parameter τ_f

Eigenvalues/eigenvectors vs. bifurcation parameter



Top eigenvalues of R as parameter τ_f changes

Eigenvalues/eigenvectors vs. bifurcation parameter



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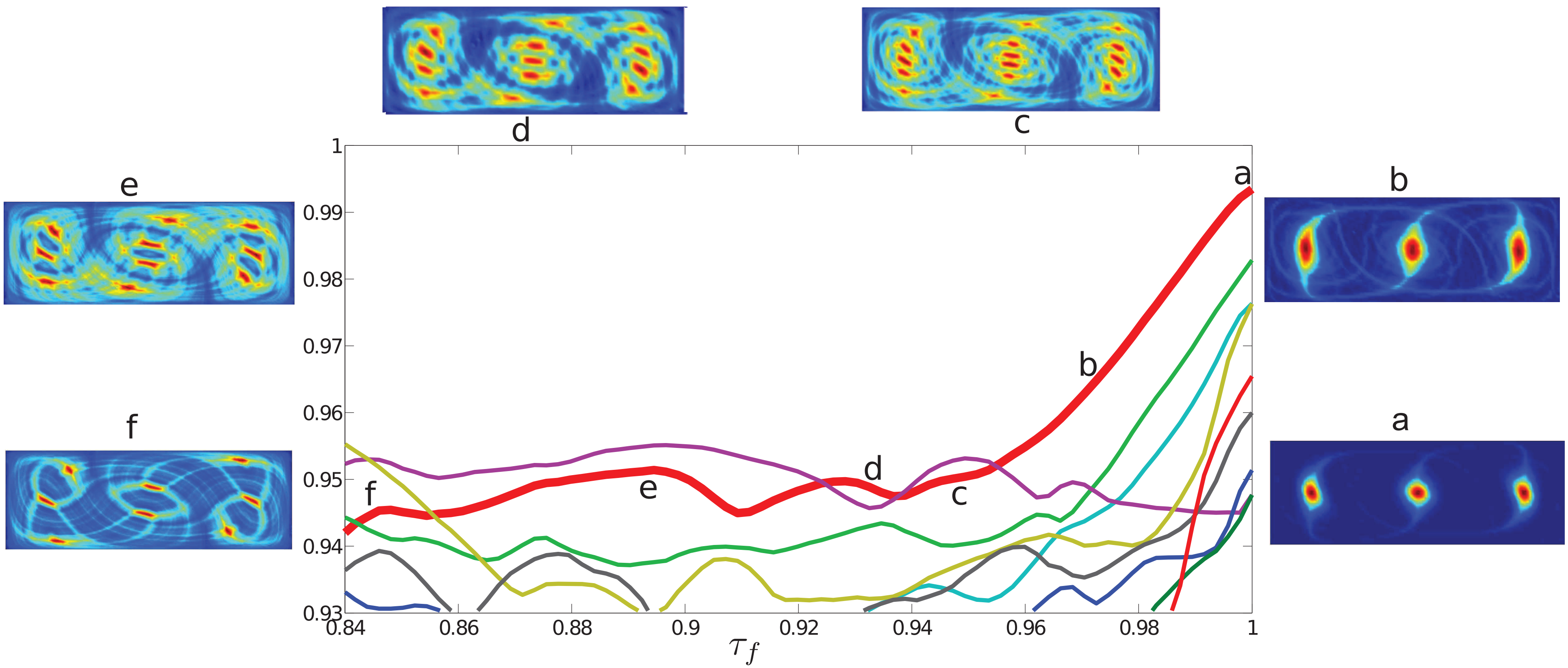


Movie shows change in eigenvector along thick red branch (a to f), as τ_f decreases.

