Stirring by braiding of coherent sets

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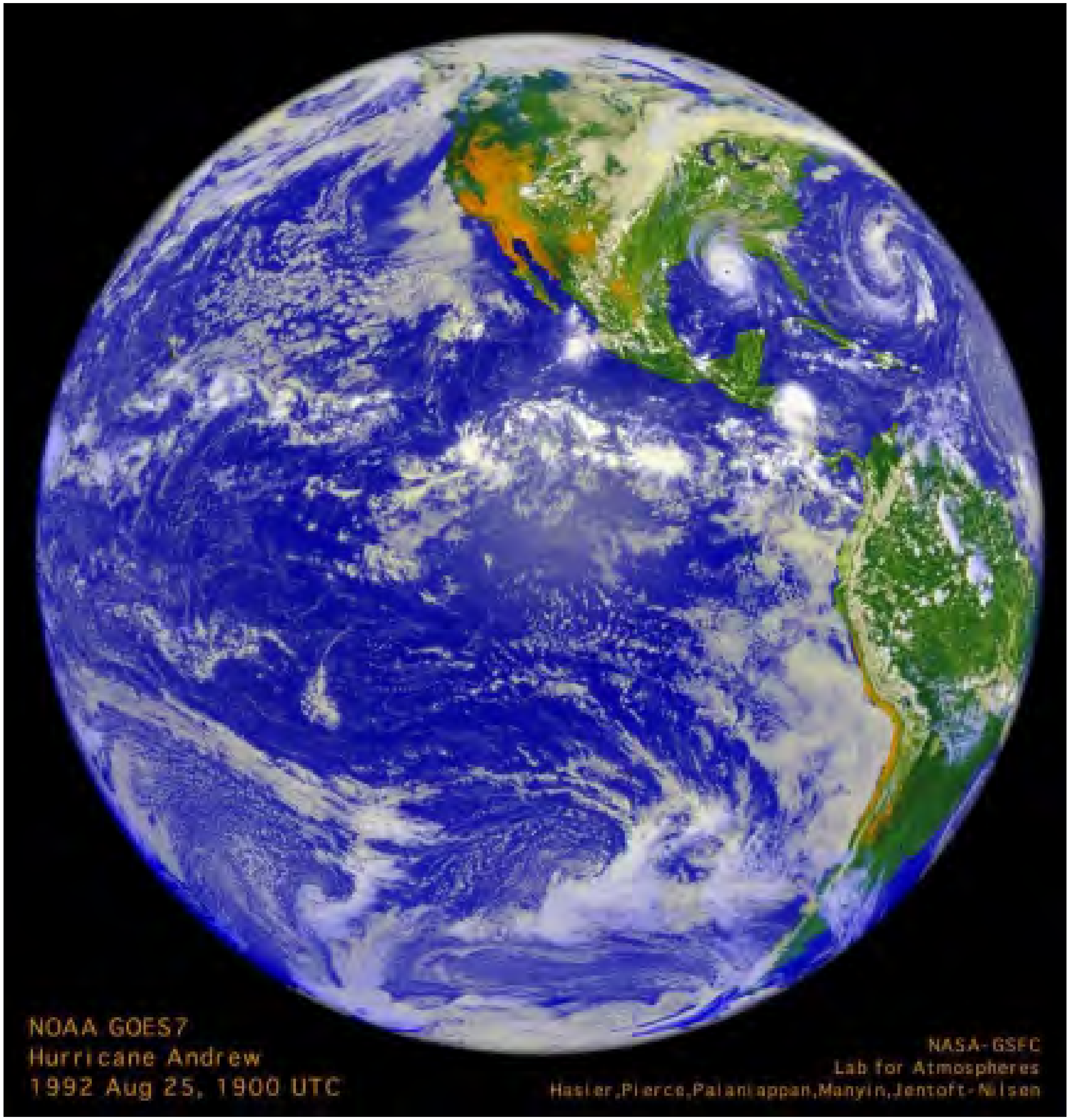
MultiSTEPS: MultiScale Transport in Environmental & Physiological Systems, www.multisteps.esm.vt.edu

Motivation: complex fluid mixing



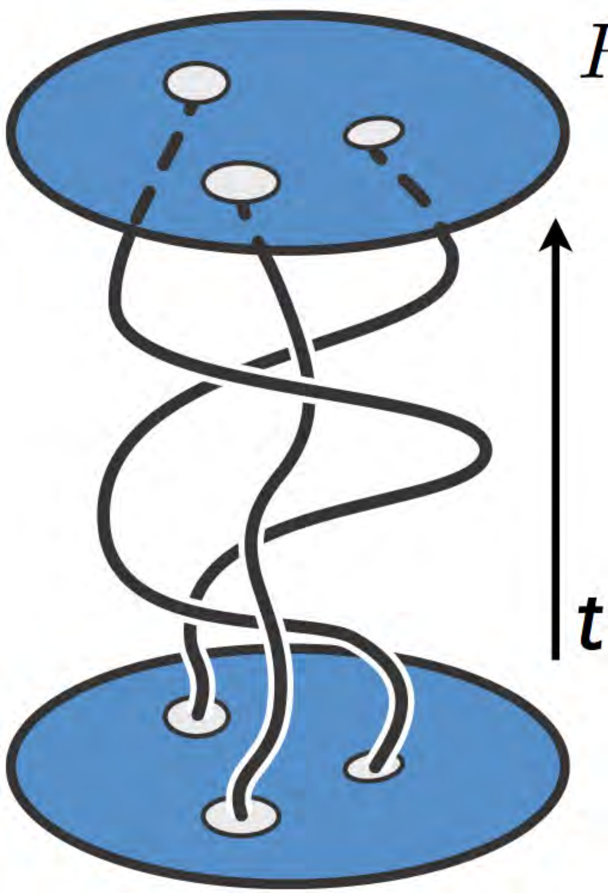
Stirring fluids with solid rods

turbulent mixing
spoon in coffee

laminar mixing
3 ‘braiding’ rods in glycerin

Topological chaos through braiding of stirrers

- Topological chaos is 'built in' the flow due to the topology of boundary motions



R_N : 2D fluid region with *N* stirring 'rods'

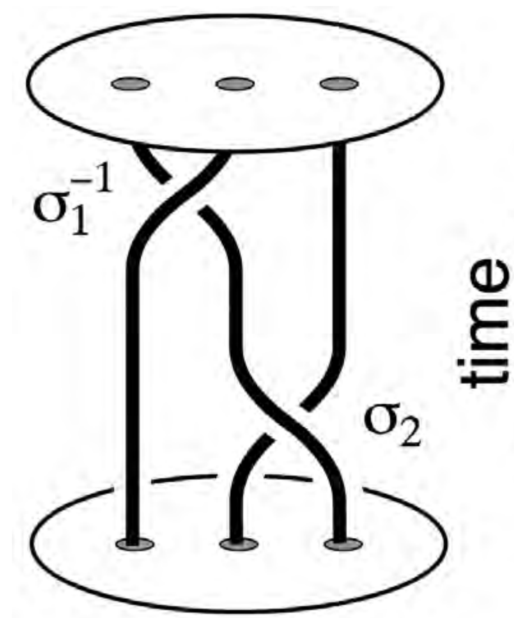
- stirrers move on periodic orbits
- stirrers = solid objects or *fluid particles*
- stirrer motions generate diffeomorphism

$$f : R_N \to R_N$$

- stirrer trajectories generate braids in 2+1 dimensional space-time

Thurston-Nielsen classification theorem

- Thurston (1988) Bull. Am. Math. Soc.
- A stirrer motion f is isotopic to a stirrer motion g of one of three types (i) finite order (f.o.): the nth iterate of g is the identity (ii) pseudo-Anosov (pA): g has dense orbits, Markov partition with transition matrix A, topological entropy $h_{\mathrm{TN}}(g) = \log(\lambda_{PF}(A))$, where $\lambda_{\mathrm{PF}}(A) > 1 =$ Perron-Frobenius eigenvalue of A (iii) reducible: g contains both f.o. and pA regions
- h_{TN} computed from 'braid word', e.g., $\sigma_1^{-1}\sigma_2$
- $\log(\lambda_{PF}(A))$ provides a **lower bound** on the true topological entropy



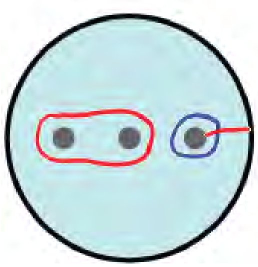
Topological chaos in a viscous fluid experiment

Move 3 rods on 'figure-8' paths through glycerin

Boyland, Aref & Stremler (2000) *J. Fluid Mech.*

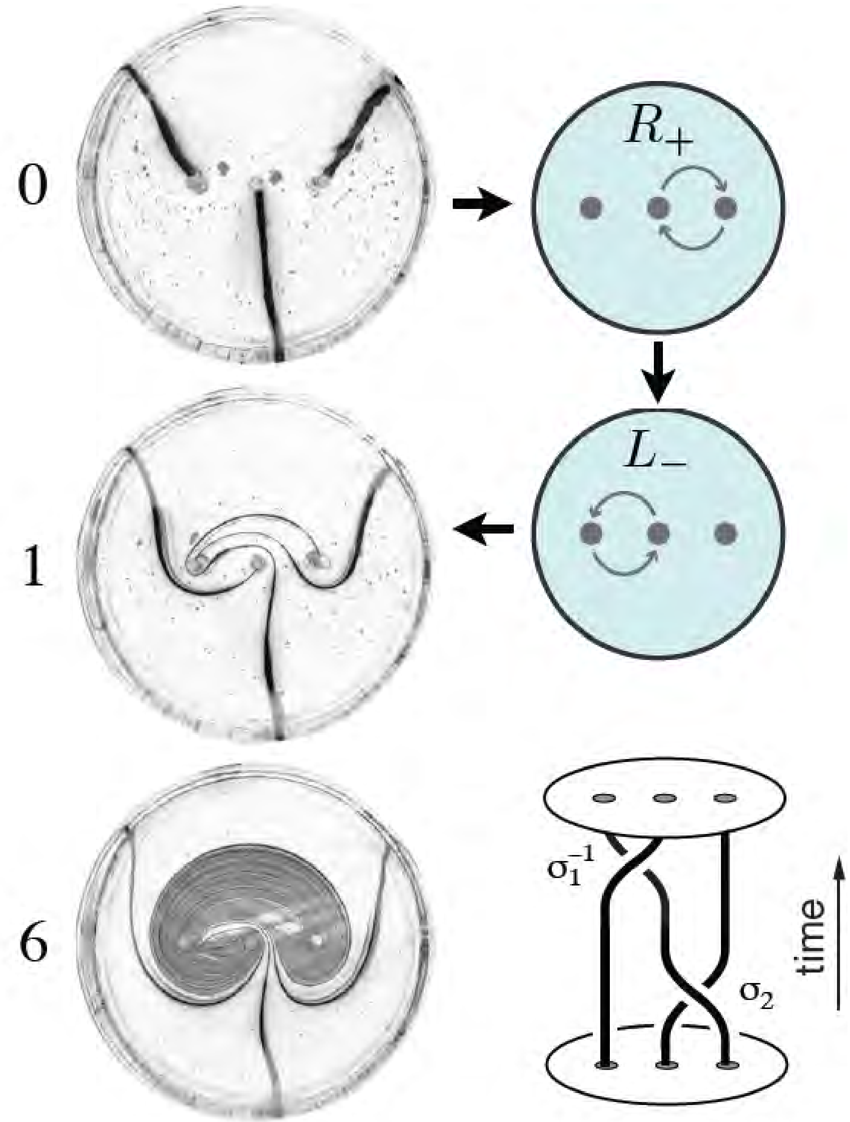
- stirrers move on periodic orbits in two steps
- Thurston-Nielsen theorem gives a lower bound on stretching:

$$\lambda_{\mathrm{TN}} = \tfrac{1}{2}\left(3 + \sqrt{5}\right)$$
$$h_{\mathrm{TN}} = \log(\lambda_{\mathrm{TN}}) = 0.962\ldots$$

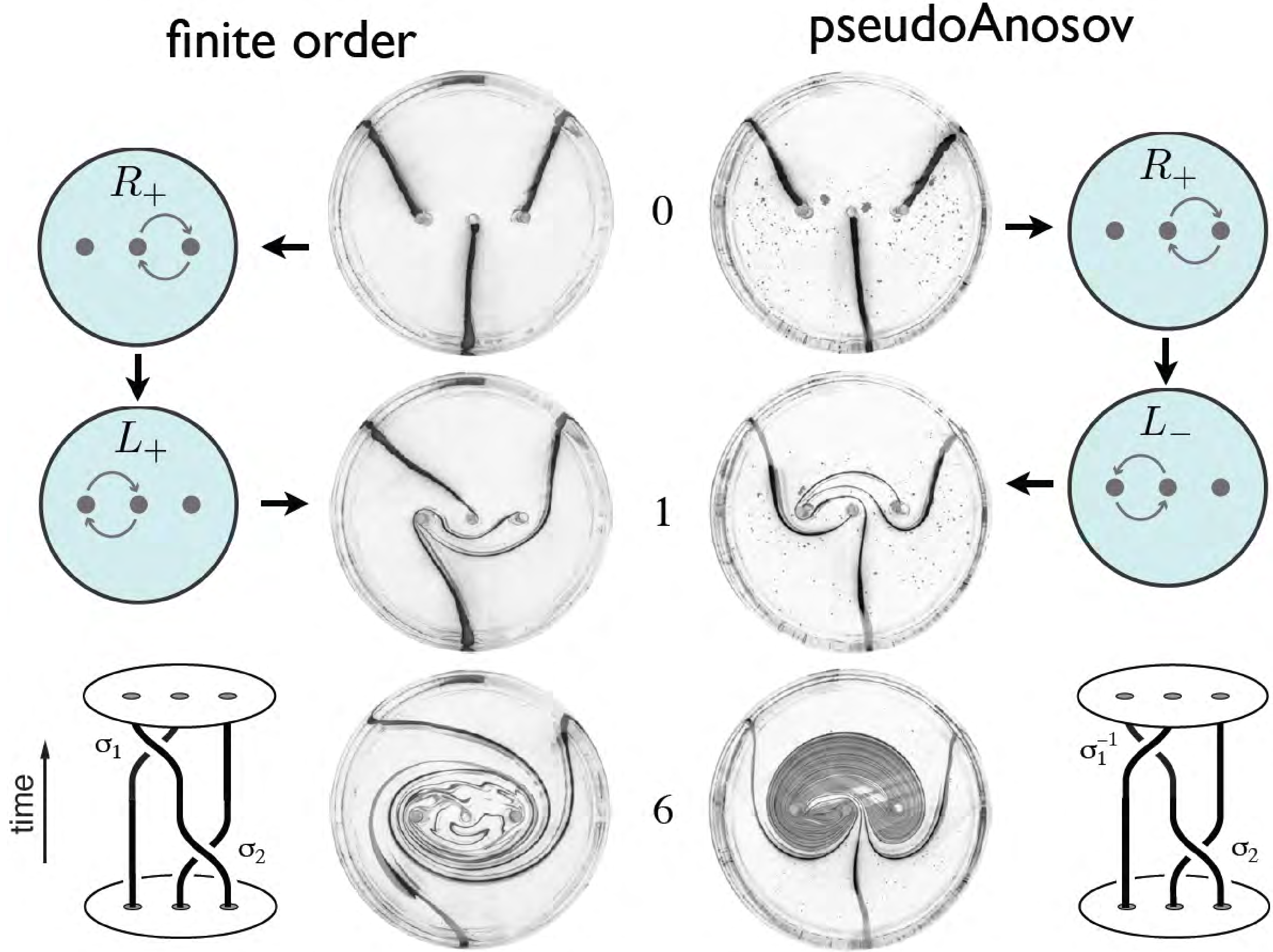


non-trivial material lines grow like $l \sim l_0\ \lambda^n$

$$\lambda \geq \lambda_{\mathrm{TN}}$$



Topological chaos in a viscous fluid experiment



‘Stirring’ with fluid particles

point vortices in a periodic domain

Boyland, Stremler & Aref (2003) *Physica* D

one rod moving on an epicyclic trajectory

Gouillart, Thiffeault & Finn (2006) *Phys. Rev.* E

‘ghost rods’ solid rods

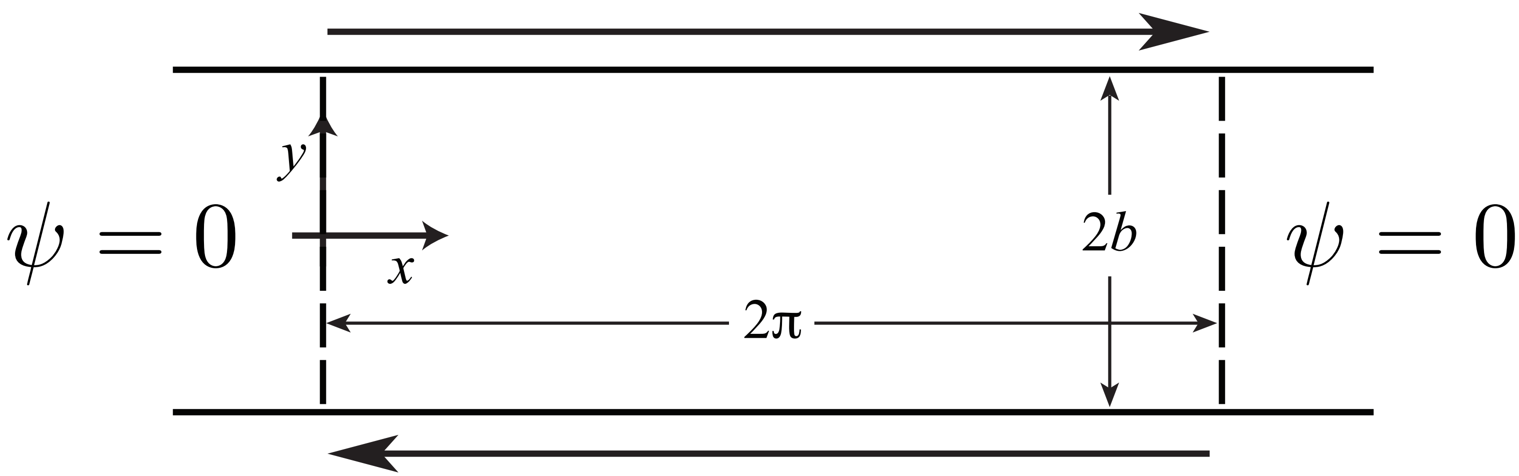
Fluid is wrapped around ‘ghost rods’ in the fluid

