Geometry of phase space transport in dynamical systems

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MultiSTEPS: MultiScale Transport in Environmental & Physiological Systems, www.multisteps.esm.vt.edu

The tale of a confused comet

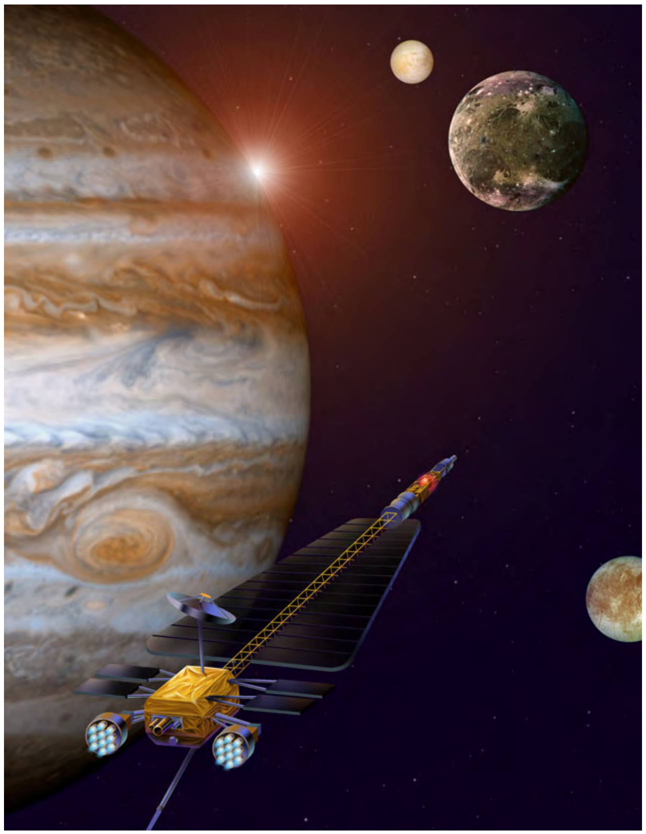
- comet Oterma from 1910 to 1980
- Rapid transition: outside to inside Jupiter's orbit; temporarily captured.

The tale of a confused comet

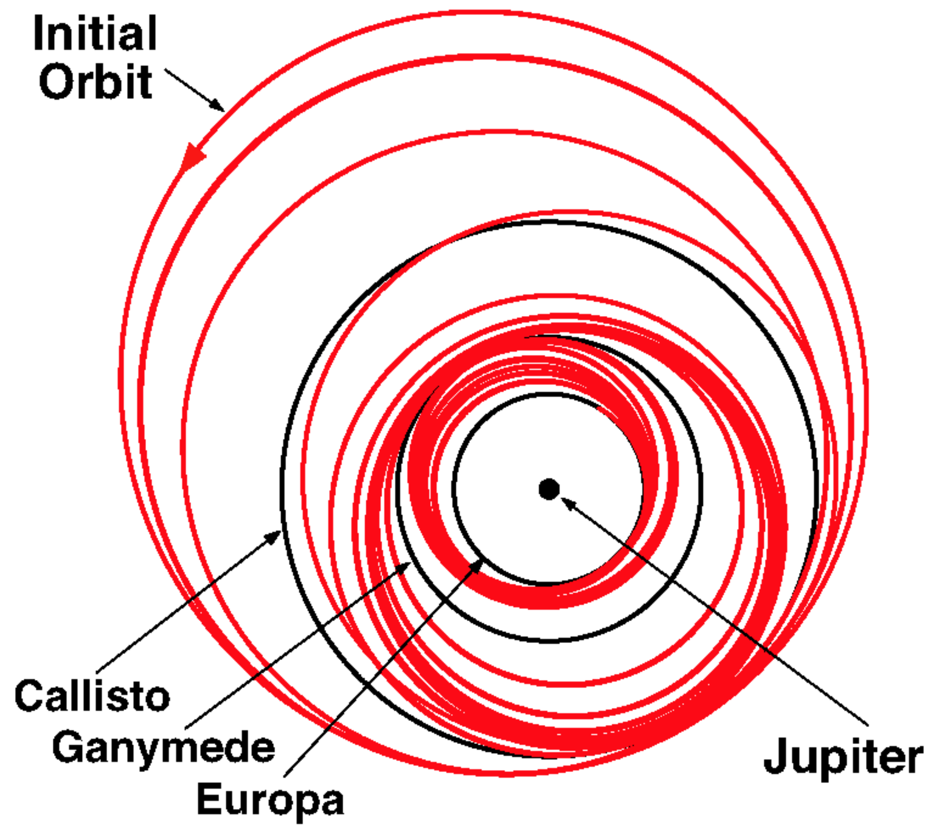
- Oterma's orbit in rotating frame
  with special nearby orbits (green)