Grover, Ross, Stremler, Kumar [2012] Chaos

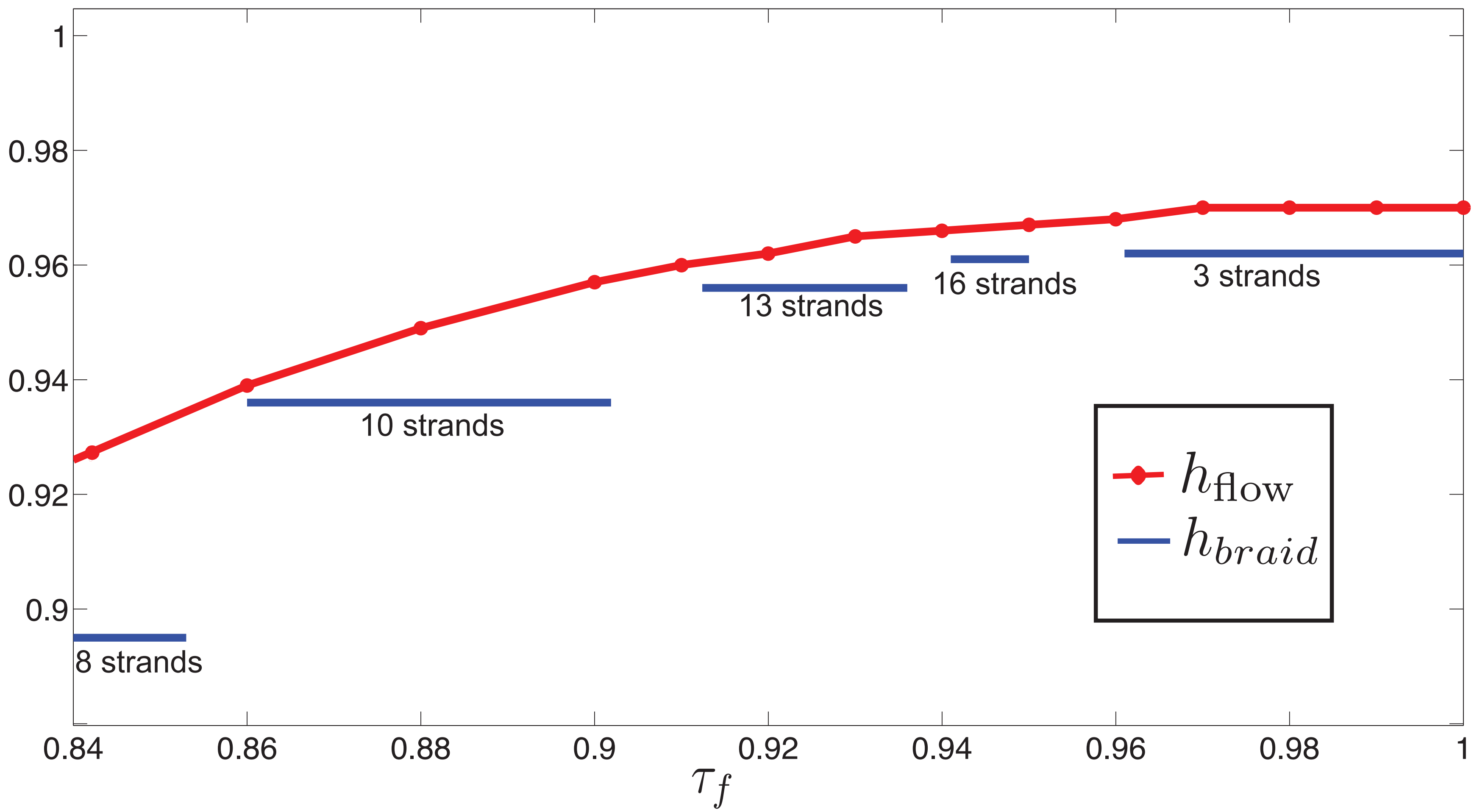
Bifurcation of ACSs

For example, braid on 13 strands for $\tau_f = 0.93$

Movie shown is second eigenvector for $P_t^{t+\tau_f}$ for $t \in [0, \tau_f)$

Thurson-Nielsen for this braid provides lower bound on topological entropy

Sequence of ACS braids bounds entropy



For various braids of ACSs, the calculated entropy is given, bounding from below the true topological entropy over the range where the braid exists

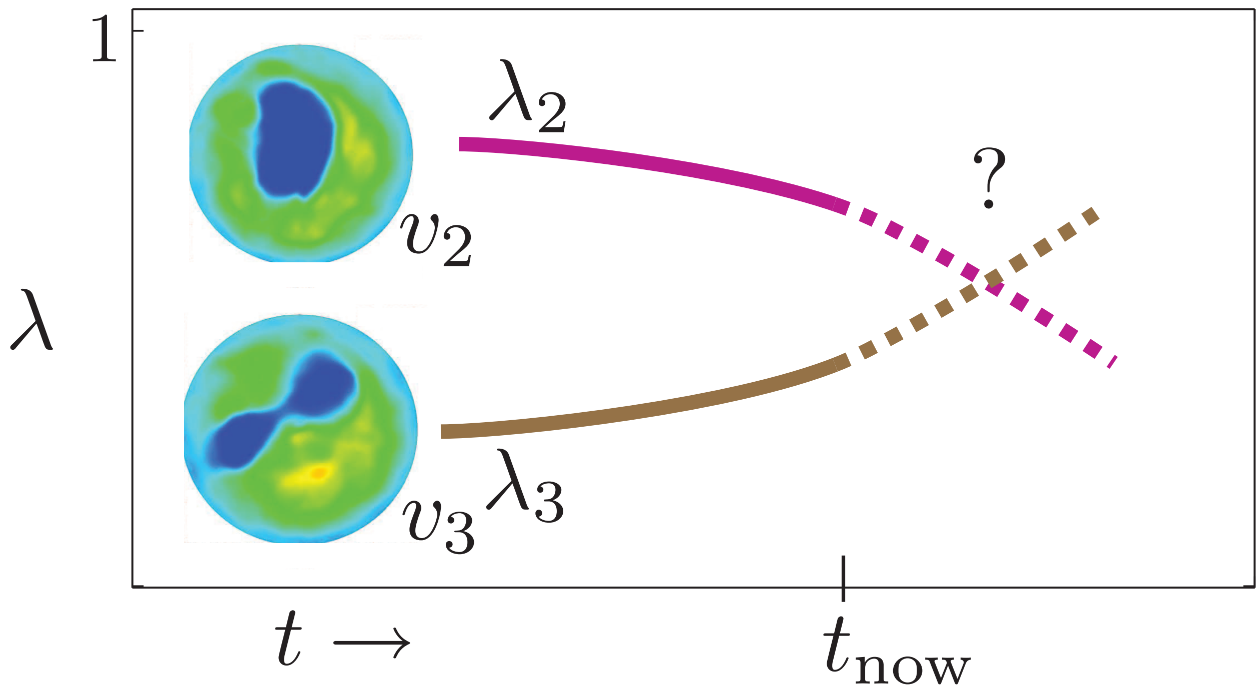
Grover, Ross, Stremler, Kumar [2012] Chaos