– *flow structure assists in the stirring*

Designing for ghost rods

- Lid-driven cavity flow (Stokes flow)
  - design a flow with ghost rods that appear when and where we want e.g., for an optimal micro-scale mixer

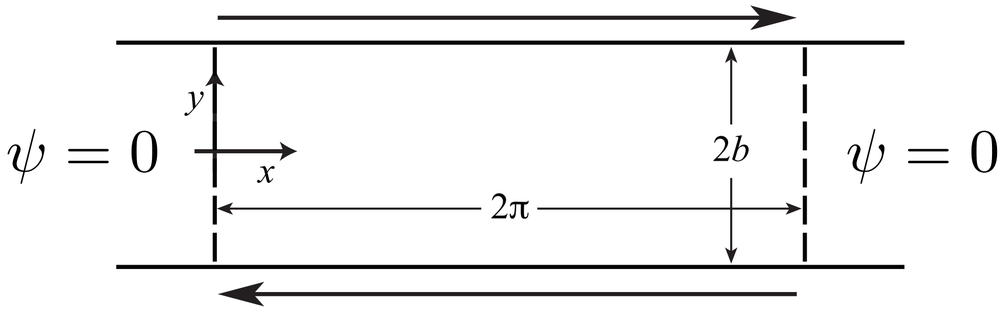
$$u = \frac{\partial \psi}{\partial y} = \pm \sum_{n=1}^{N} U_n \sin(nx/2)$$



$$u = \frac{\partial \psi}{\partial y} = \mp \sum_{n=1}^{N} U_n \sin(nx/2)$$

Solving for the streamfunction ψ

$$\frac{\partial\psi}{\partial y} = \pm\sum_{n=1}^{N} U_n \, \sin(nx/2)$$



$$\frac{\partial\psi}{\partial y} = \mp\sum_{n=1}^{N} U_n \, \sin(nx/2)$$

Assume a streamfunction

$$\psi(x,y) = \sum_{n=1}^{N} U_n \, \psi_n(x,y)$$

$$= \sum_{n=1}^{N} U_n \, \mathbb{C}_n \, f_n(y) \, \sin(nx/2)$$

Stokes flow: $\nabla^4\psi_n = 0 \;\Rightarrow\; f_n''''(y) - 2\,(n/2)^2\, f_n''(y) + (n/2)^4\, f(y) = 0$

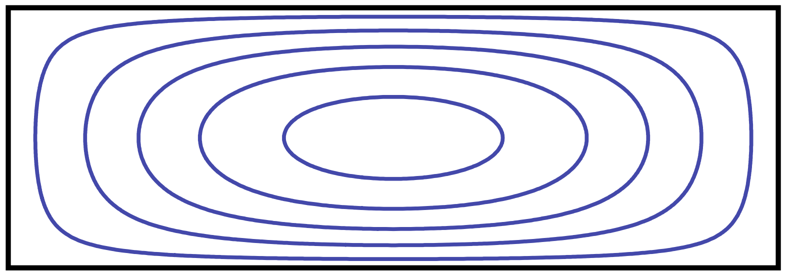
Boundary conditions: $f_n\,(\pm b) = 0 \qquad \mathbb{C}_n\, f_n'\,(\pm b) = \pm 1$

$$f_n(y) = y\,\cosh(nb/2)\,\sinh(ny/2) - b\,\sinh(nb/2)\,\cosh(ny/2)$$

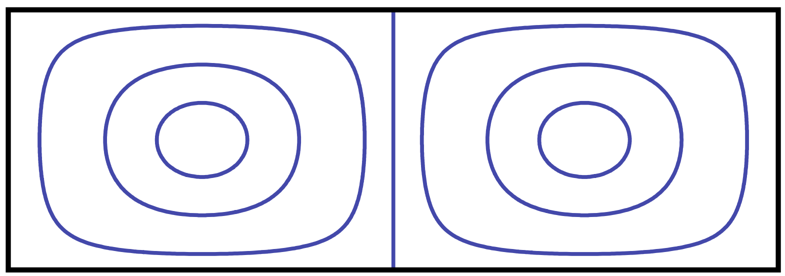
$$\mathbb{C}_n = 2\left[\sinh(nb) + nb\right]^{-1}$$

(exact solution)

Solving for the streamfunction ψ



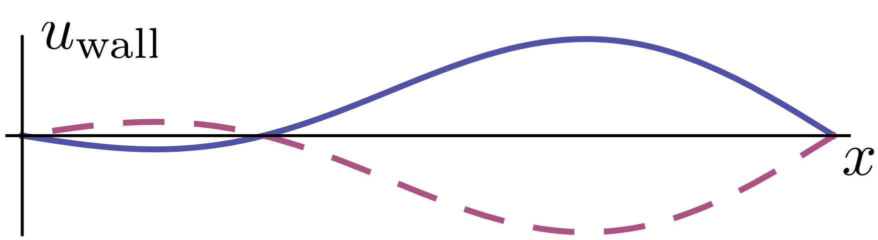
$$\psi_1 = \mathbb{C}_1\, f_1(y)\, \sin(x/2)$$



$$\psi_2 = \mathbb{C}_2\, f_2(y)\, \sin(x)$$

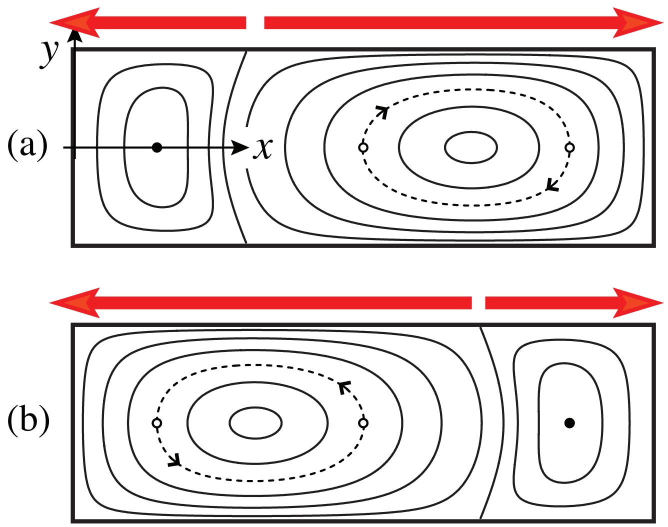
Assume: $\psi = U_1\, \mathbb{C}_1\, f_1(y)\, \sin(x/2) + U_2\, \mathbb{C}_2\, f_2(y)\, \sin(x)$

$$= U\left[\sqrt{1-\beta}\,\psi_1 + \sqrt{\beta}\,\psi_2\right]$$



Pick parameters to get desired stirring protocol

- Alternate boundary motions to generate braid



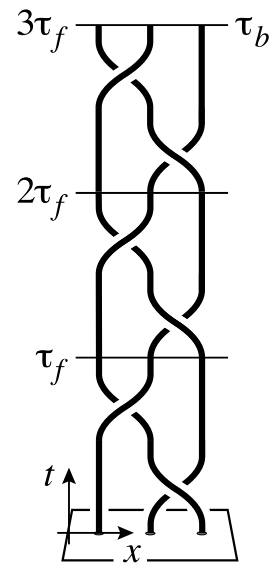
streamlines

tracer blob

- When $n\tau_f \leq t < (n+1)\tau_f/2$, right two exchange clockwise
- When $(n+1)\tau_f/2 \leq t < (n+1)\tau_f$, left two exchange counter-clockwise

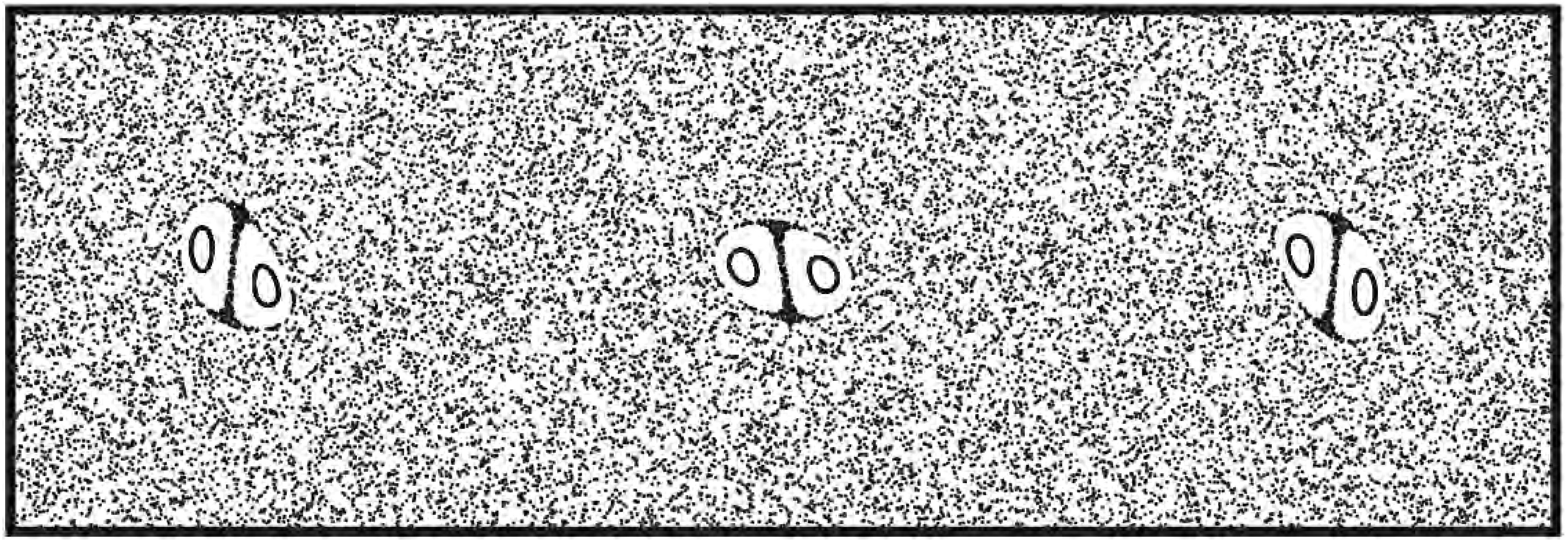
Stirring protocol ⇒ braid ⇒ topological entropy

- Periodic points of period 3 ⇒ act as 'ghost rods'
- Their braid ⇒ $h_{\mathrm{TN}} = 0.96242$ from TNCT
- Actual $h_{\mathrm{flow}} \approx 0.964$
- ⇒ h_{TN} is an excellent lower bound



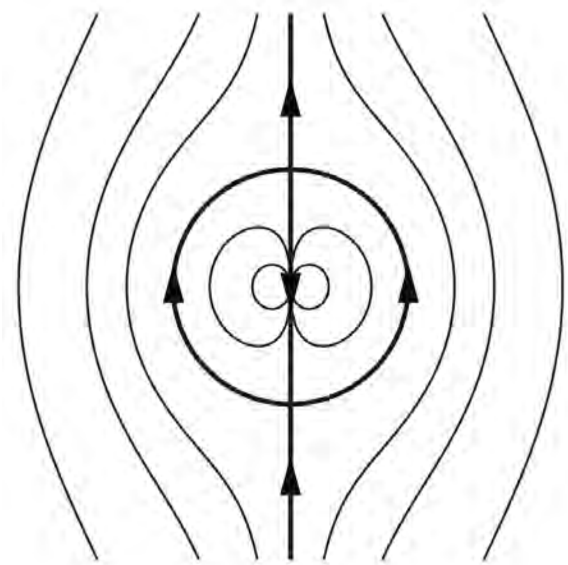
- System parameter τ_f can be treated as a bifurcation parameter critical point is $\tau_f^* = 1$; next few slides show $\tau_f > 1$

Identifying ‘ghost rods’: periodic points



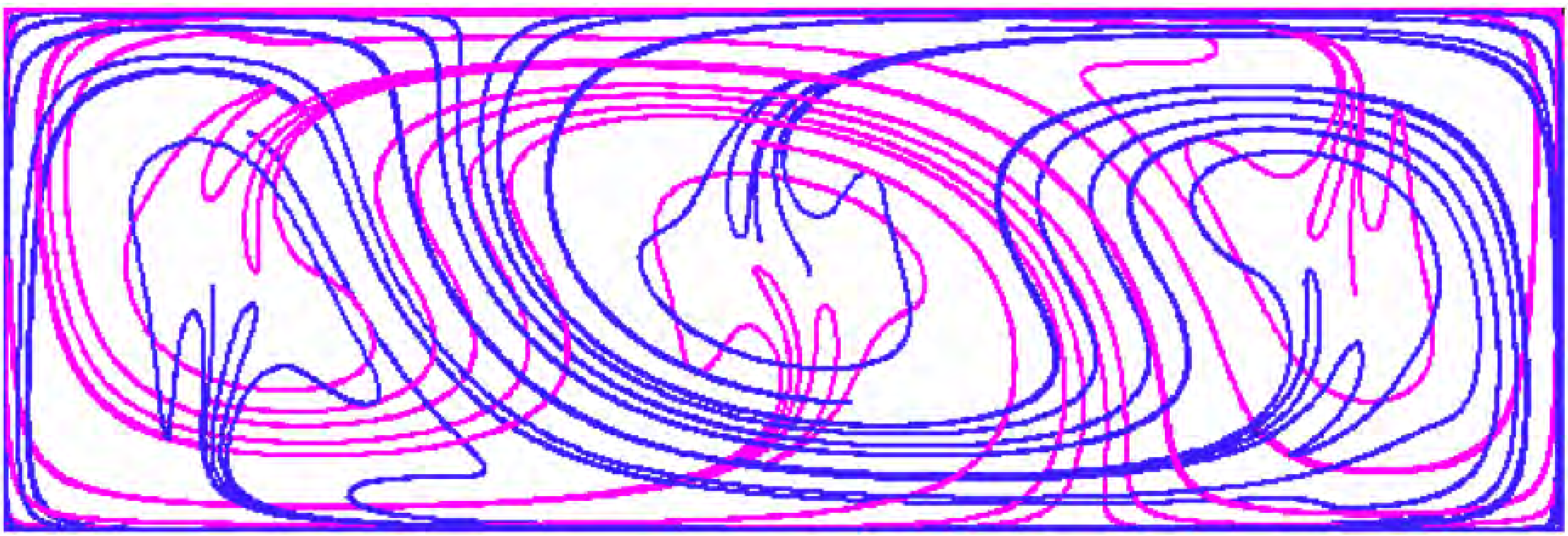
Poincaré section for $\tau_f > 1$

- At $\tau_f = 1$, parabolic points
- $\tau_f > 1$, **groups of elliptic and saddle points** of period 3
  — streamlines around groups resemble fluid motion around a solid rod ⇒
- $\tau_f < 1$, **periodic points vanish**