Natural Pathways for Fuel Efficiency

Natural Pathways for Fuel Efficiency



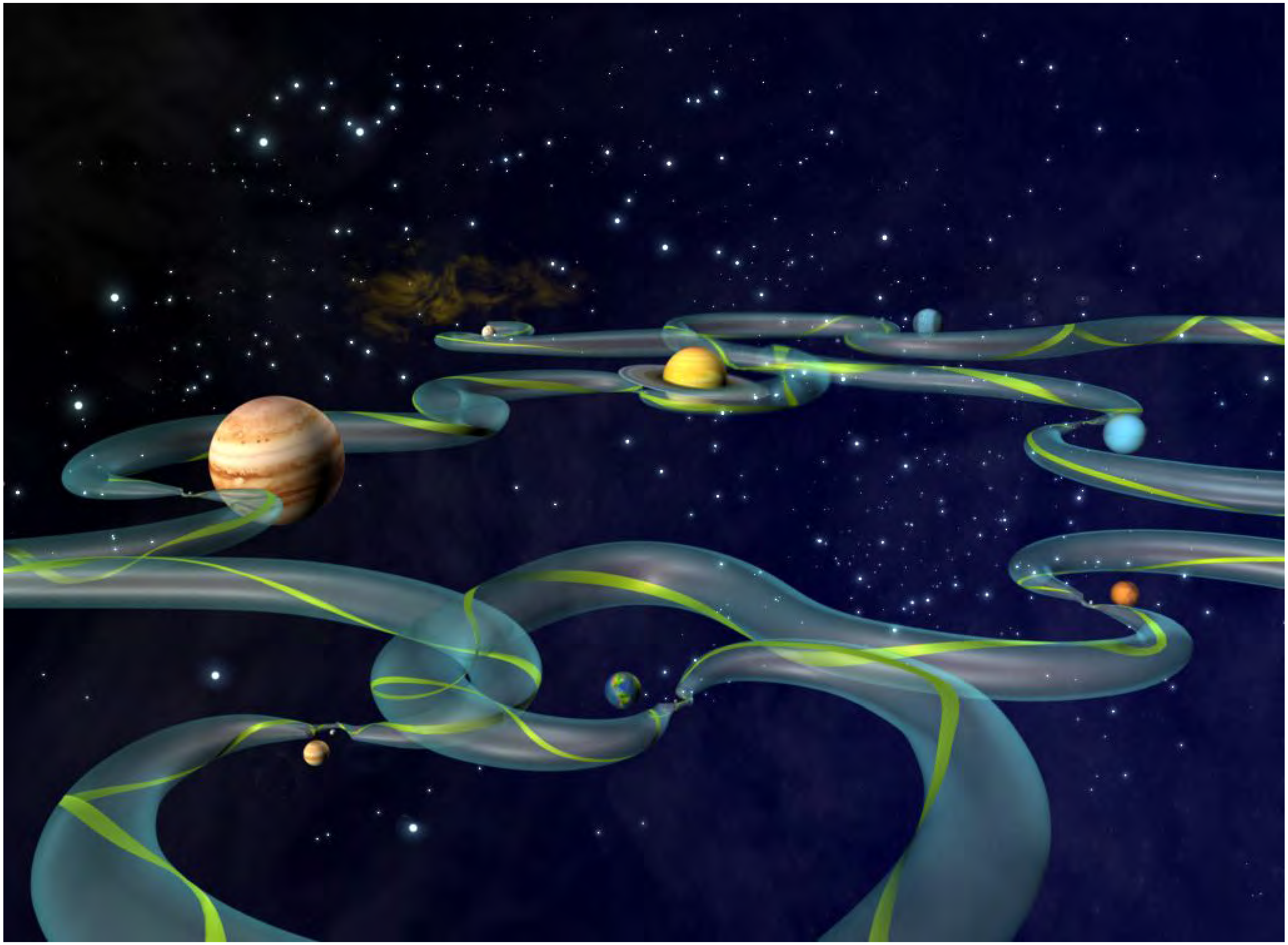