Speculation: trends in eigenvalues/modes for prediction

Predict critical transitions in geophysical transport?

Ozone data (Lekien and Ross [2010] Chaos)

Predict critical transitions in geophysical transport?



- Different eigenmodes can correspond to dramatically different behavior.
- Some eigenmodes increase in importance while others decrease
- Can we predict dramatic changes in system behavior?
- e.g., predicting major changes in geophysical transport patterns??
- work with E. Bollt, O. Junge, K. Padberg-Gehle, N. Santitissadeekorn

Coherent sets and set-based definition of FTLE

- **Definition.** The **covariance-based FTLE** of B is

$$\sigma_I(B,t,T) = \frac{1}{|T|}\log\left(\sqrt{\frac{\lambda_{max}(I(Pf))}{\lambda_{max}(I(f))}}\right)$$

  where $f = \frac{1}{\mu(B)}\mathcal{X}_B$ and $I(\cdot)$ is the covariance

- Tallapragada and Ross [2013] Comm. Nonlinear Sci. Numerical Simulation
- Reduces to usual definition of FTLE, σ, in the limit of small sets B

Coherent sets and set-based definition of FTLE

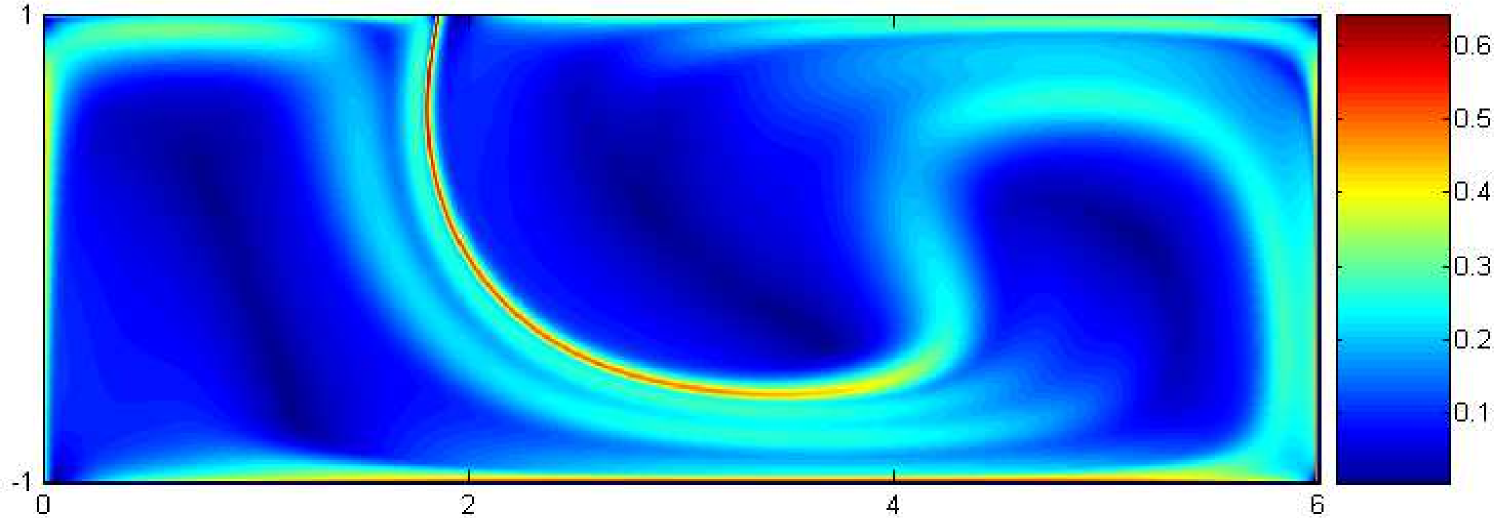
- The **coherence** of a set B during $[t, t+T]$ is $\sigma_I(B,t,T)$.
- A set B is **almost-coherent** during $[t, t+T]$ if $\sigma_I(B,t,T) \approx 0$.