Transport, mixing, and homogenization

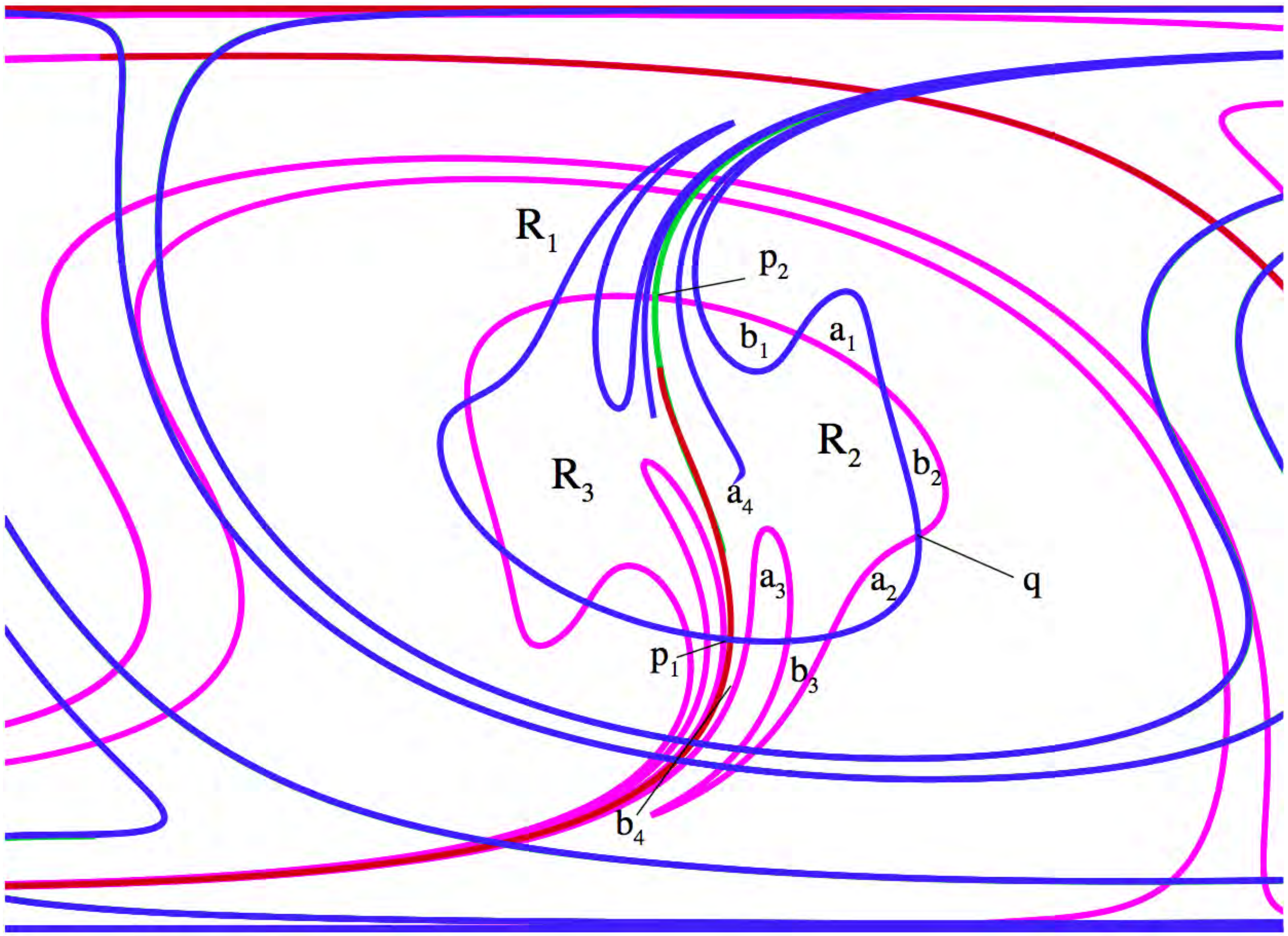
- Consider $\tau_f > 1$
- Structure associated with saddles of Poincaré map