Orbiting Jupiter's moons



zero fuel trajectory

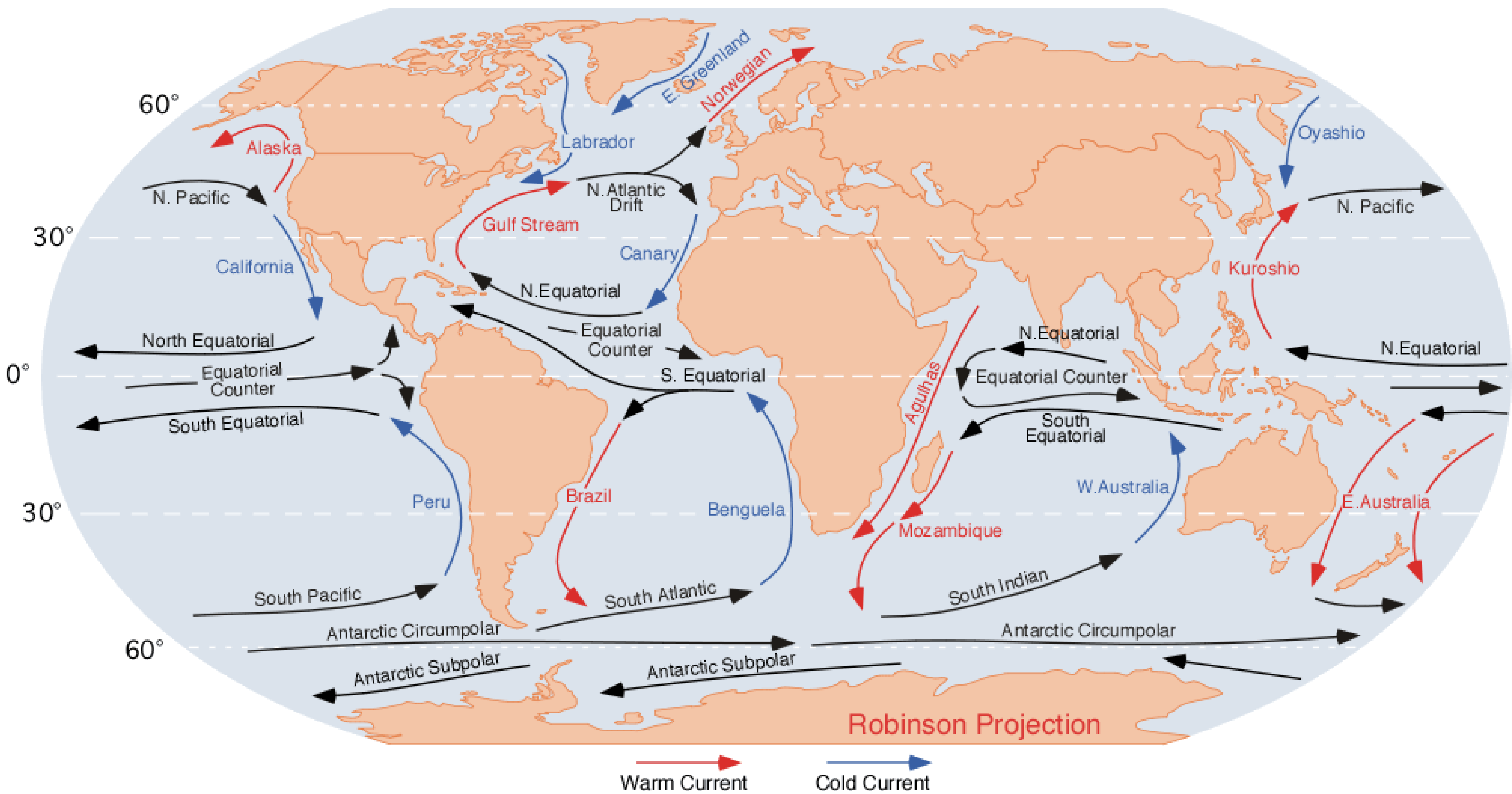
- **Fuel-efficient** tours of Jupiter's moons