- Essential feature of a coherent set: does not spread significantly.
- This definition also can identify non-mixing **translating** sets.

- Set a heuristic threshold on $\sigma_I(B,t,T)$ to identify coherent sets.
- Other methods can then be used to identify optimal coherence.
  e.g., Froyland, Santitisadeekorn, Monahan [2010], Haller, Beron-Vera [2012]

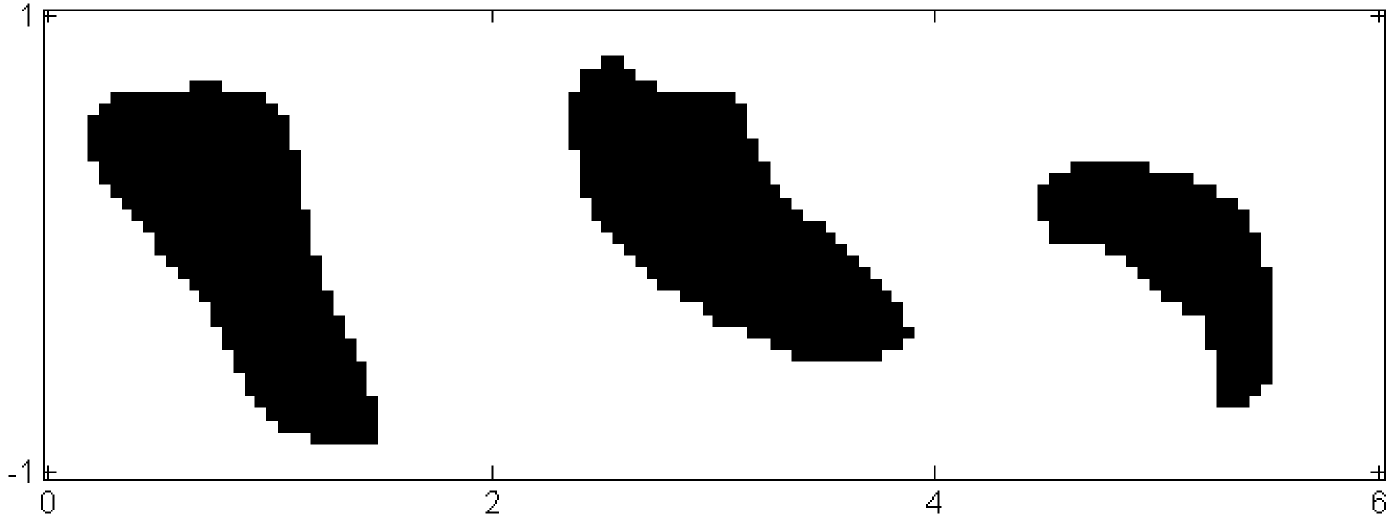
- Notice, coherent sets are valleys be separated by ridges of high FTLE, i.e., LCS