some invariant manifolds of saddles

Transport, mixing, and homogenization

- Can consider transport via **lobe dynamics**

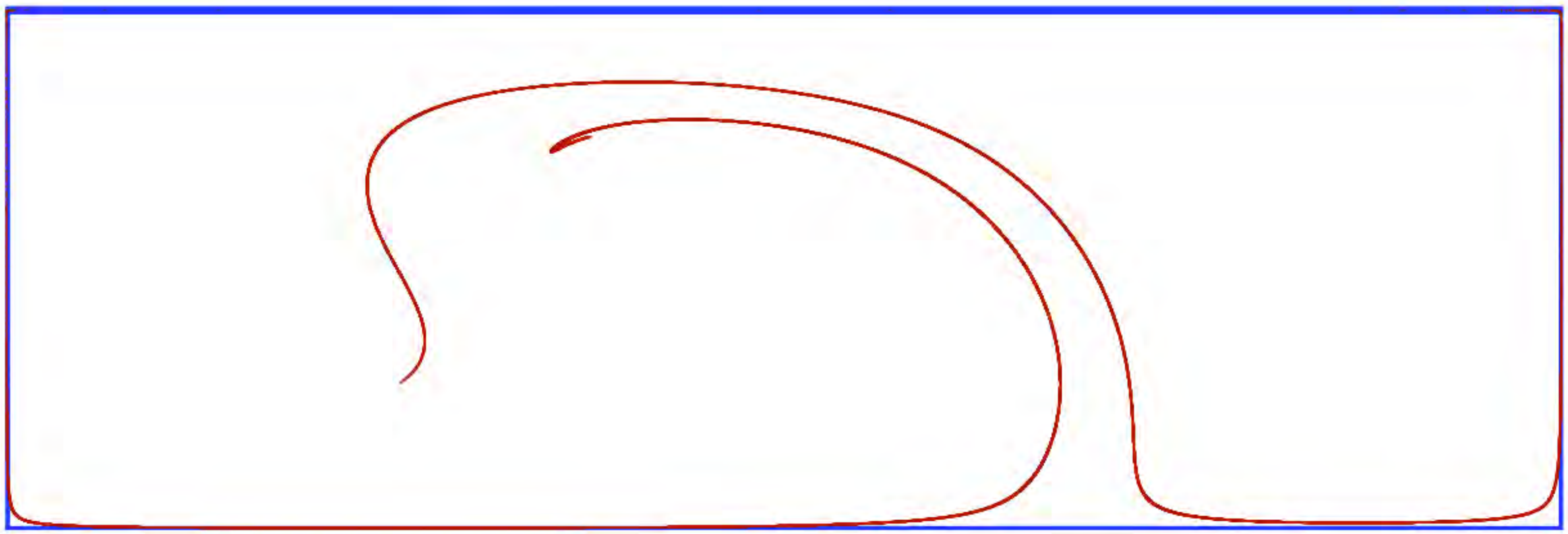


pips, regions and lobes labeled

Stable/unstable manifolds and lobes in fluids

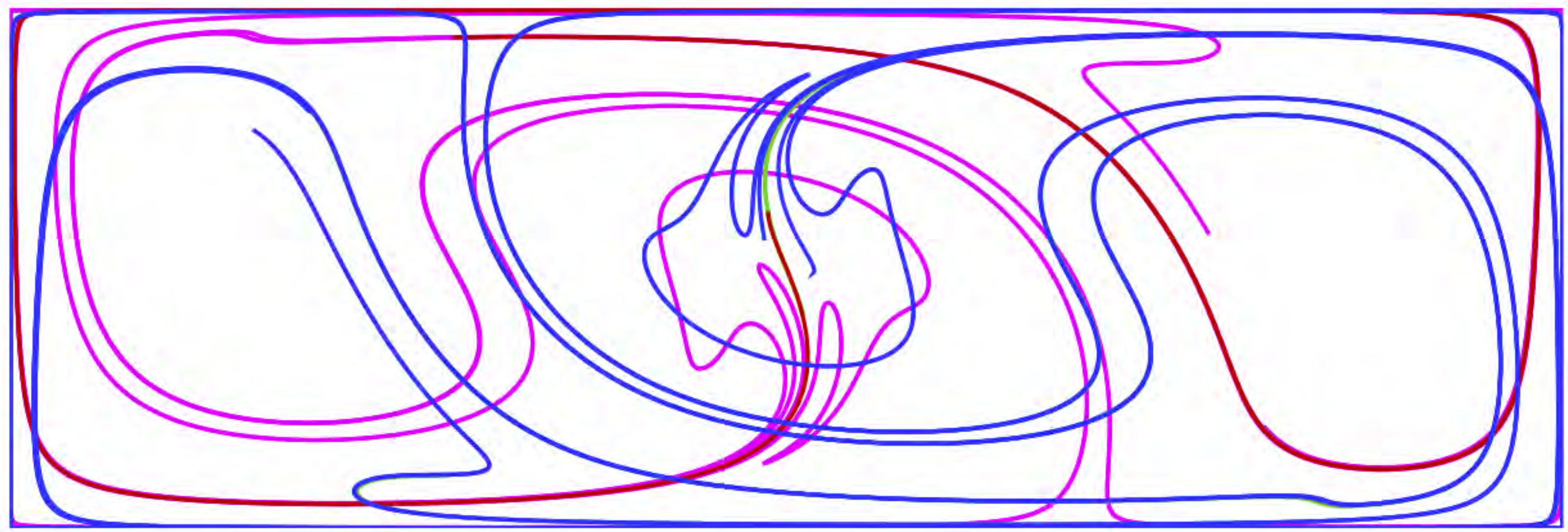
material blob at $t = 0$

Stable/unstable manifolds and lobes in fluids



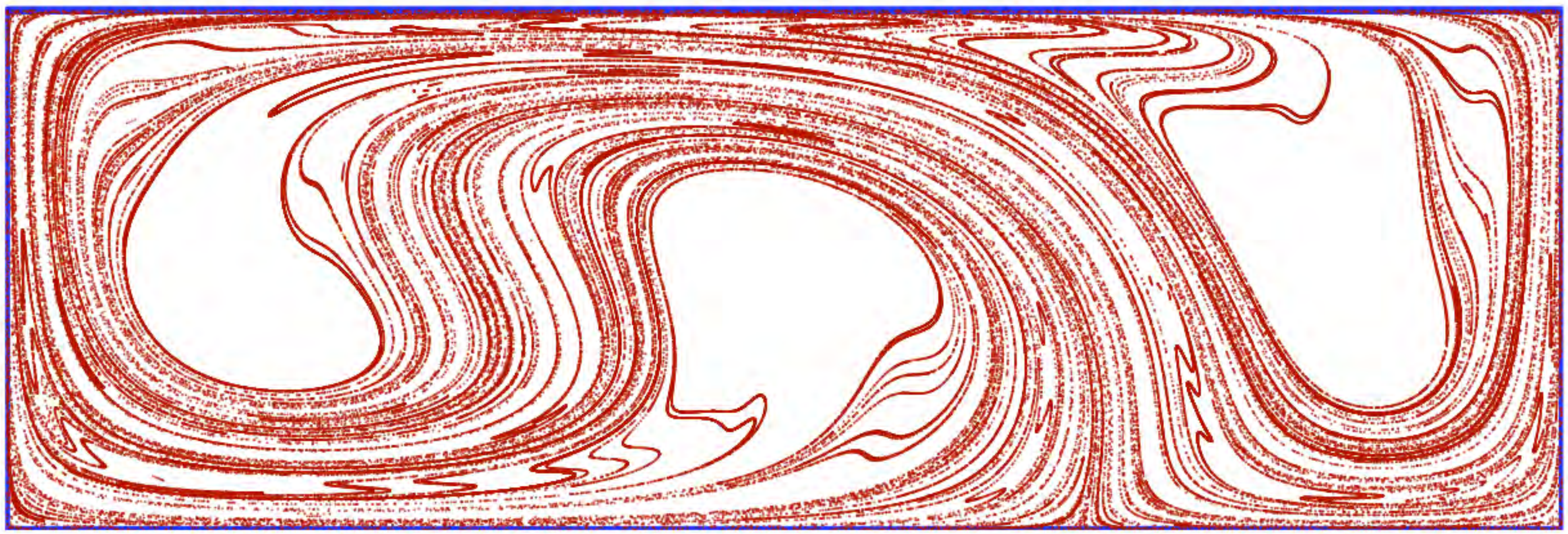
material blob at $t = 5$

Stable/unstable manifolds and lobes in fluids



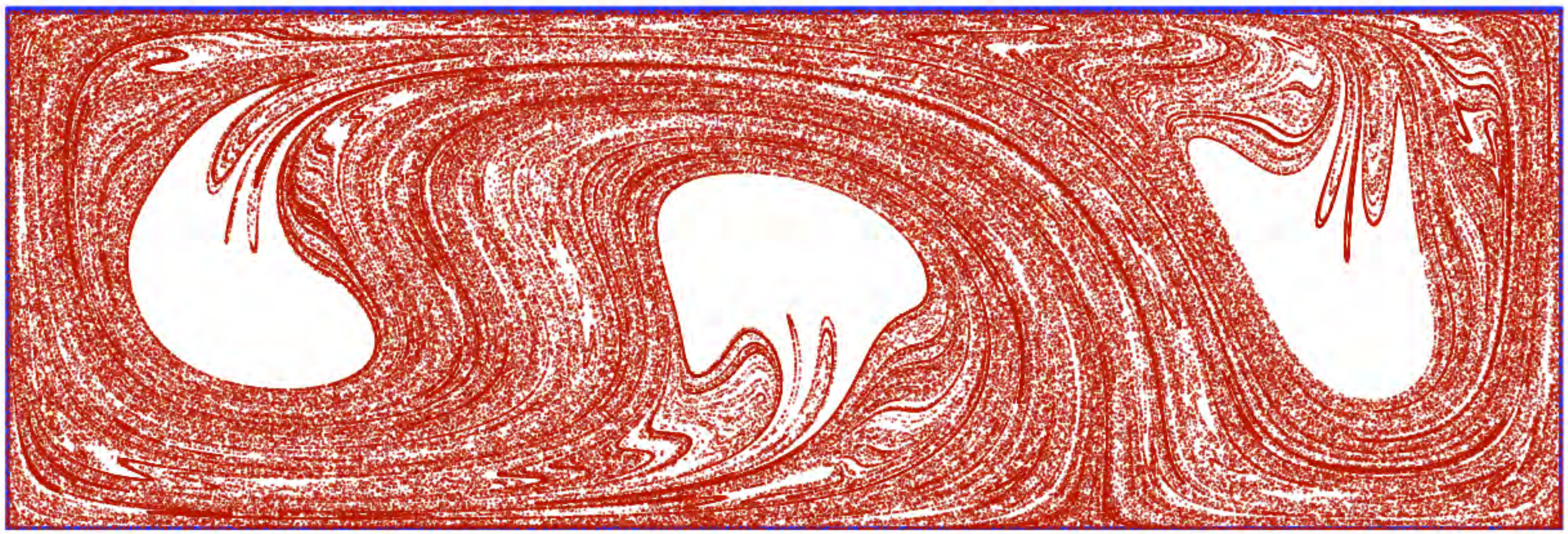
some invariant manifolds of saddles

Stable/unstable manifolds and lobes in fluids



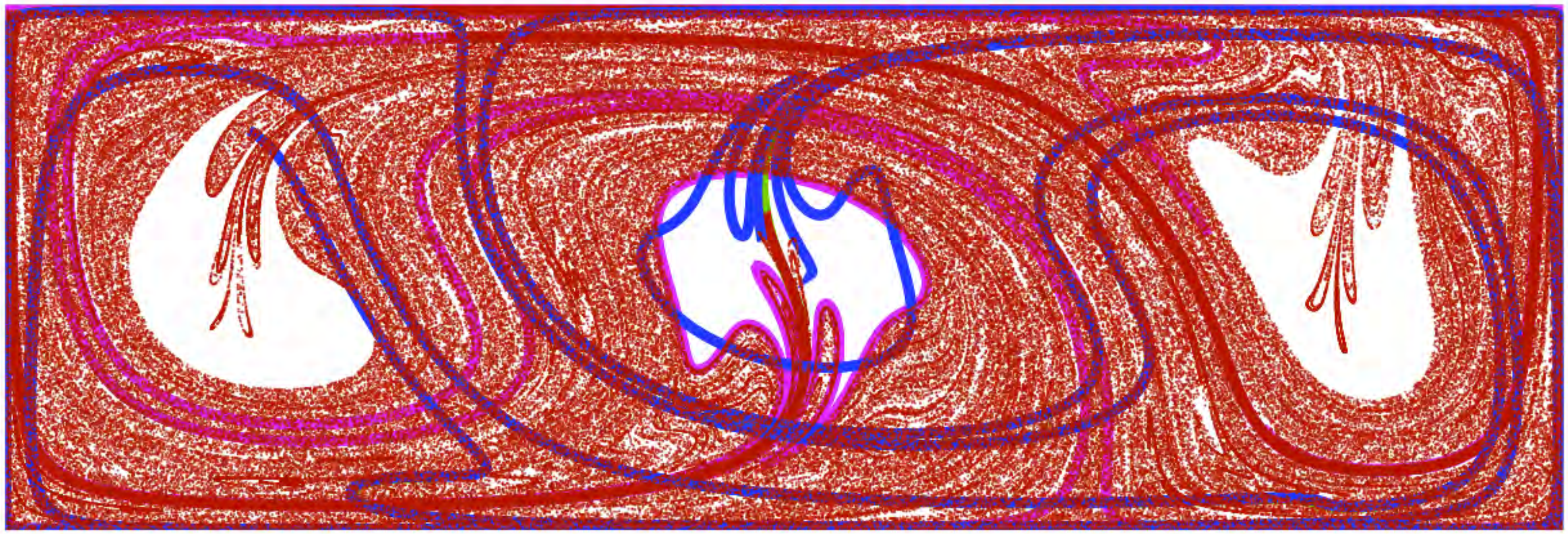
material blob at $t = 10$

Stable/unstable manifolds and lobes in fluids



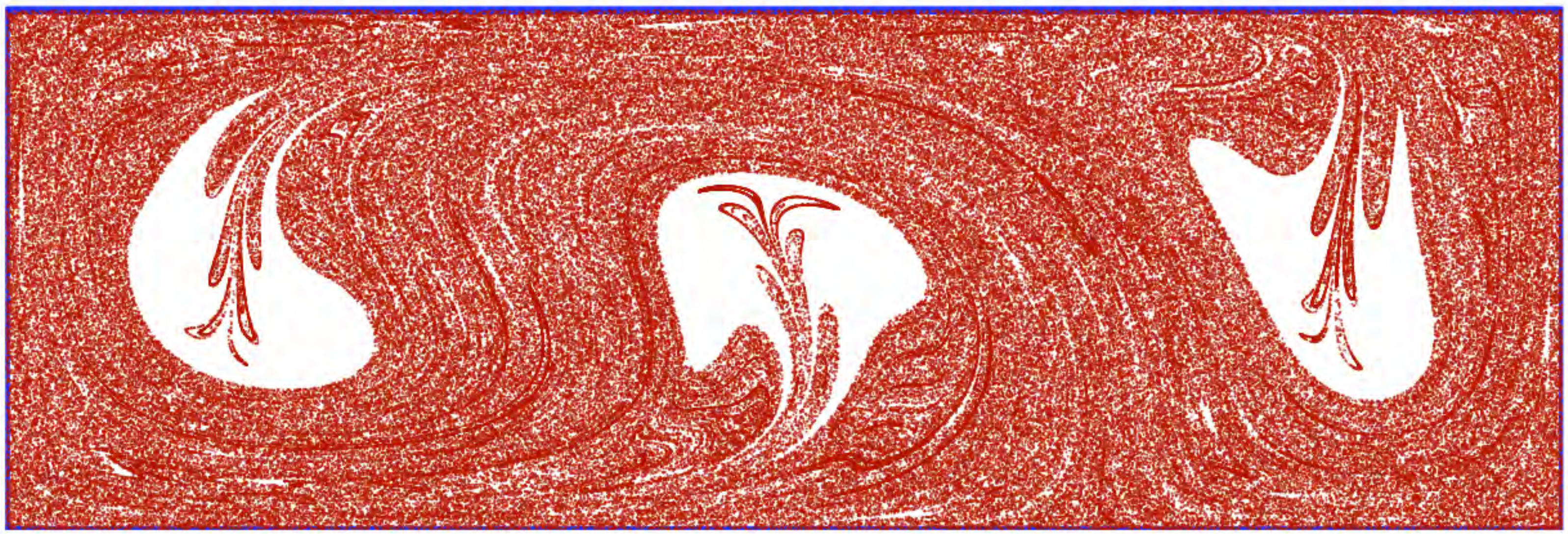
material blob at $t = 15$

Stable/unstable manifolds and lobes in fluids



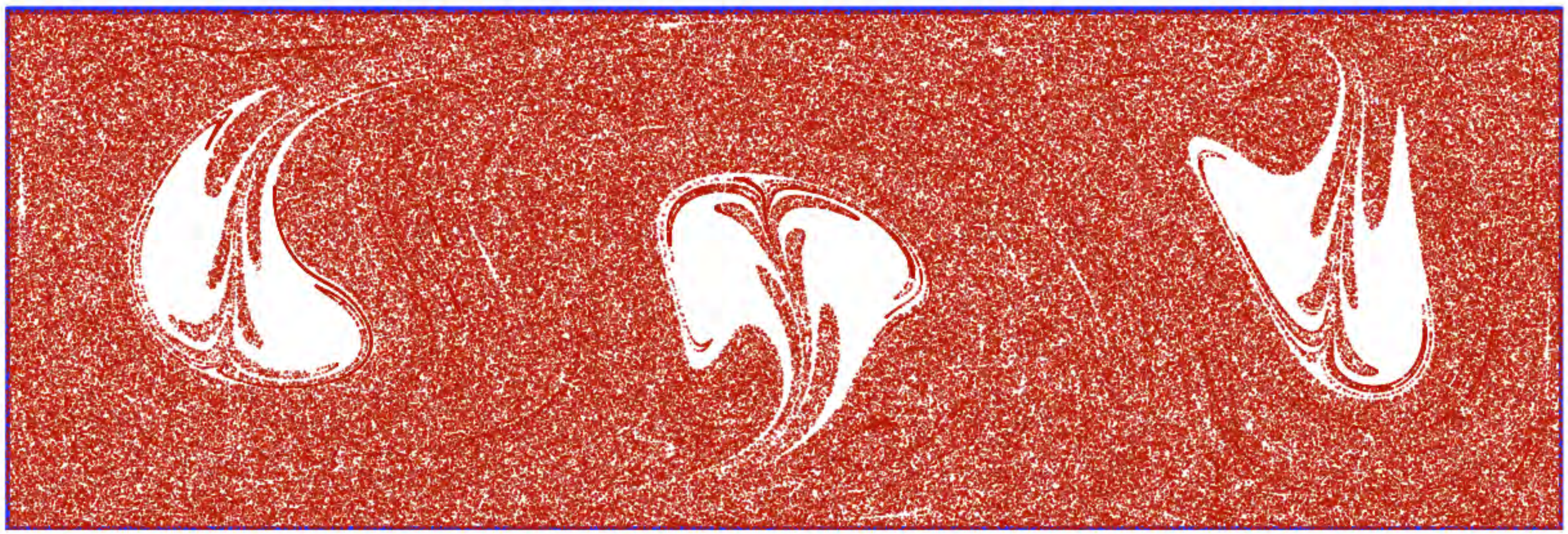
material blob and manifolds

Stable/unstable manifolds and lobes in fluids



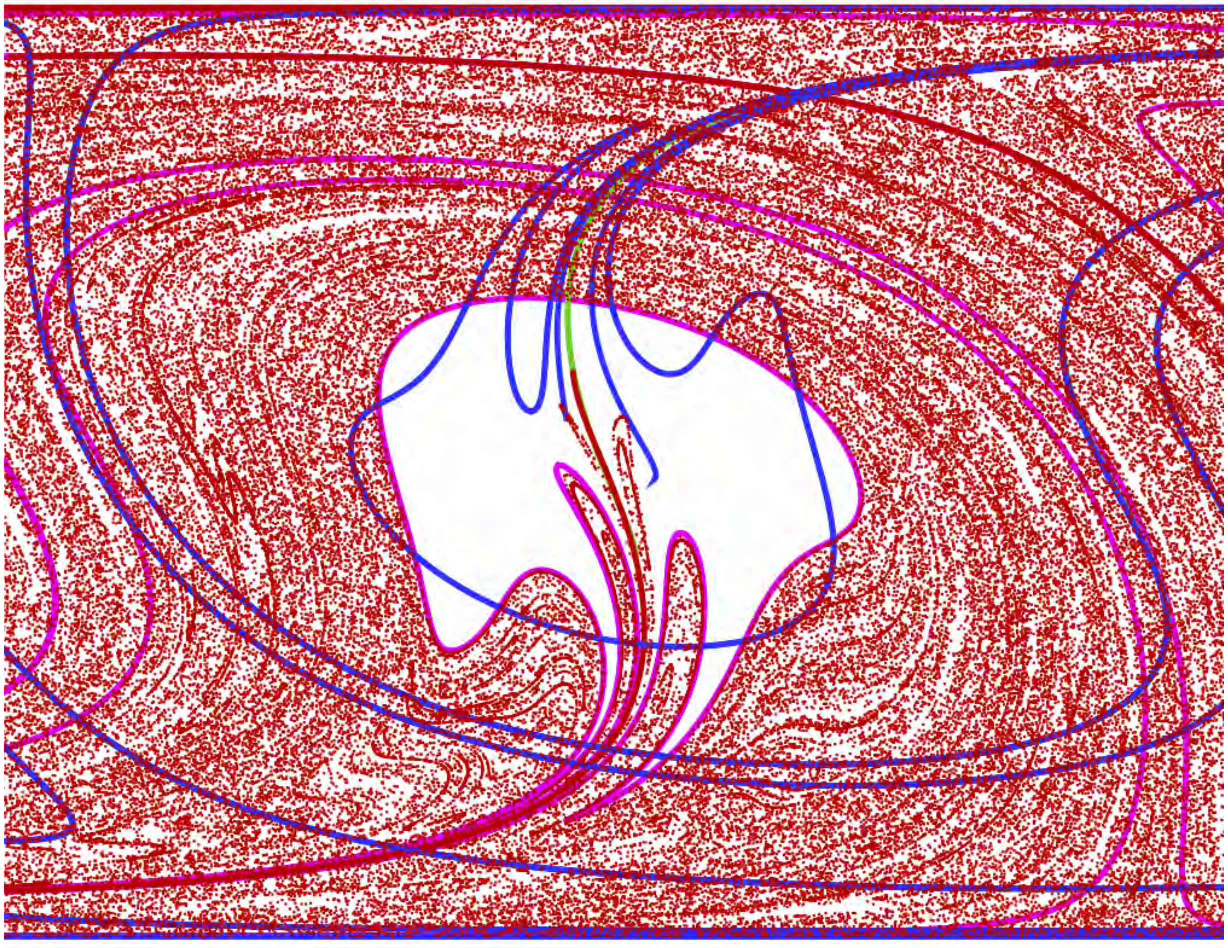
material blob at $t = 20$

Stable/unstable manifolds and lobes in fluids



material blob at $t = 25$

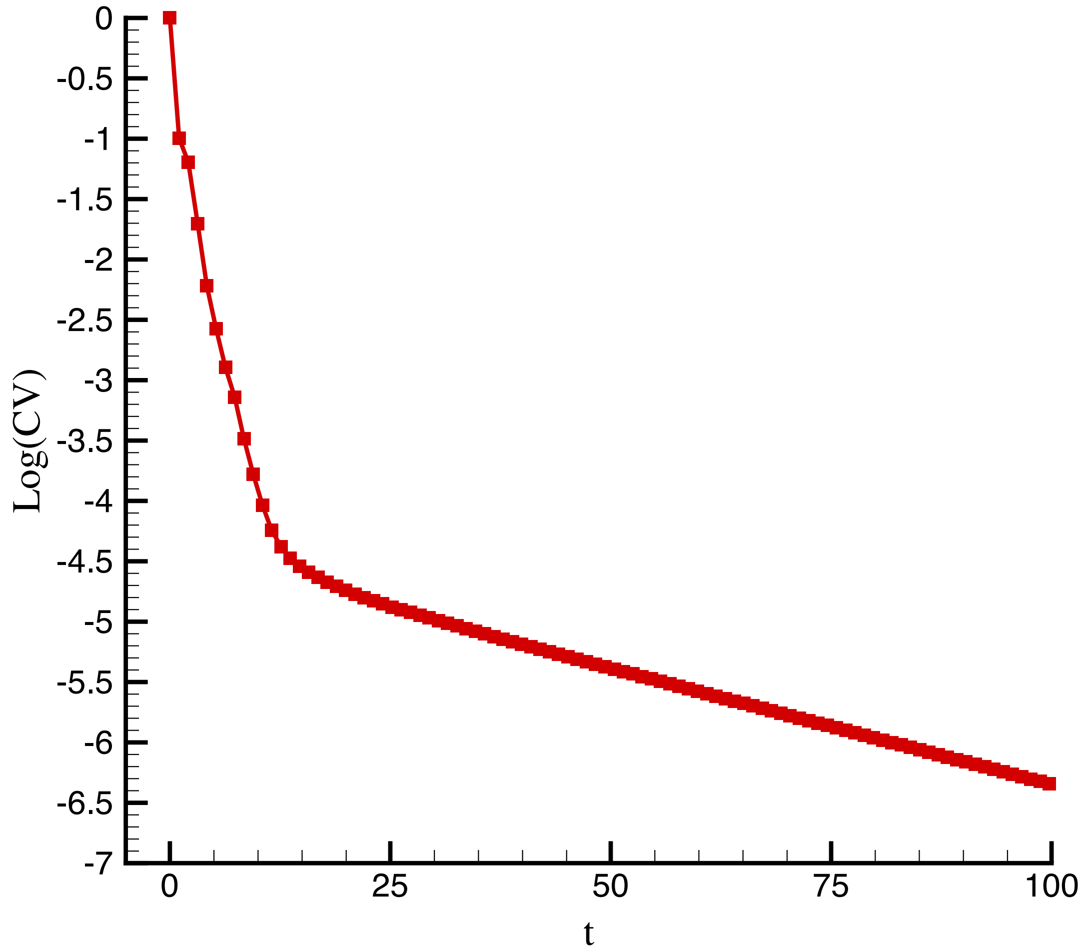
Stable/unstable manifolds and lobes in fluids



- Saddle manifolds and lobe dynamics provide template for motion

Stable/unstable manifolds and lobes in fluids

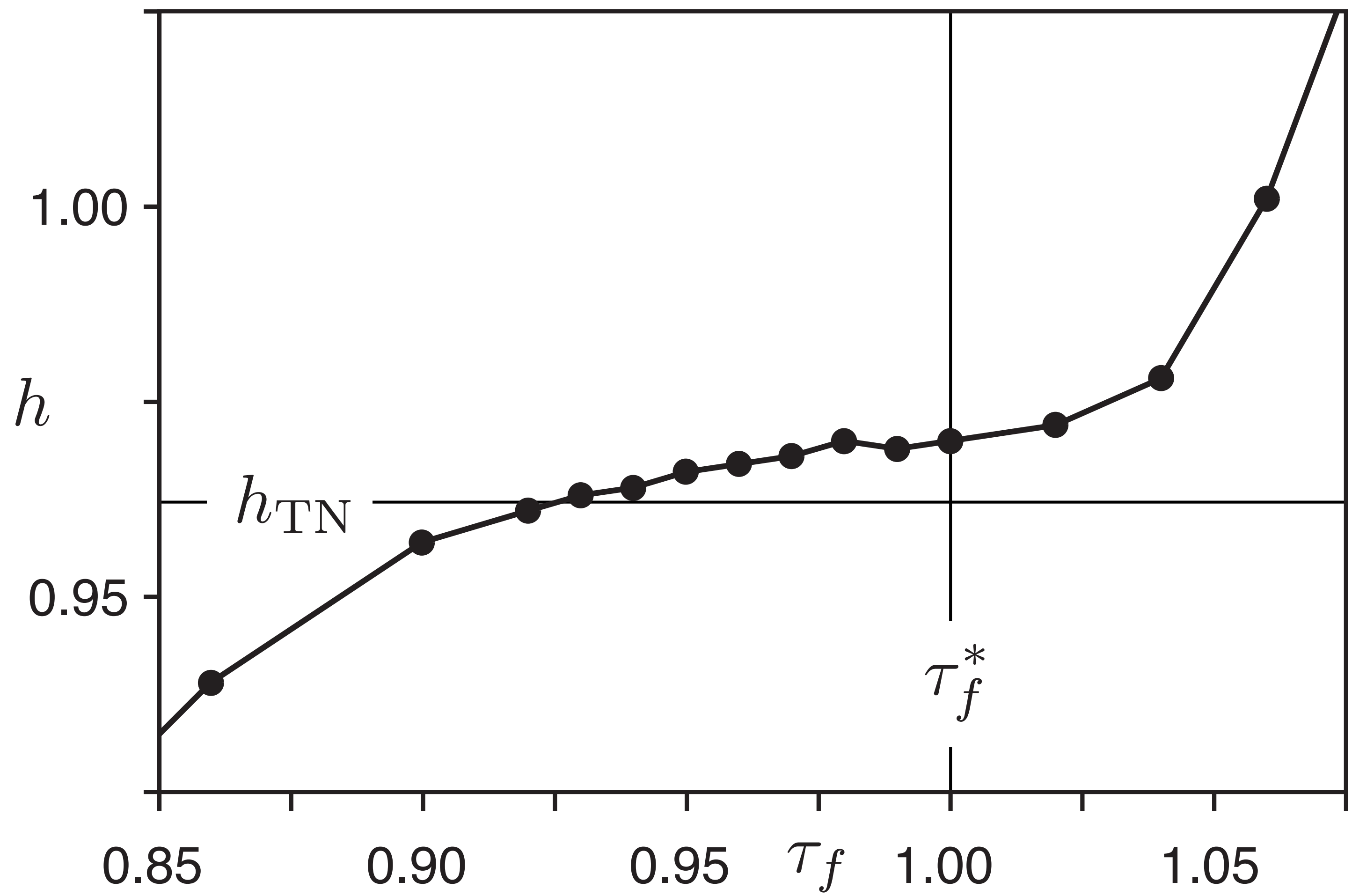
- Concentration variance; a measure of homogenization



- Homogenization has two exponential rates: slower one related to lobes
- Fast rate due to braiding of ‘ghost rods’