Interplanetary transport network



Natural pathways winding through the solar system

Oceanic transport network



Ocean currents: natural pathways on Earth

Atmospheric transport network

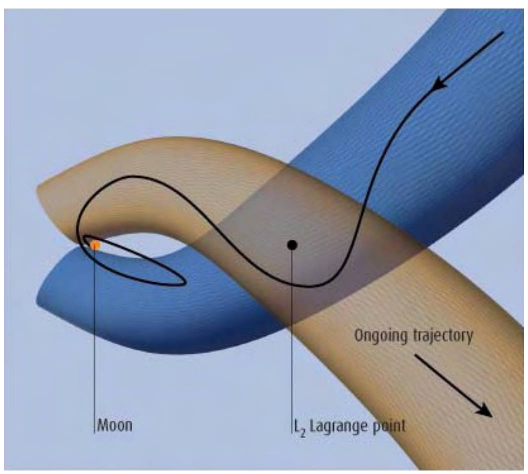
Atmospheric transport network

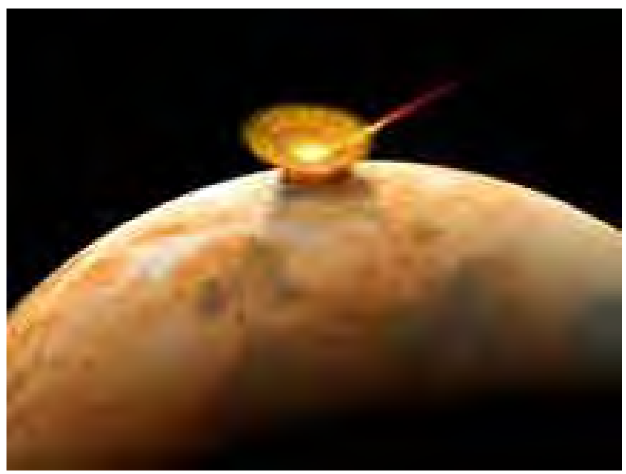
Transport networks: overview

- Main objective: geometric description of transport
  - insight into phase space mixing and regions of further interest
  - efficient control schemes
- Motivating principle: structures guiding transport
  — especially systems with symmetry, e.g., Hamiltonian
- celestial mechanics example
- geophysical flow example

Interplanetary transport: main ideas

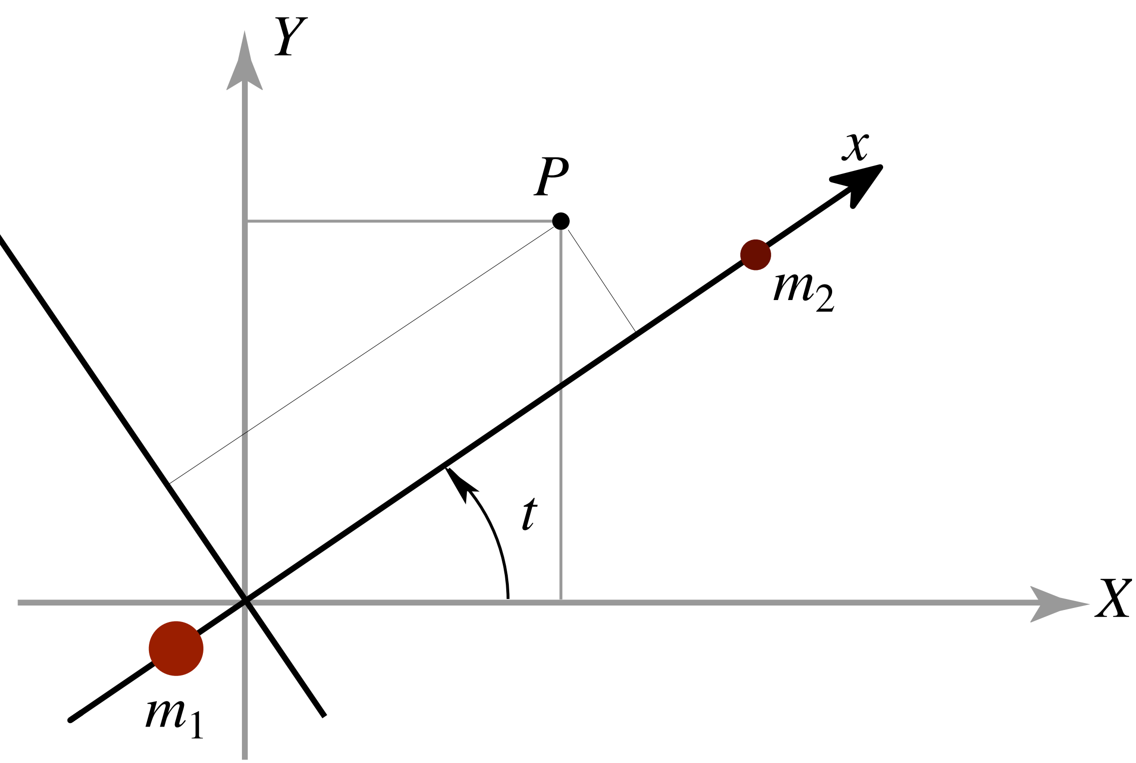
- Break N-body problem into several 3-body problems
- Invariant manifolds of unstable bound orbits act as **separatrices** (codimension 1 surfaces)
- Determine **transport**, e.g., collisions, transitions