Coherent sets in the microfluidic mixer from before



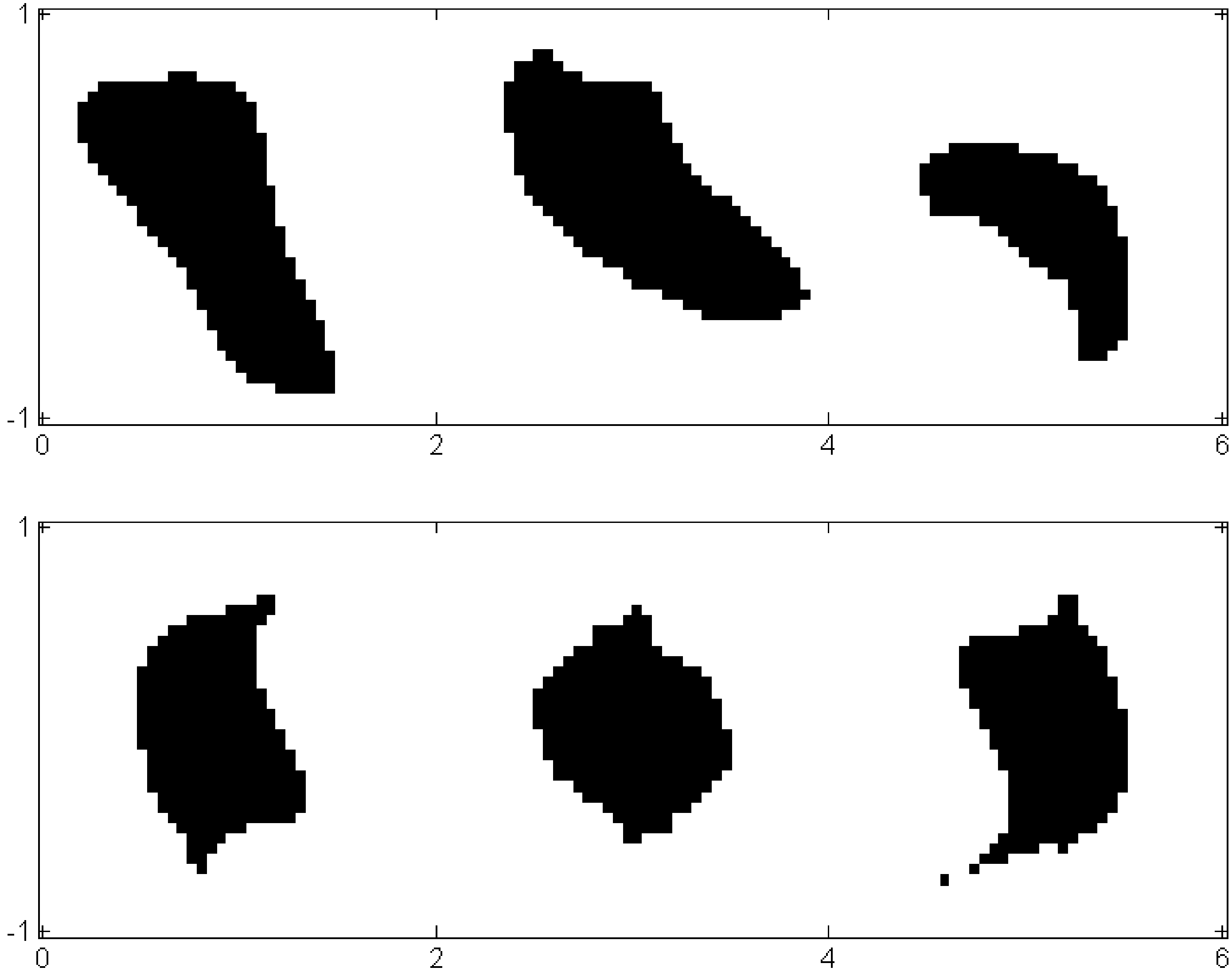
FTLE during $[0, \tau_f]$

Coherent sets in the microfluidic mixer from before



Sets of coherence $\sigma_I(0, \tau_f) < 0.06$

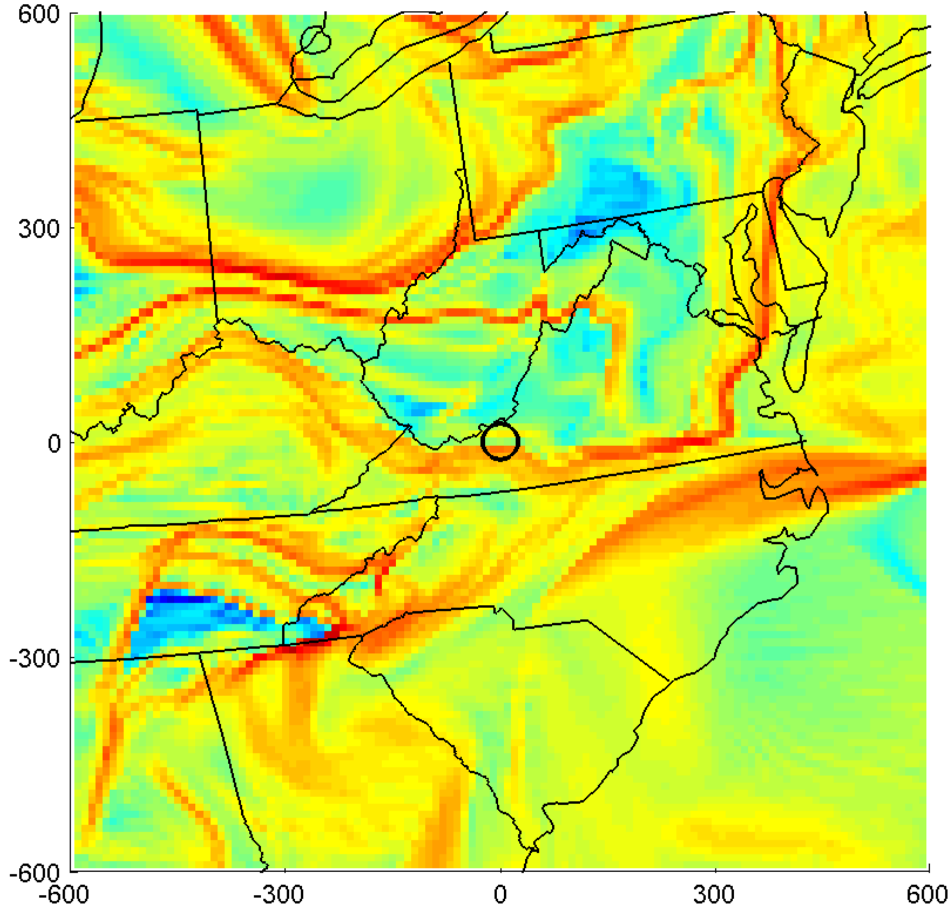
Coherent sets in the microfluidic mixer from before