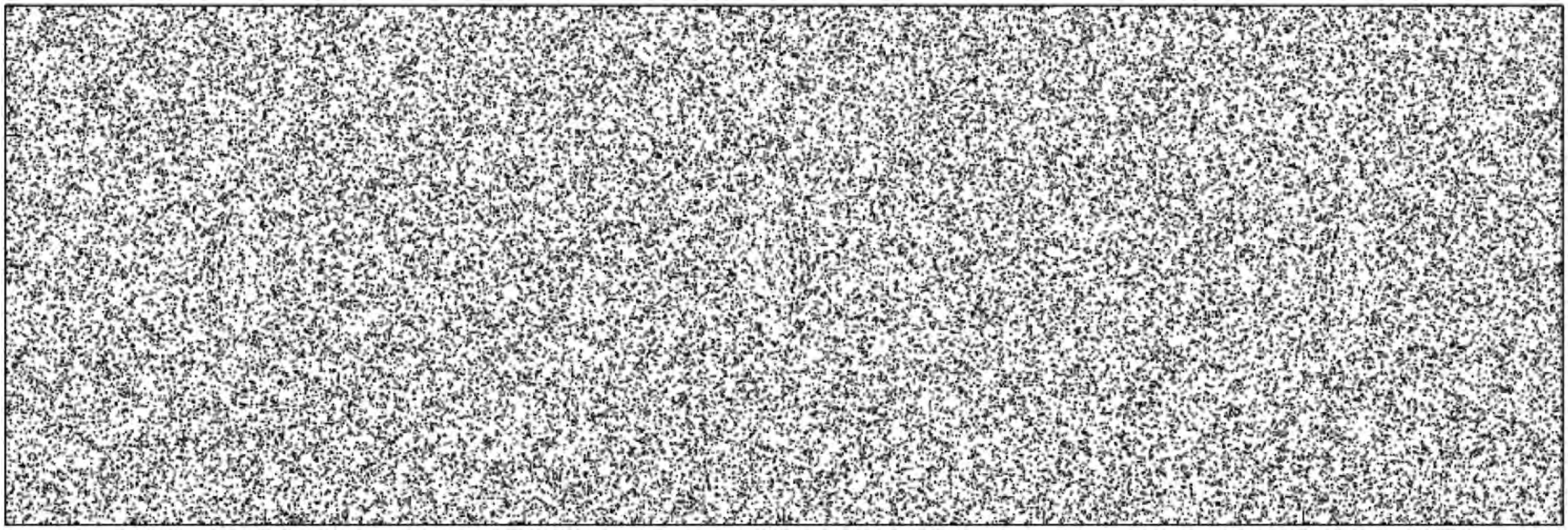
Topological entropy continuity across critical point

- Consider $\tau_f < 1$



topological entropy as a function of τ_f

Identifying 'ghost rods'?



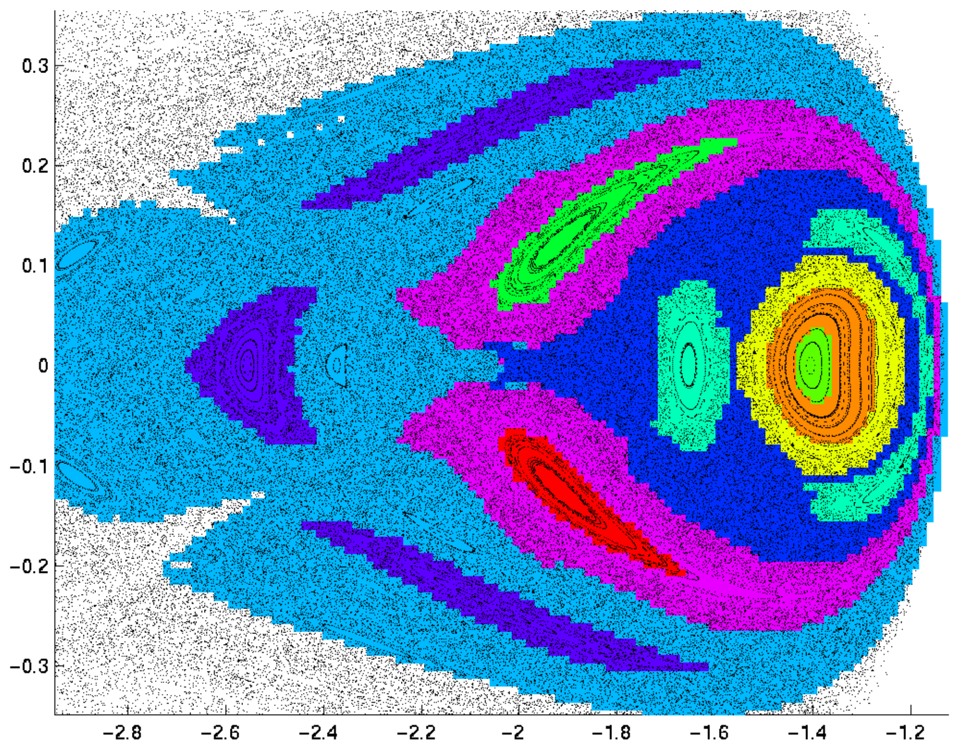
Poincaré section for $\tau_f < 1 \Rightarrow$ no obvious structure!

- Note the absence of any elliptical islands
- No periodic orbits of low period were found
- Is the phase space featureless?

Almost-cyclic set approach

- Take probabilistic point of view
- Partition phase space into **loosely coupled regions**

Almost-invariant sets ≈ "Leaky" regions with a long residence time[1]

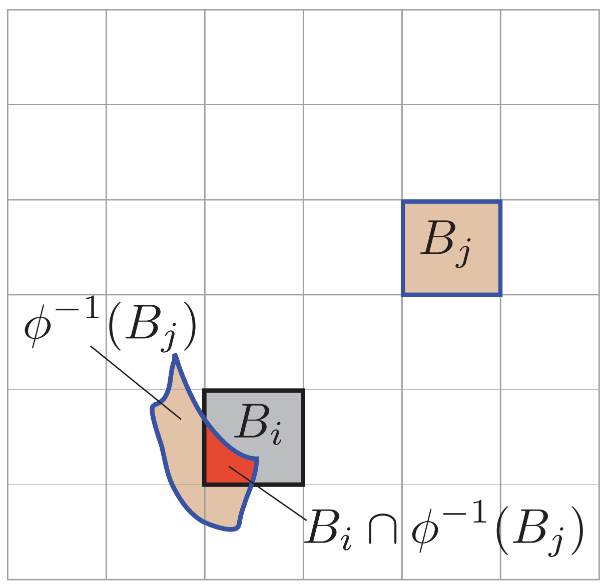


3-body problem phase space is divided into several invariant and almost-invariant sets.

[1] work of Dellnitz, Junge, Deuflhard, Froyland, Schütte, et al

Almost-cyclic set approach

- Create box partition of phase space $\mathcal{B} = \{B_1, \ldots B_q\}$, with q large
- Consider a q-by-q **transition (Ulam) matrix**, P, for our dynamical system, where
$$P_{ij} = \frac{m(B_i \cap f^{-1}(B_j))}{m(B_i)},$$
the *transition probability* from B_i to B_j using, e.g., $f = \phi_t^{t+T}$

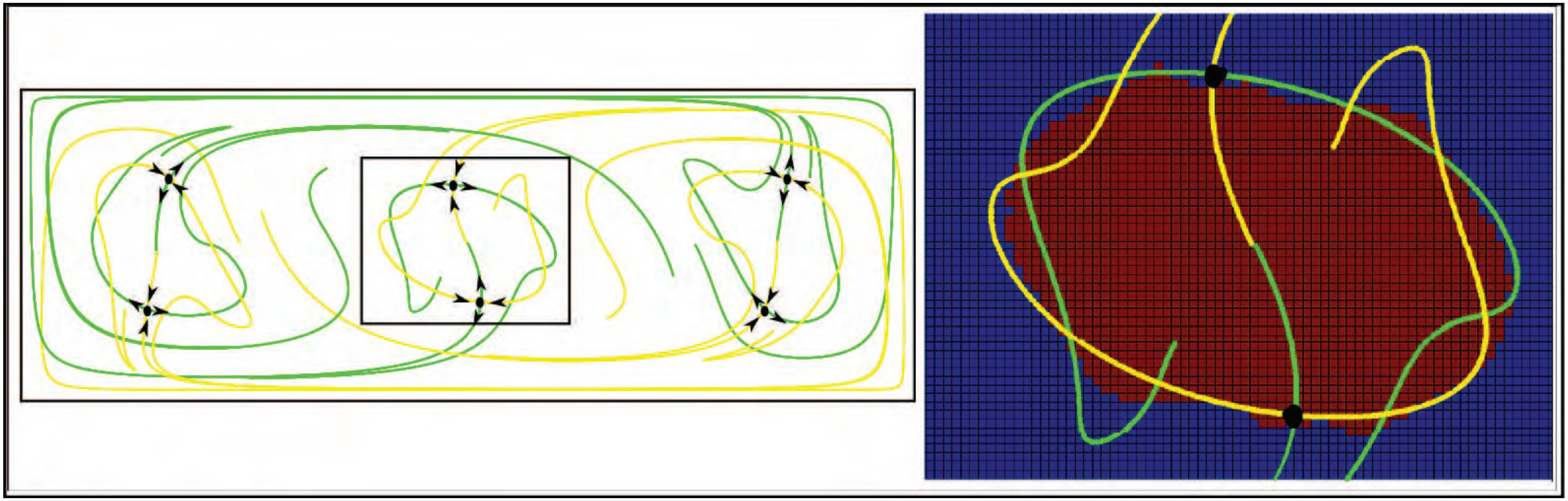


- P approximates our dynamical system via a finite state Markov chain.

Almost-cyclic set approach

- A set B is called almost invariant over the interval $[t, t+T]$ if
$$\rho(B) = \frac{m(B \cap \phi^{-1}(B))}{m(B)} \approx 1.$$
- Can maximize value of ρ over all possible combinations of sets $B \in \mathcal{B}$.
- In practice, AISs or relatedly, almost-cyclic sets (ACSs), identified via **eigenvectors** (of eigenvalues with $|\lambda| \approx 1$) of P or graph-partitioning

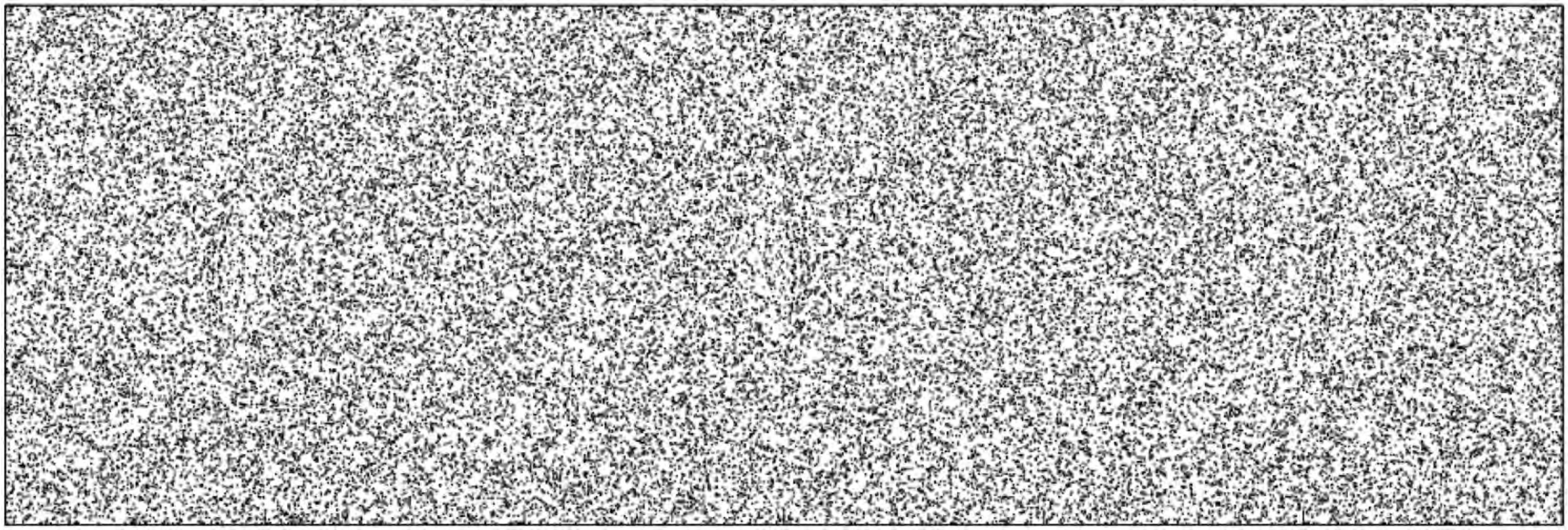
Identifying 'ghost rods': almost-cyclic sets



- Return to $\tau_f > 1$ case, where periodic points and manifolds exist
- Agreement between AIS boundaries and manifolds of periodic points
- Known previously[2] and applies to more general objects than periodic points, i.e. normally hyperbolic invariant manifolds (NHIMs)

[2] Dellnitz, Junge, Lo, Marsden, Padberg, Preis, Ross, Thiere [2005] Phys. Rev. Lett.; Dellnitz, Junge, Koon, Lekien, Lo, Marsden, Padberg, Preis, Ross, Thiere [2005] Int. J. Bif. Chaos

Identifying 'ghost rods': almost-cyclic sets



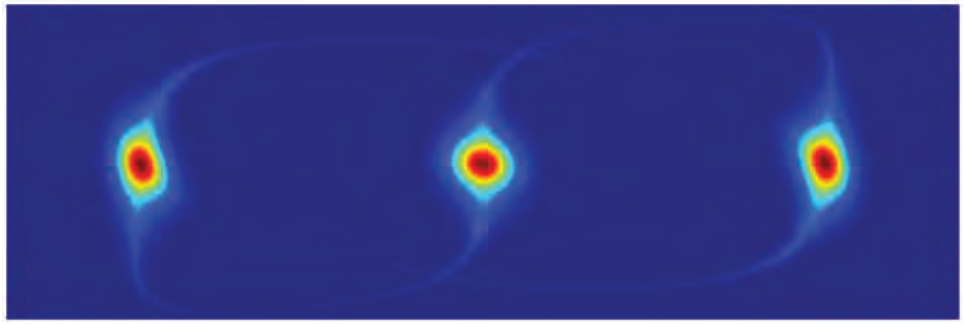
Poincaré section for $\tau_f < 1 \Rightarrow$ no obvious structure!

- Return to $\tau_f < 1$ case, where no periodic orbits of low period known
- Is the phase space featureless?