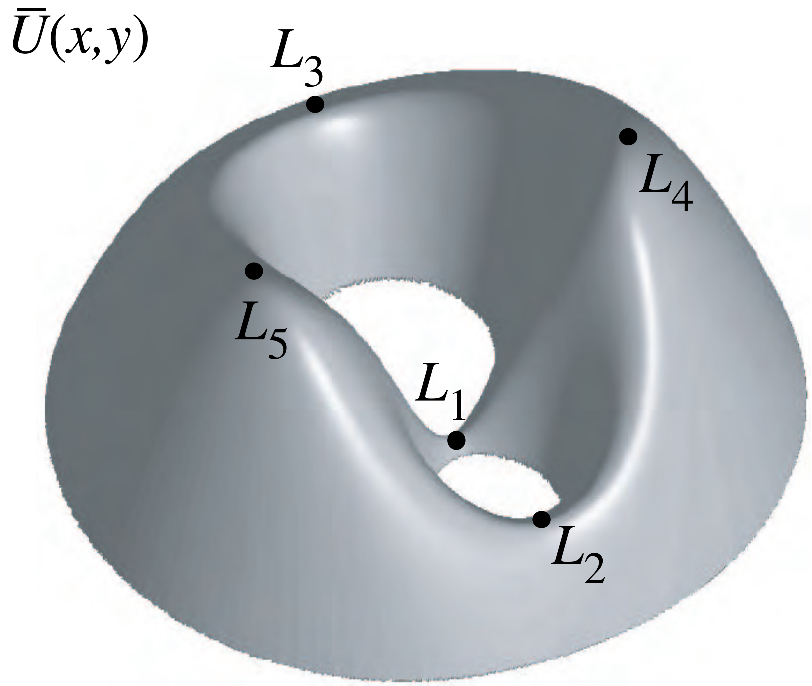
3-Body Problem

- ***Restricted 3-body approximation***
  - P in field of two massive bodies, m_1 and m_2
  - x-y frame rotates w.r.t. X-Y inertial frame



3-Body Problem

- Equations of motion in **rotating frame** describe P moving in effective potential plus a coriolis force (goes back to work of Jacobi, Hill, etc)



Effective Potential

Hamiltonian system

- Hamiltonian function (2 d.o.f.) — time-independent

$$H(x, y, p_x, p_y) = \frac{1}{2}((p_x + y)^2 + (p_y - x)^2) + \bar{U}(x, y),$$

where p_x and p_y are the conjugate momenta, and

$$\bar{U}(x, y) = -\frac{1}{2}(x^2 + y^2) - \frac{1-\mu}{r_1} - \frac{\mu}{r_2}$$