Compare coherent set with AIS (from second eigenvector of R)

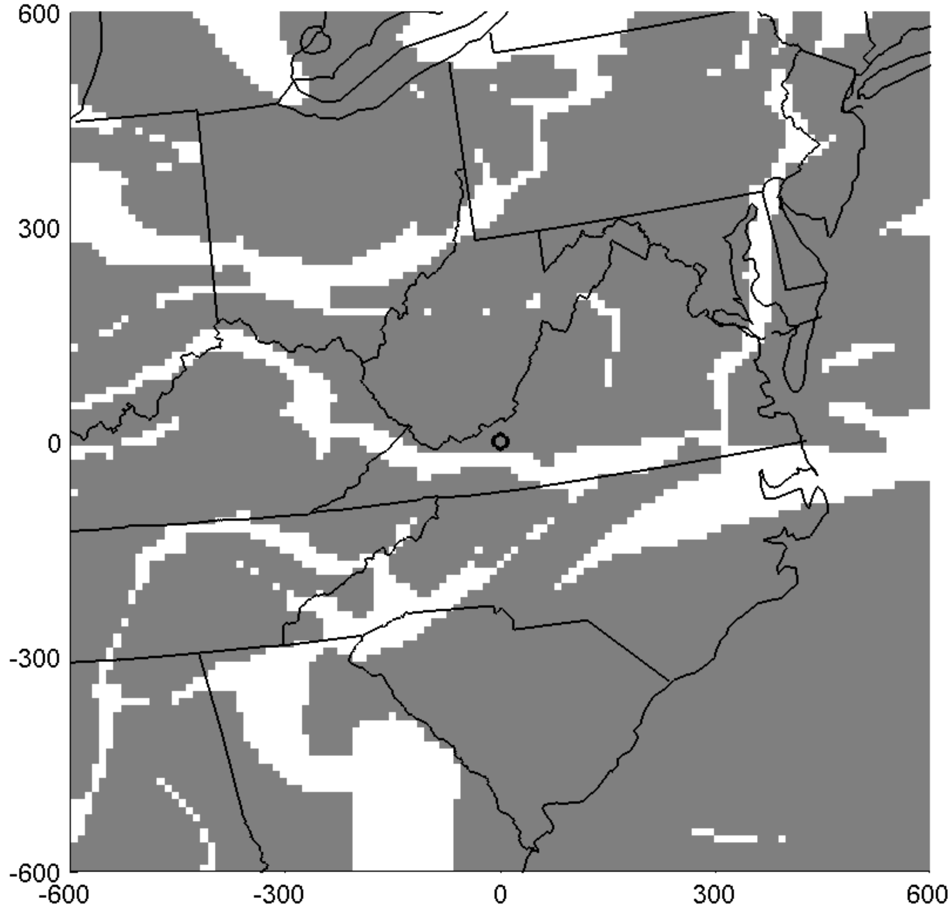
Coherent sets in fluid experiments

Coherent sets in the atmosphere



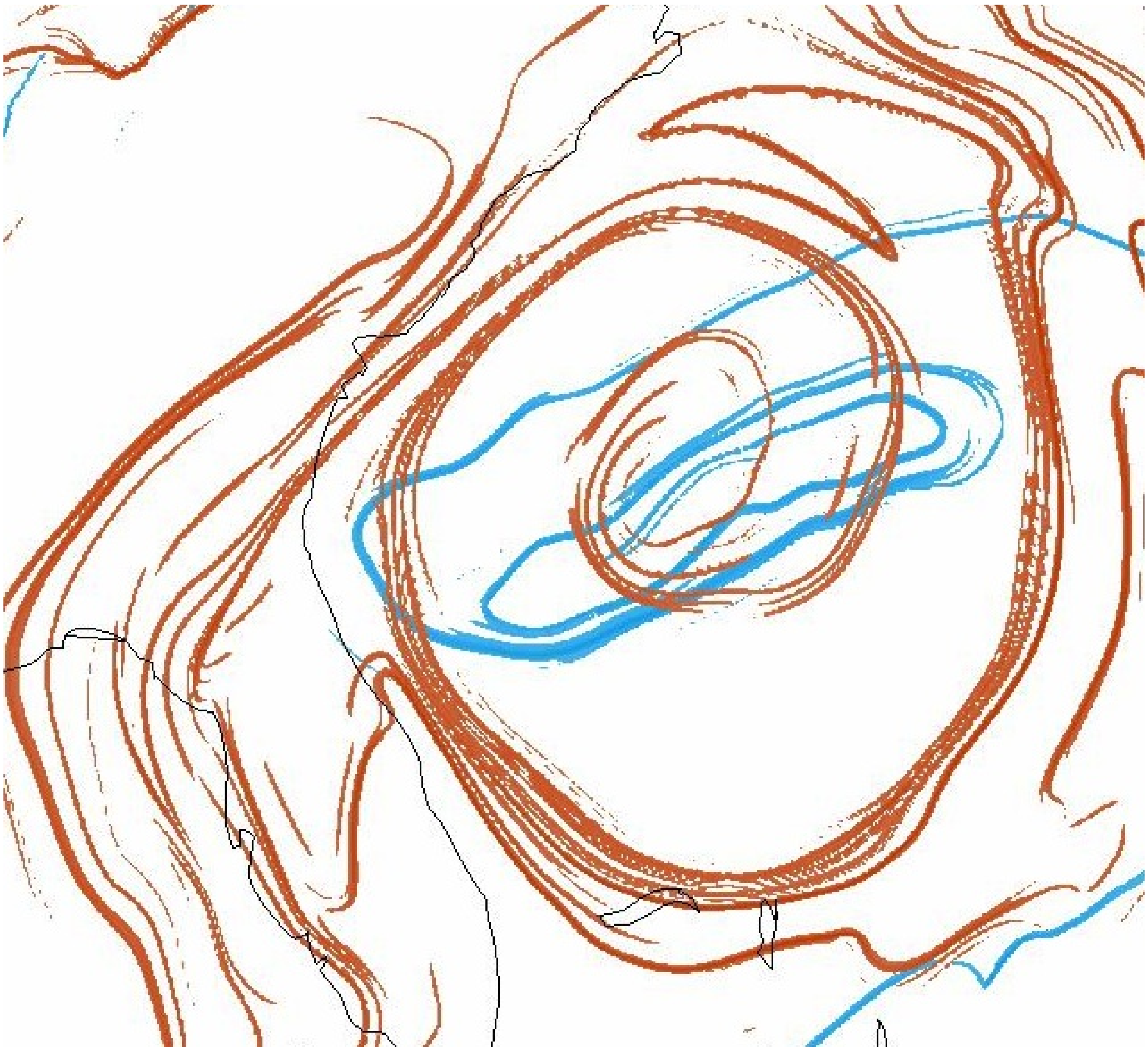