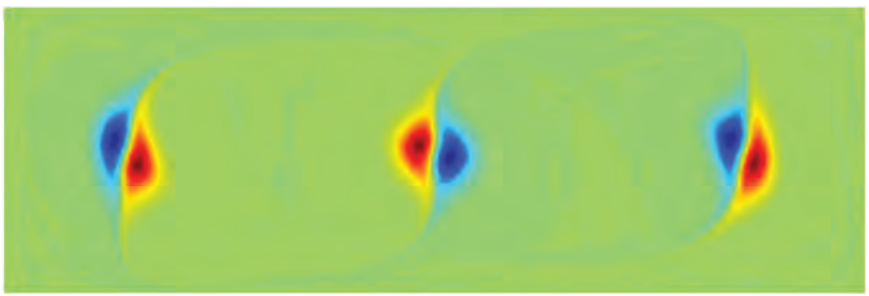
- Consider transition matrix $P_t^{t+\tau_f}$ induced by Poincaré map $\phi_t^{t+\tau_f}$

Identifying 'ghost rods': almost-cyclic sets

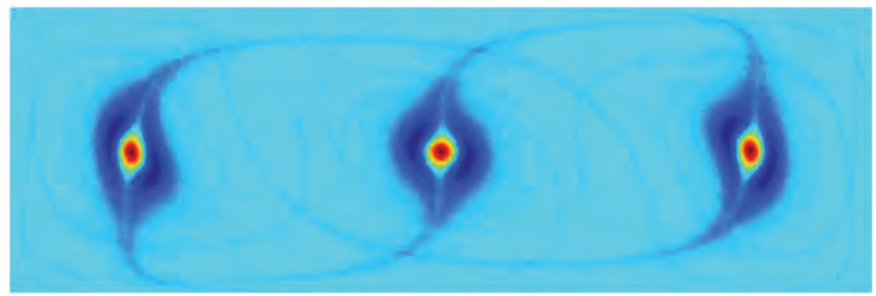
Top eigenvectors for $\tau_f = 0.99$ reveal hierarchy of phase space structures



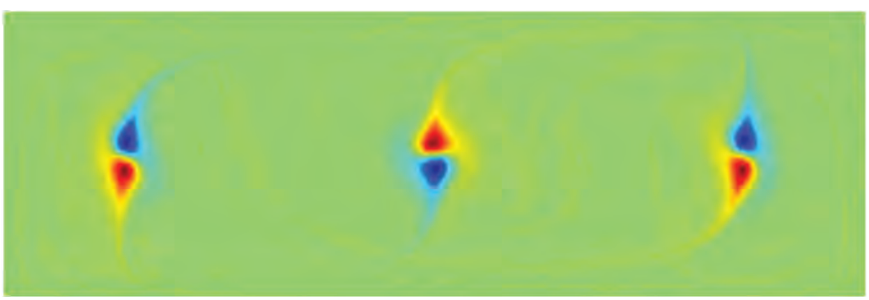
ν_2



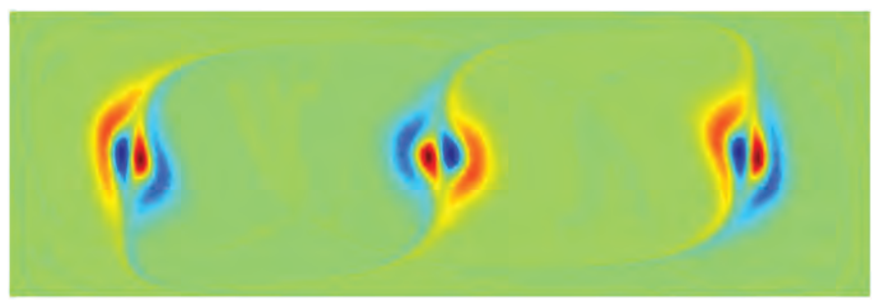
ν_3



ν_4

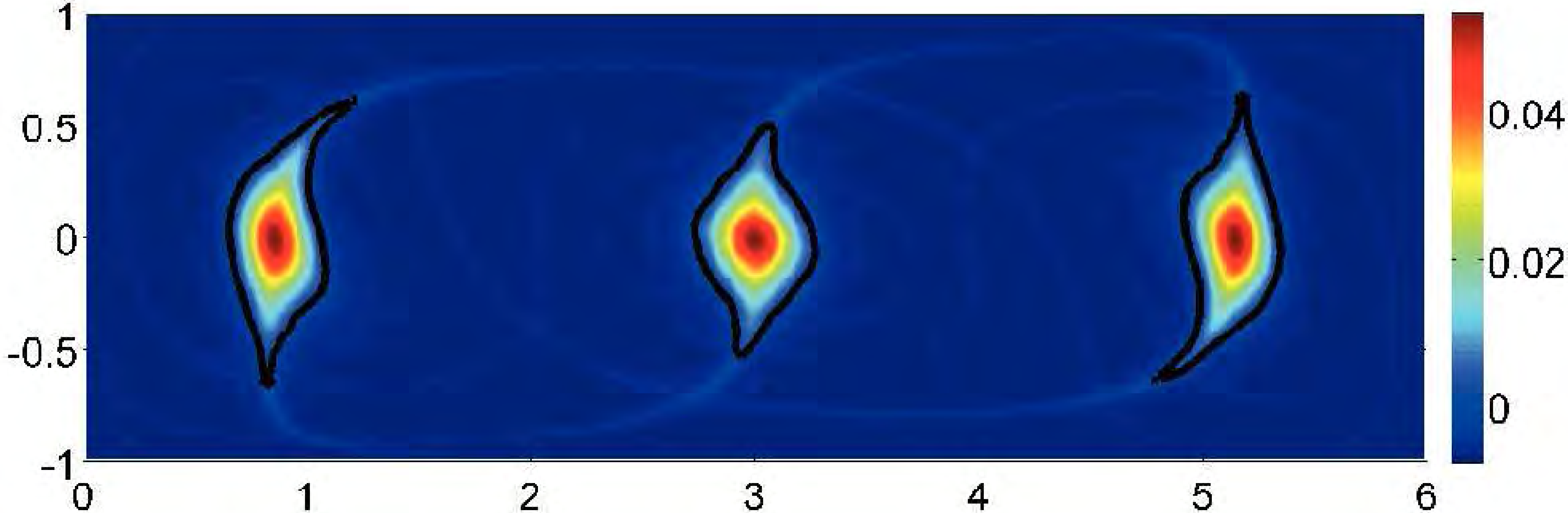


ν_5



ν_6

Identifying 'ghost rods': almost-cyclic sets



The zero contour (black) is the boundary between the two almost-invariant sets.

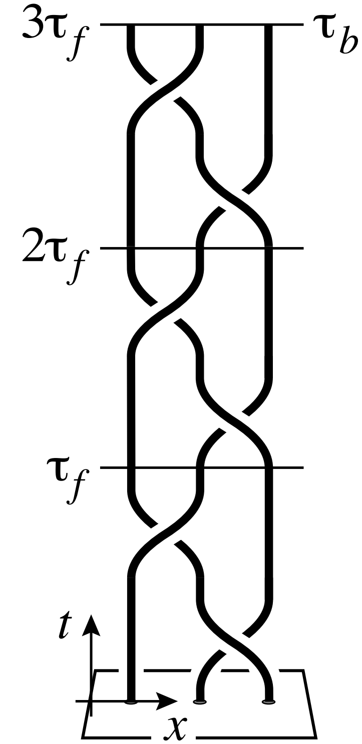
- Three-component AIS made of 3 almost-cyclic sets (ACSs) of period 3
- ACS effectively replace compact region bounded by saddle manifolds
- Also: we see a **dynamical remnant of the global 'stable and unstable manifolds' of the saddle points**, even there are no more saddle points

Identifying ‘ghost rods’: almost-cyclic sets

Almost-cyclic sets stirring the surrounding fluid like ‘ghost rods’
— **works even when periodic orbits are absent!**

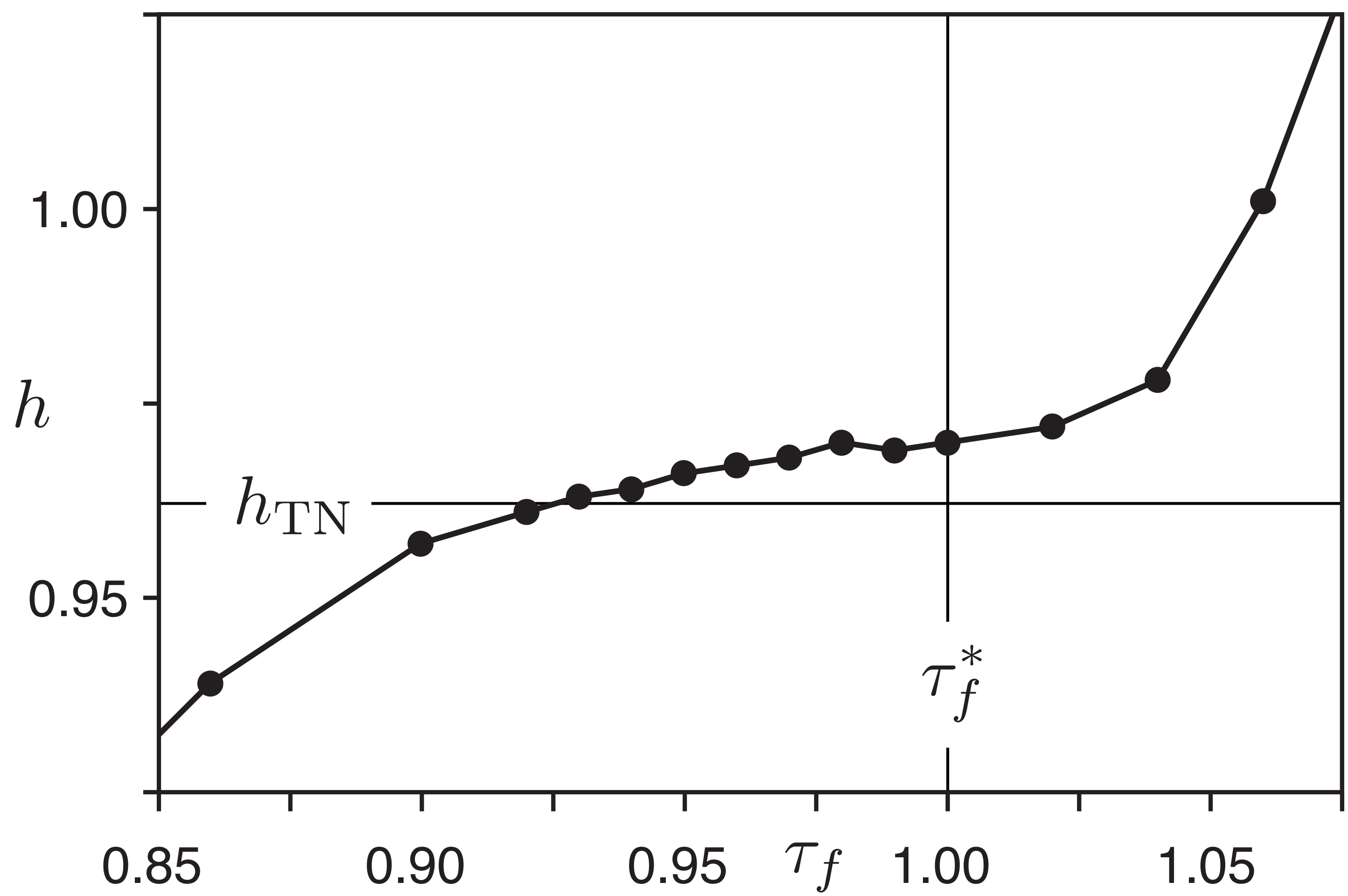
Movie shown is second eigenvector for $P_t^{t+\tau_f}$ for $t \in [0, \tau_f)$

Identifying ‘ghost rods’: almost-cyclic sets



Braid of ACSs gives lower bound of entropy via Thurston-Nielsen
— One only needs approximately cyclic blobs of fluid
— Even though the theorems require exactly periodic points!
— Stremler, Ross, Grover, Kumar [2011] Phys. Rev. Lett.

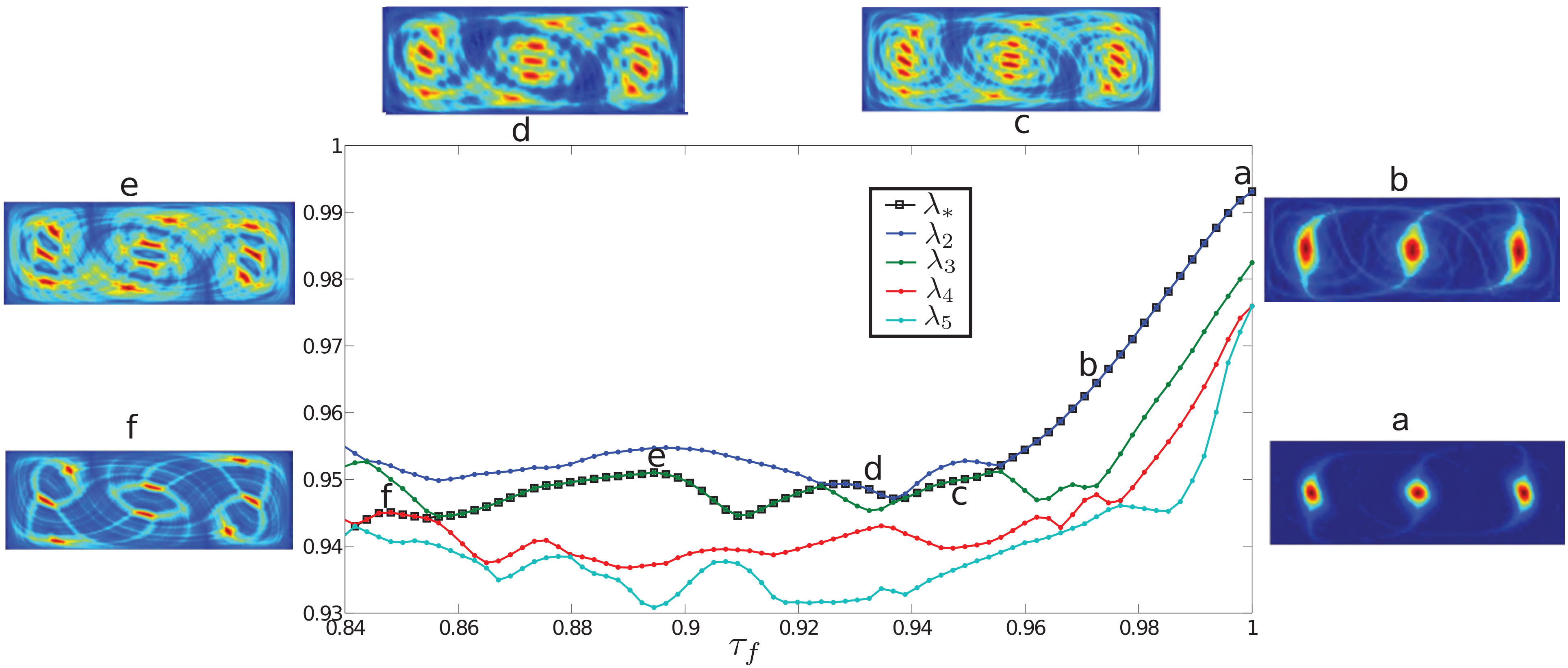
Topological entropy vs. bifurcation parameter



topological entropy as a function of τ_f

- $h_{\rm TN}$ shown for ACS braid on 3 strands

Eigenvalues/eigenvectors vs. bifurcation parameter



Movie shows change in eigenvector along branch marked with '–□–' above (from a to f), as τ_f decreases $\Rightarrow$

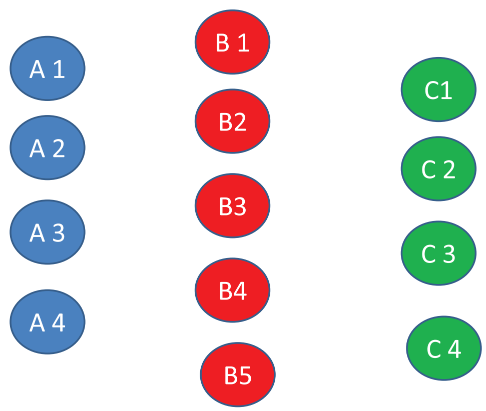
Bifurcation of ACSs

For example, braid on 13 strands for $\tau_f = 0.92$

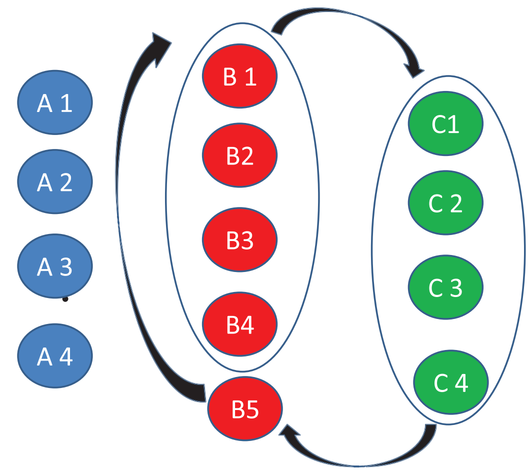
Movie shown is second eigenvector for $P_t^{t+\tau_f}$ for $t \in [0, \tau_f)$