- FTLE from covariance during 24 hours starting 09:00 1 May 2007

Coherent sets in the atmosphere



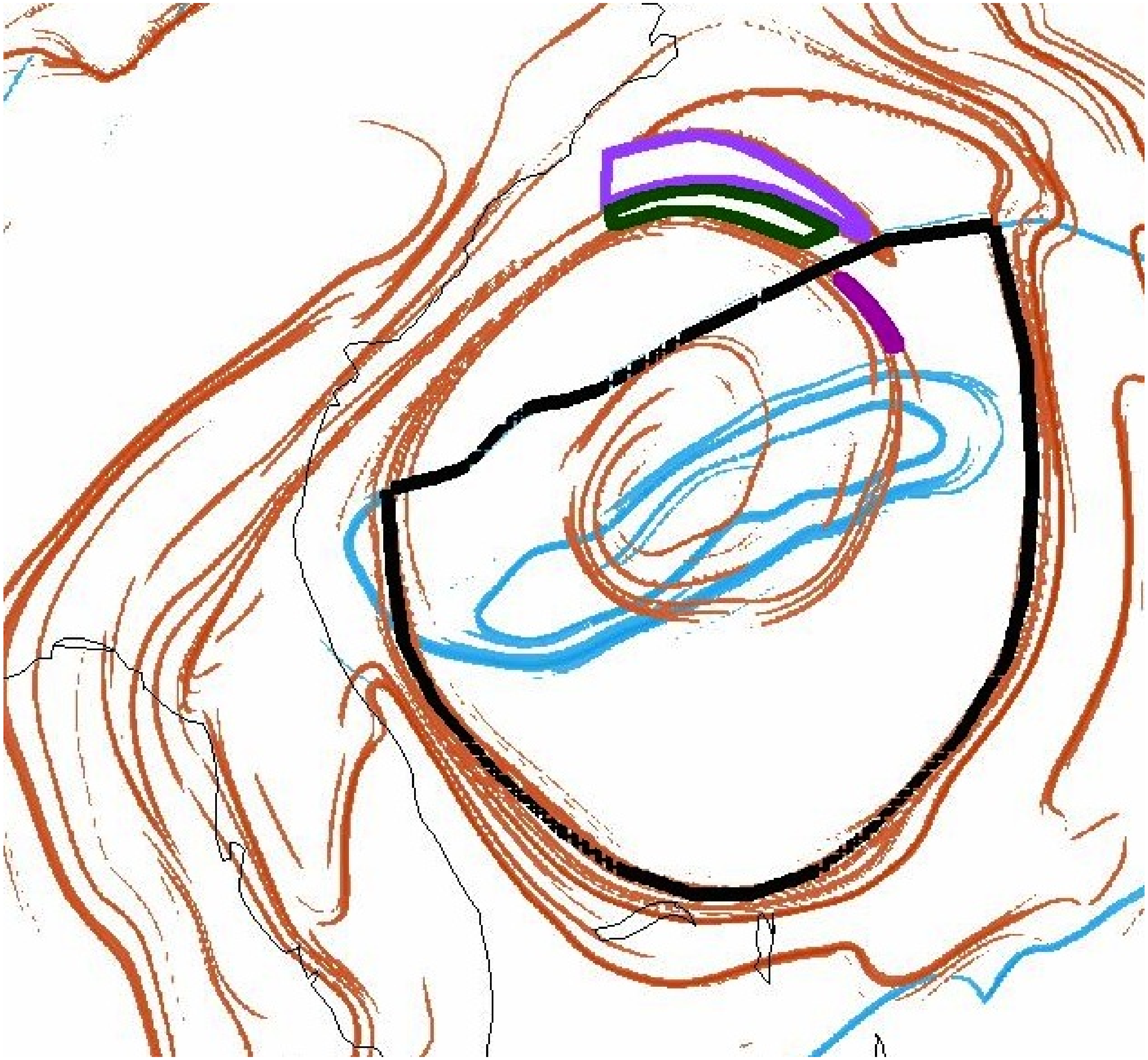
- Coherent sets during 24 hours starting 09:00 1 May 2007

Coherent sets in the atmosphere that braid



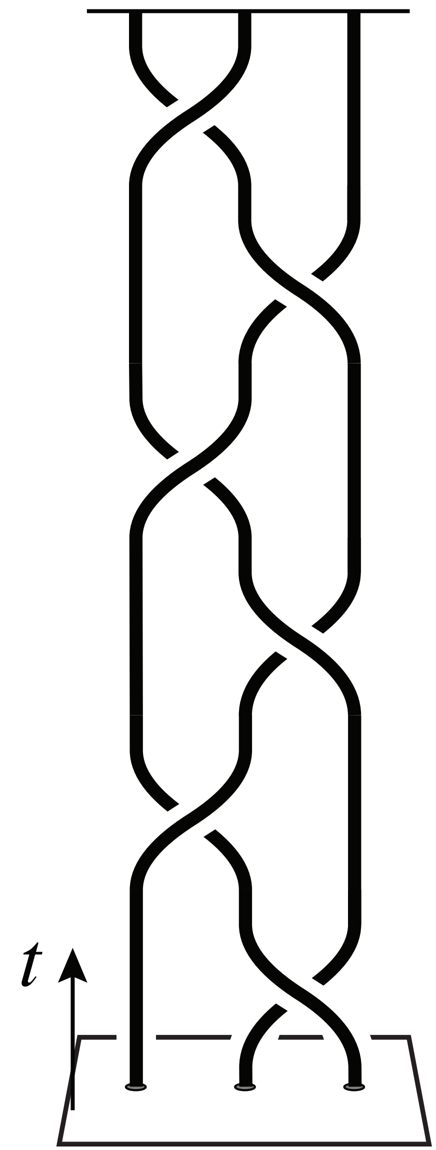
Andrea at one snapshot; LCS shown (orange = repelling, blue = attracting)

Coherent sets in the atmosphere that braid



three sets: magenta, green, purple

Coherent sets in the atmosphere that braid



Sets form braid on three strands

Final words on coherent sets from data

- From sequences of particle images, e.g., from PIV fluid experiments, can directly compute flow map and coherent structures (skipping the velocity field). Useful for inertial particles which don't follow fluid velocity.

- From FTLE, get first order picture of coherent sets, the 'valleys' as opposed to the ridges.

- Links between geometric, probabilistic and topological methods.

- Future work: predicting bifurcations in transport structure from transfer operator trends

- **For more, see Shibabrat Naik's poster tonight and Amir BozorgMagham's talk on Thursday, MS126**

The End

For papers, movies, etc., visit:
www.shaneross.com

Main Papers:

- Tallapragada & Ross [2013] A set oriented definition of the finite-time Lyapunov exponent and coherent sets. *Communications in Nonlinear Science and Numerical Simulation* 18(5), 1106-1126.
- Grover, Ross, Stremler, Kumar [2012] Topological chaos, braiding and breakup of almost-invariant sets. *Chaos* 22, 043135.
- Stremler, Ross, Grover, Kumar [2011] Topological chaos and periodic braiding of almost-cyclic sets. *Physical Review Letters* 106, 114101.
- Tallapragada, Ross, Schmale [2011] Lagrangian coherent structures are associated with fluctuations in airborne microbial populations. *Chaos* 21, 033122.
- Senatore & Ross [2011] Detection and characterization of transport barriers in complex flows via ridge extraction of the finite time Lyapunov exponent field, *International Journal for Numerical Methods in Engineering* 86, 1163.
- Lekien & Ross [2010] The computation of finite-time Lyapunov exponents on unstructured meshes and for non-Euclidean manifolds. *Chaos* 20, 017505.