Thurson-Nielsen for this braid provides lower bound on topological entropy

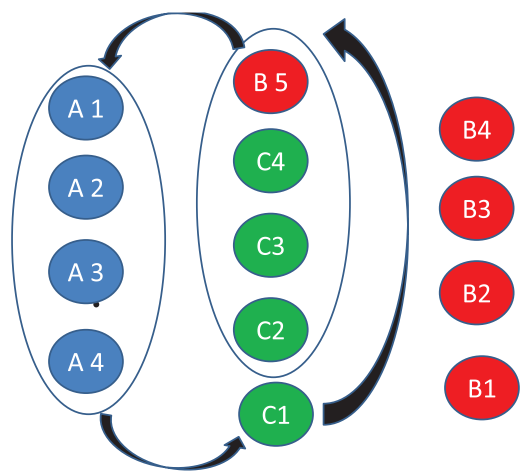
Bifurcation of ACSs



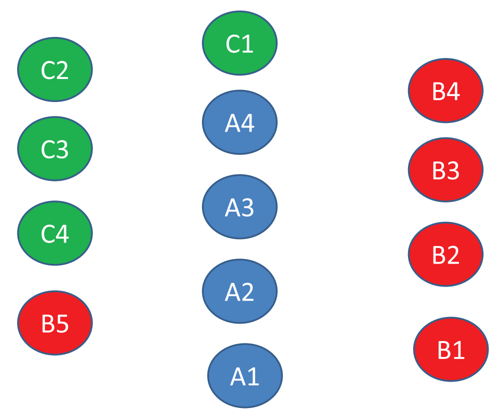
(a) Initial state



(b) First half-period

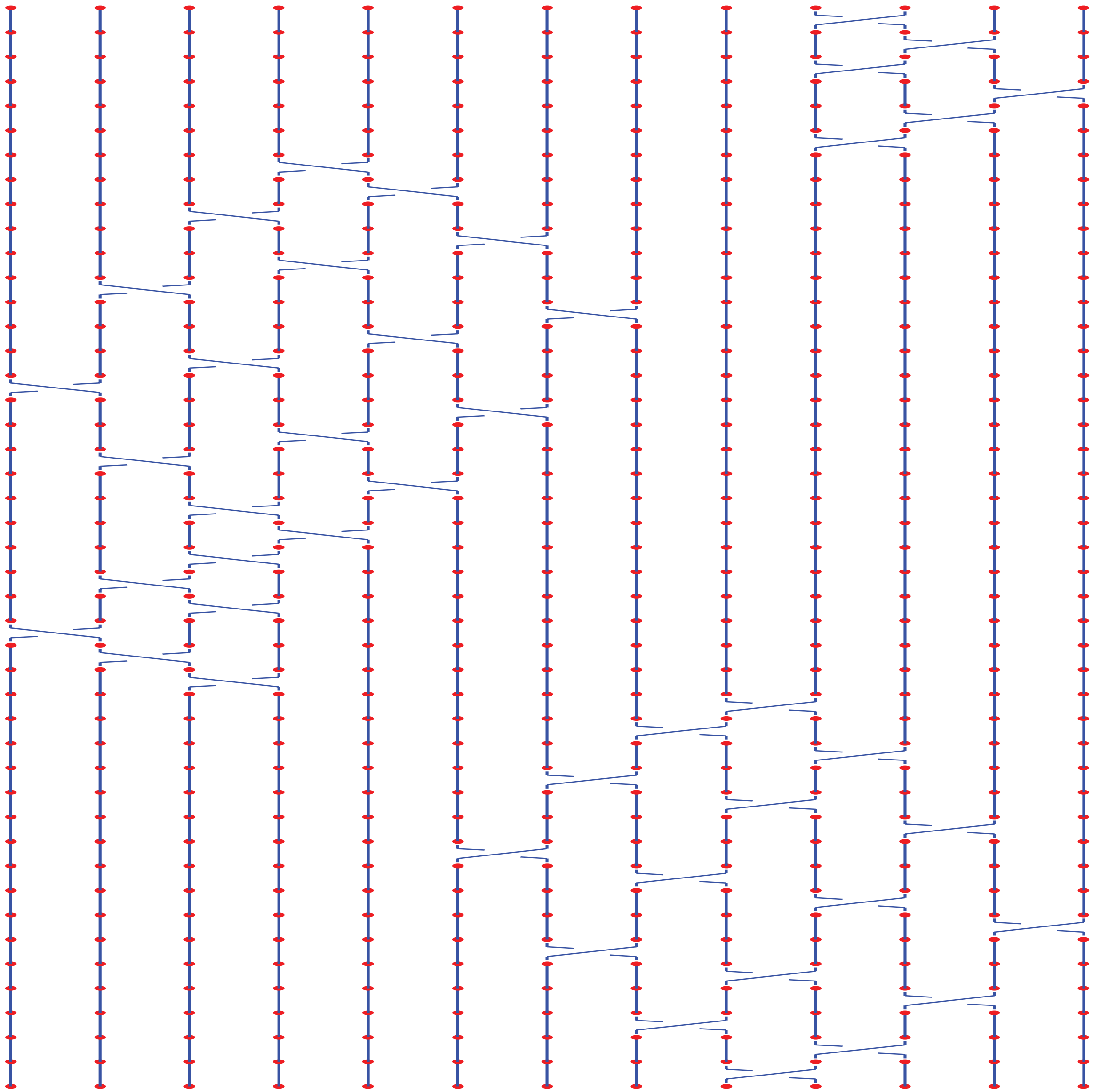


(c) Second half-period



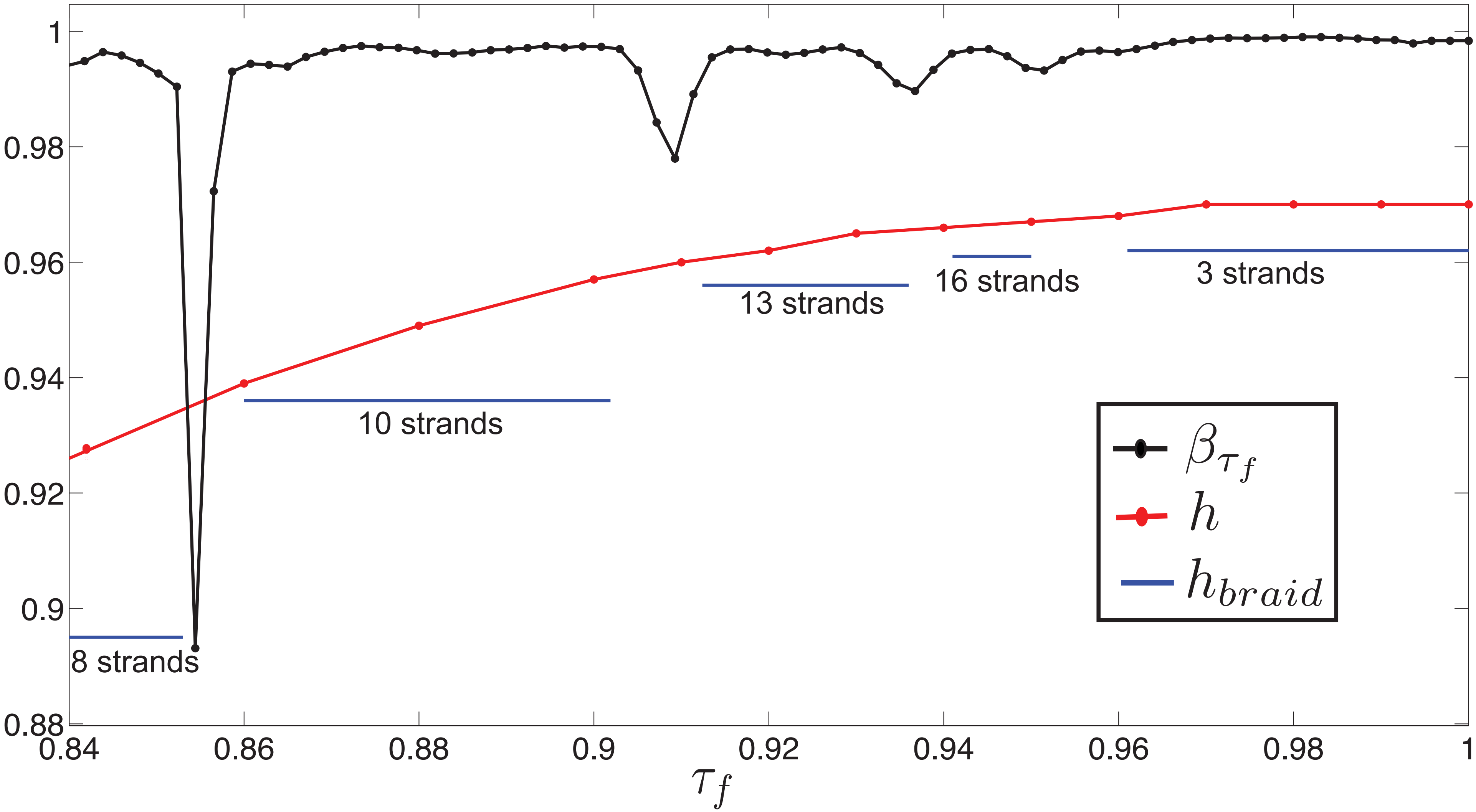
(d) State after 1 period

Bifurcation of ACSs



representation of braid

Sequence of ACS braids bounds entropy

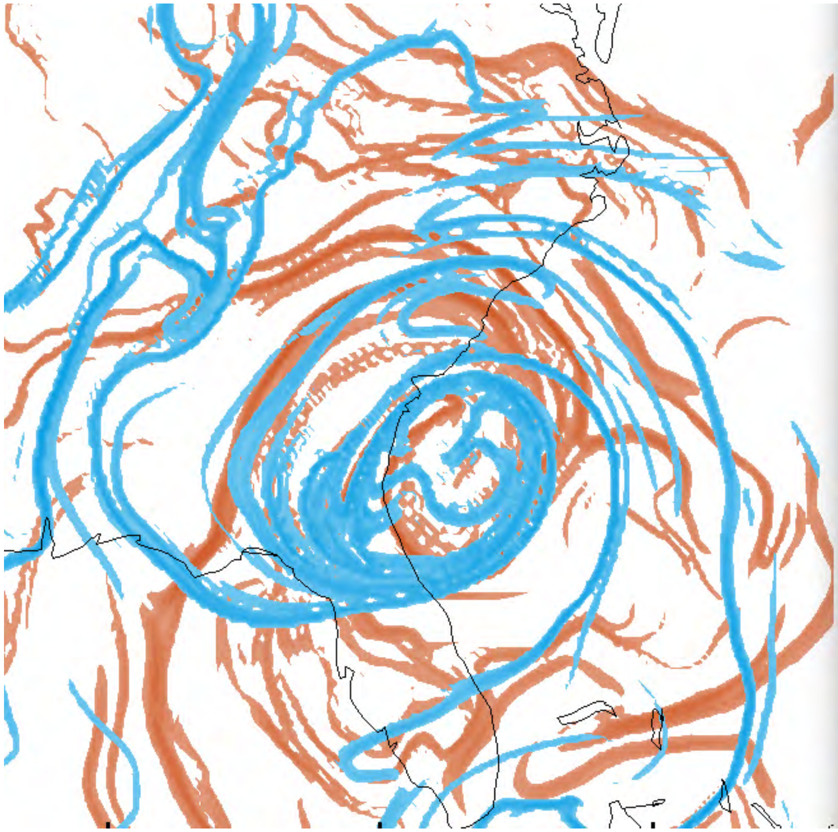


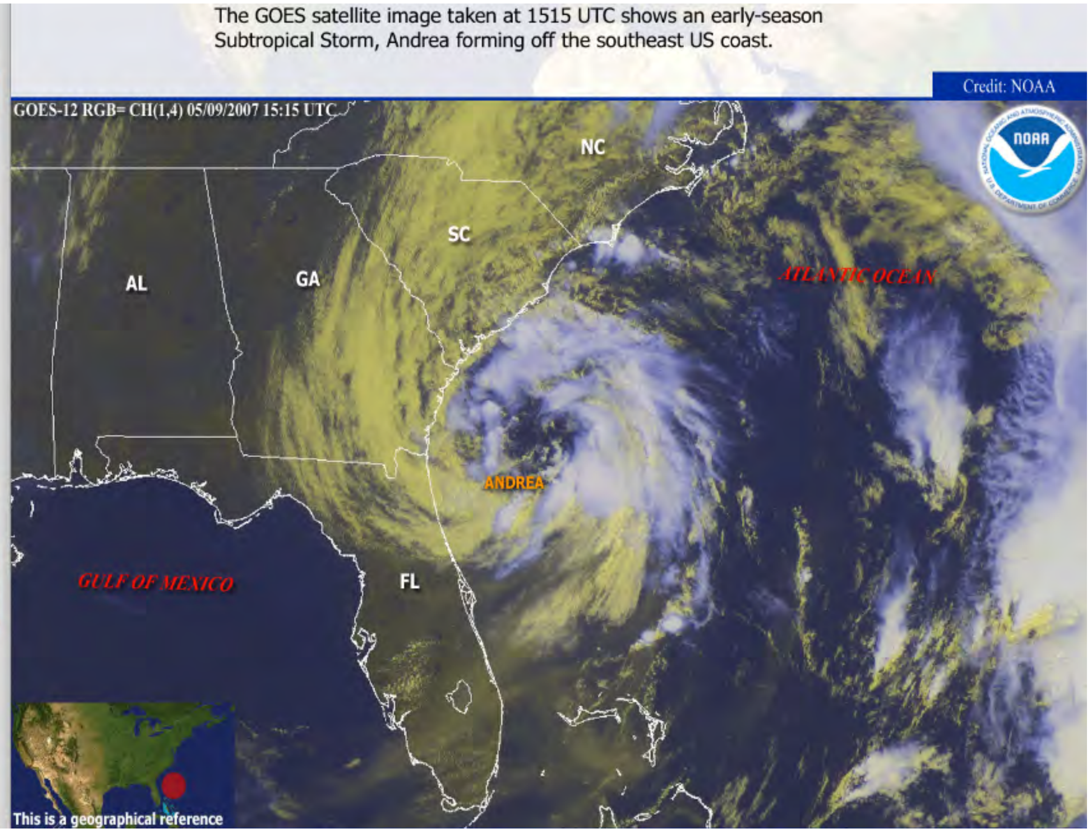
For various braids of ACSs, the calculated entropy is given, bounding from below the true topological entropy over the range where the braid exists

Aperiodic, finite-time setting

- Data-driven, finite-time, aperiodic setting
- How do we get at transport?
- Are there, e.g., braids in realistic fluid flows?

Atmospheric flows





orange = repelling LCSs, blue = attracting LCSs satellite

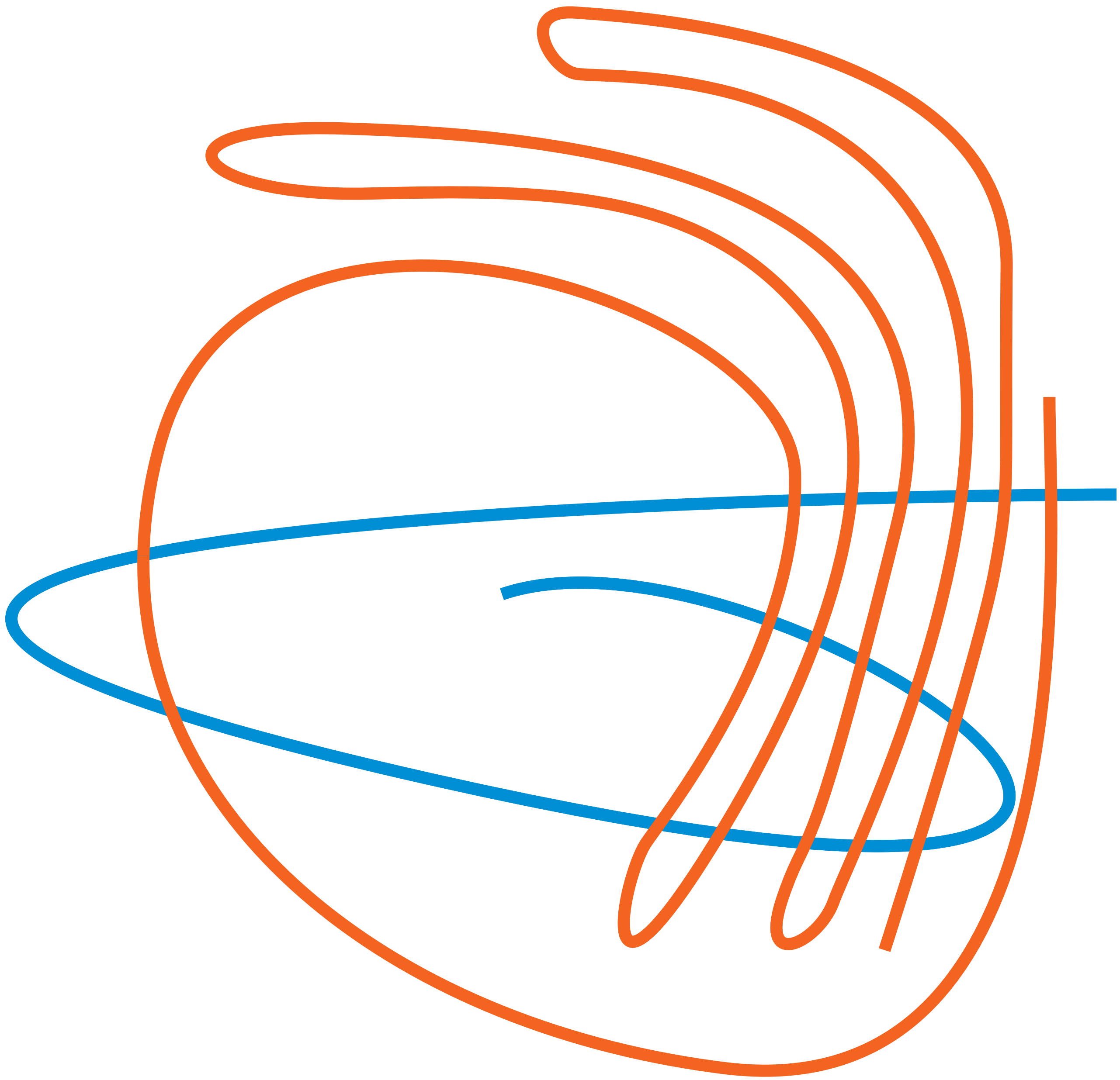
Andrea, first storm of 2007 hurricane season

cf. Sapsis & Haller [2009], Du Toit & Marsden [2010], Lekien & Ross [2010], Tallapragada & Ross [2011]

Atmospheric flows

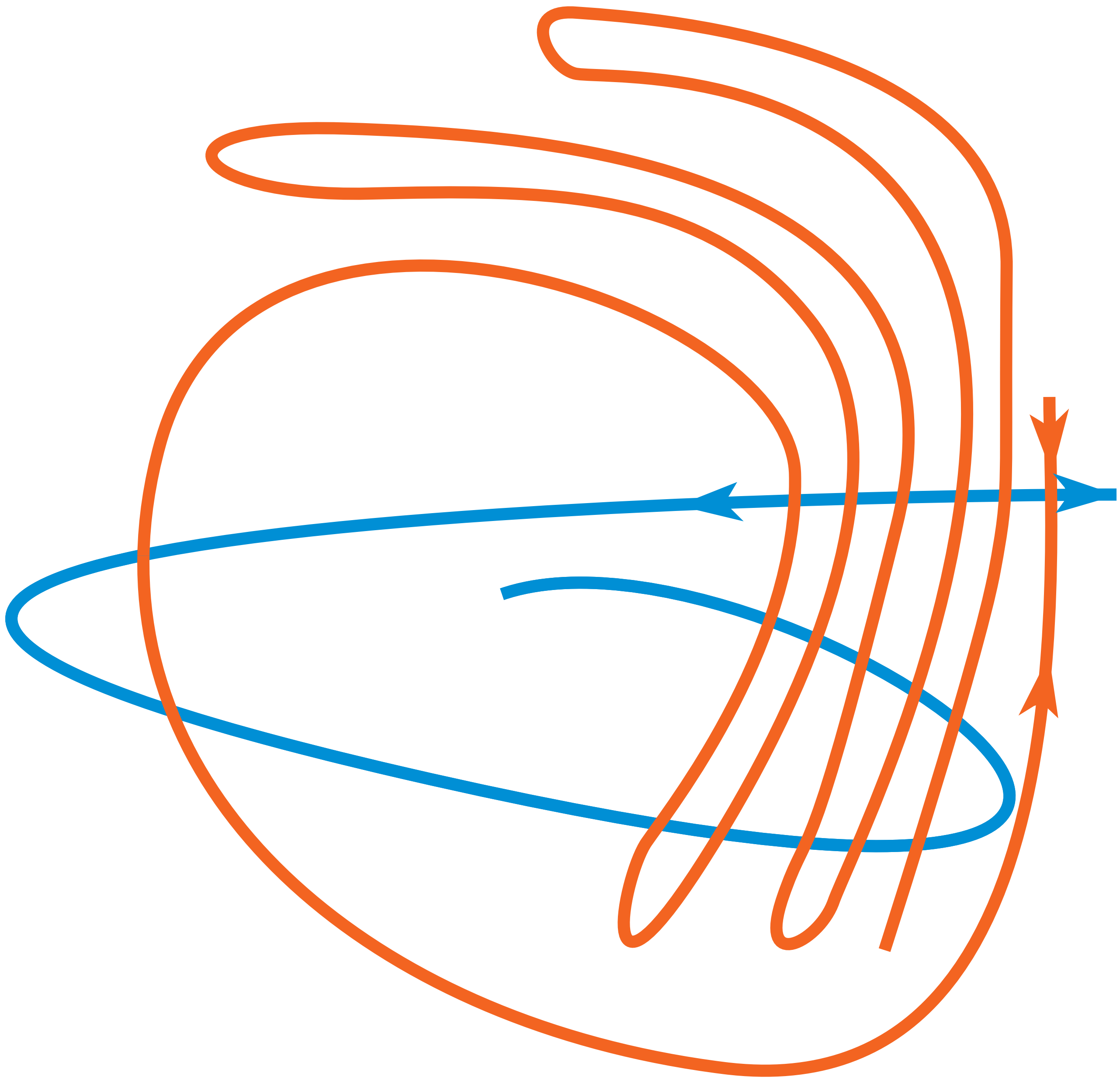
Andrea at one snapshot; LCS shown (orange = repelling, blue = attracting)

Atmospheric flows: lobe dynamics to find braids



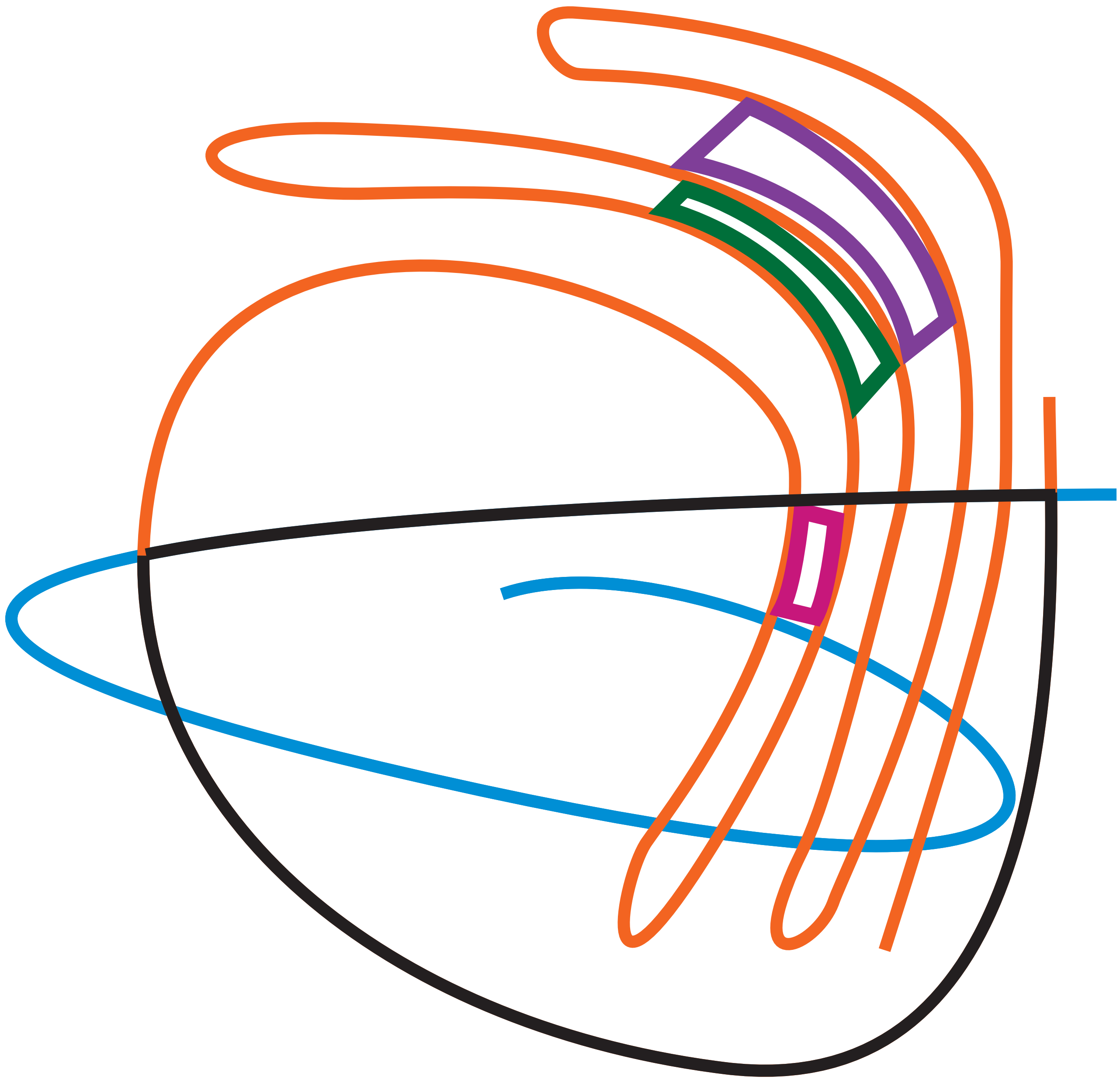
orange = repelling (stable manifold), blue = attracting (unstable manifold)

Atmospheric flows: lobe dynamics to find braids



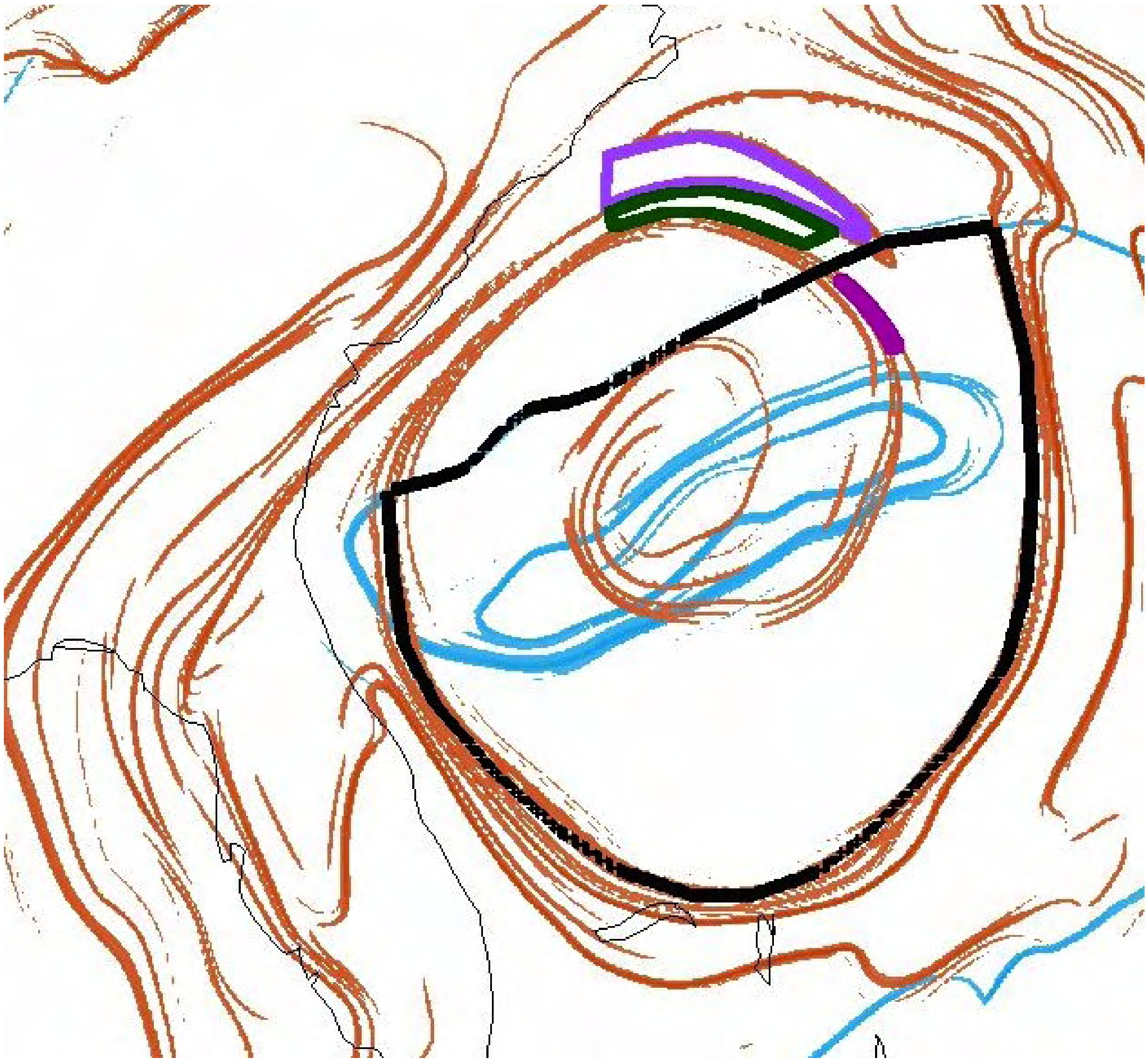
orange = repelling (stable manifold), blue = attracting (unstable manifold)

Atmospheric flows: lobe dynamics to find braids



Portions of lobes colored; magenta = outgoing, green = incoming, purple = stays out

Atmospheric flows: lobe dynamics to find braids

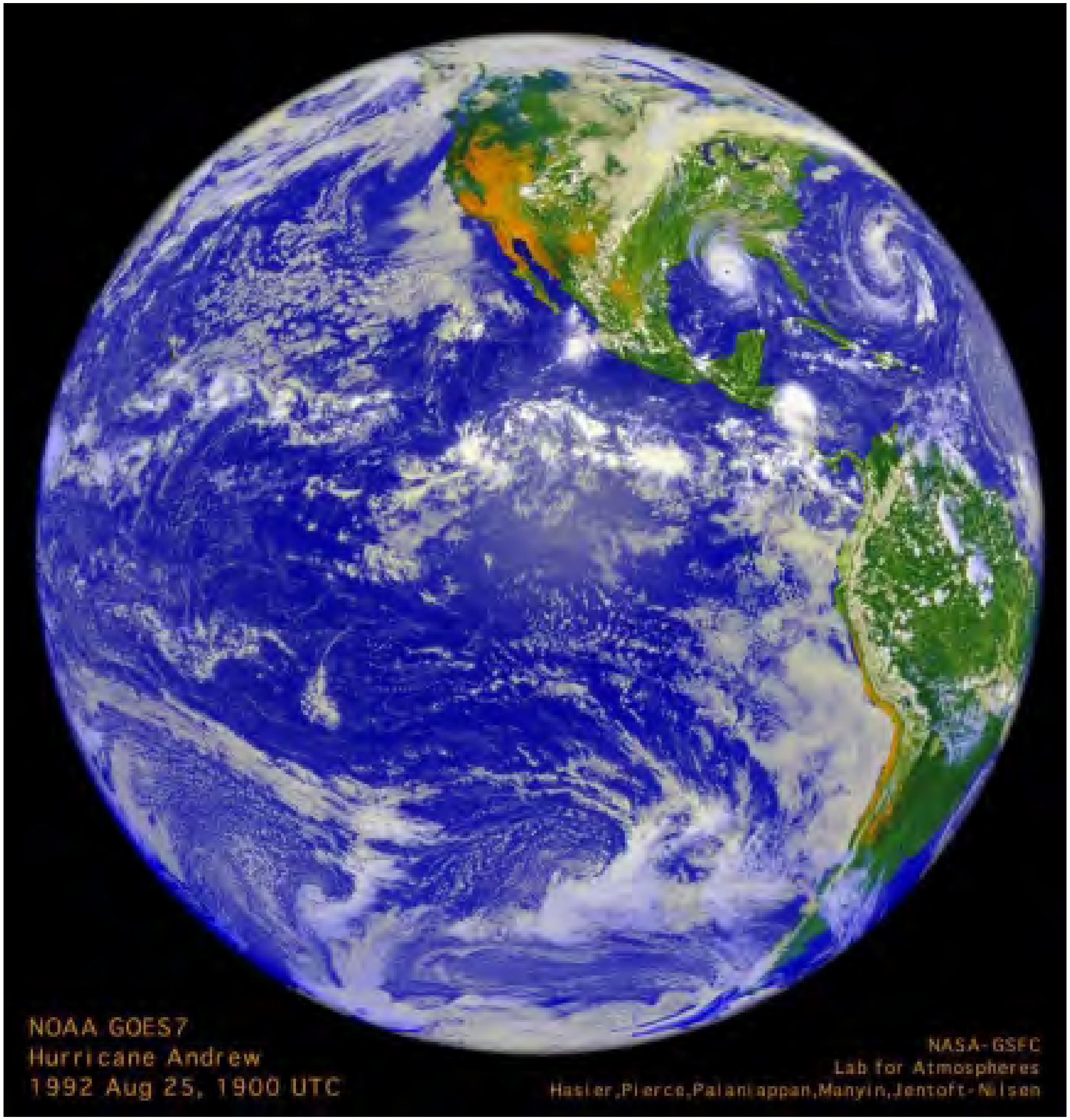


Portions of lobes colored; magenta = outgoing, green = incoming, purple = stays out

Atmospheric flows: lobe dynamics to find braids

Sets behave as lobe dynamics dictates $\Rightarrow$ form braid, but no periodicity

Can you find the ghost rods stirring the Earth?



Final words on stirring by braiding of coherent sets

- For engineering systems, it makes sense to design for mixing with ghost rods in mind
- For natural systems, ghost rod paradigm may aid interpretation
- What qualifies as a ghost rod?
  - We can apply Thurston-Nielsen classification theorem not only to solid rods and periodic orbits, but also almost-cyclic coherent regions
  - Where might this break down?
- Probabilistic approach provides extension of useful dynamical systems notions
  - periodic points $\Rightarrow$ almost-cyclic sets
  - their 'stable/unstable invariant manifolds' $\Rightarrow$???

The End

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Main Papers:

- Stremler, Ross, Grover, Kumar [2011] Topological chaos and periodic braiding of almost-cyclic sets. *Physical Review Letters* 106, 114101.
- Tallapragada & Ross [2011] A geometric and probabilistic description of coherent sets. Preprint.
- Lekien & Ross [2010] The computation of finite-time Lyapunov exponents on unstructured meshes and for non-Euclidean manifolds. *Chaos* 20, 017505.
- Senatore & Ross [2011] Detection and characterization of transport barriers in complex flows via ridge extraction of the finite time Lyapunov exponent field, *International Journal for Numerical Methods in Engineering* 86, 1163.
- Grover, Ross, Stremler, Kumar [2011] Topological chaos, braiding and breakup of almost-invariant sets. Preprint.
- Tallapragada & Ross [2008] Particle segregation by Stokes number for small neutrally buoyant spheres in a fluid, *Physical Review E* 78, 036308.