where r_1 & r_2 are the distances of P from m_1 & m_2 and

$$\mu = \frac{m_2}{m_1 + m_2} \in (0, 0.5]$$

- For systems of interest, $\mu \approx 10^{-6}$-10^{-2}

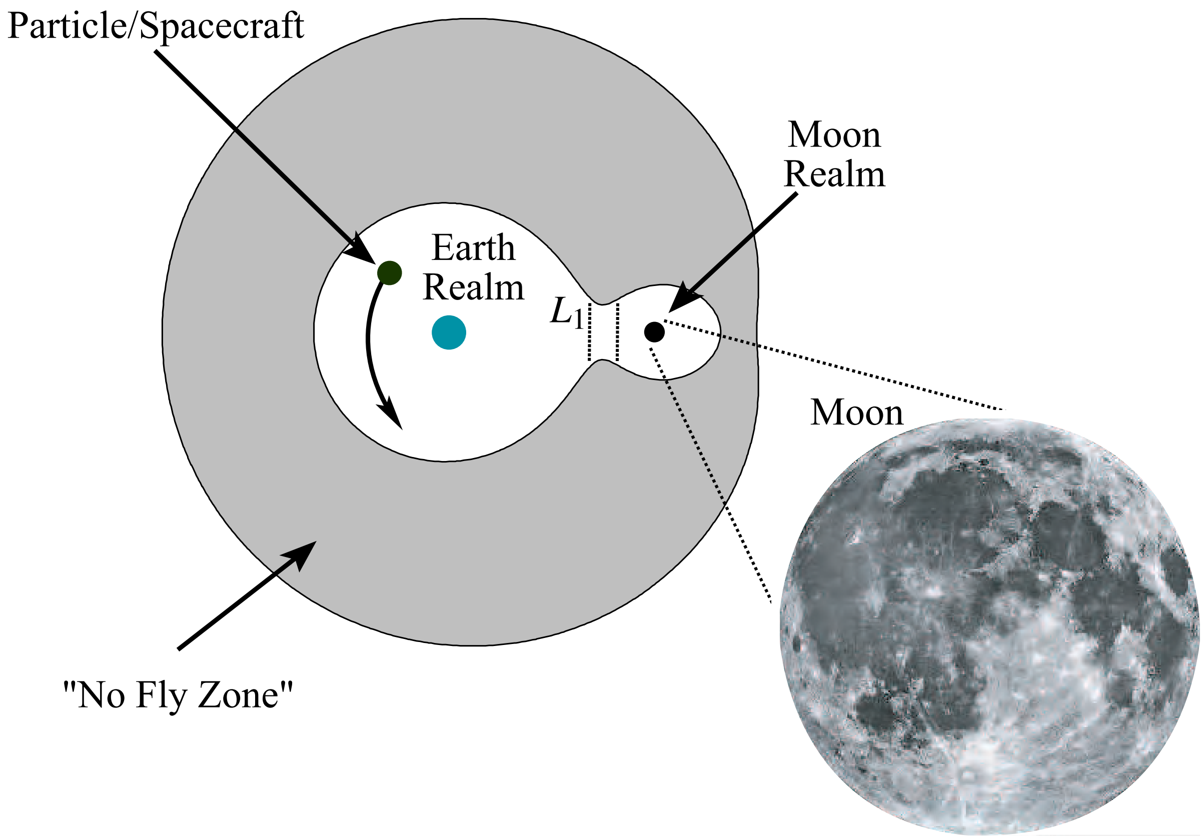
Motion in energy surface

- **Energy surface** of energy E is codim-1 surface

$$\mathcal{M}(E) = \{(\mathrm{q}, \mathrm{p}) \mid H(\mathrm{q}, \mathrm{p}) = E\}.$$

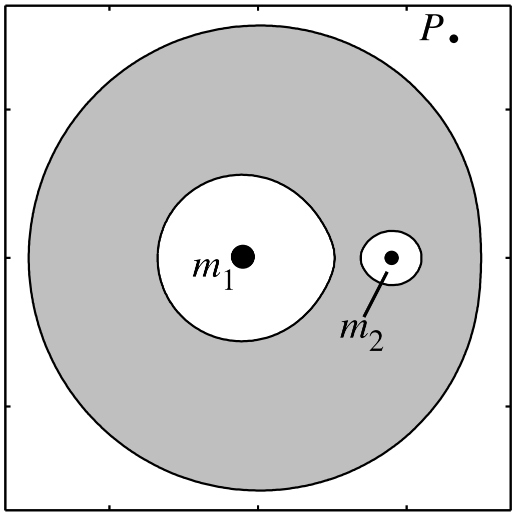
- In 2 d.o.f., 3D surfaces foliating the 4D phase space (in 3 d.o.f., 5D energy surfaces)

Realms of possible motion

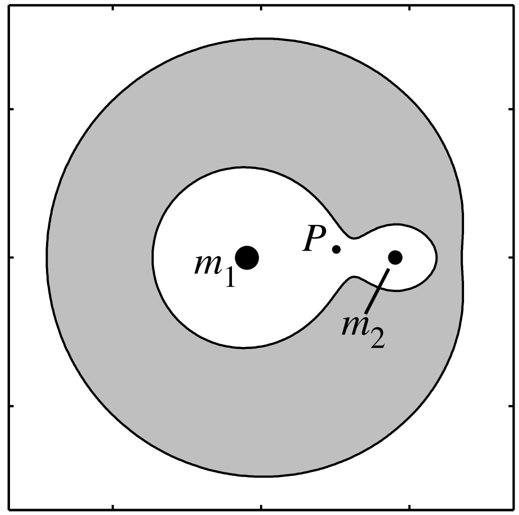


- $\mathcal{M}_\mu(E)$ partitioned into three **realms**
  e.g., Earth realm = phase space around Earth
- Energy E determines their connectivity

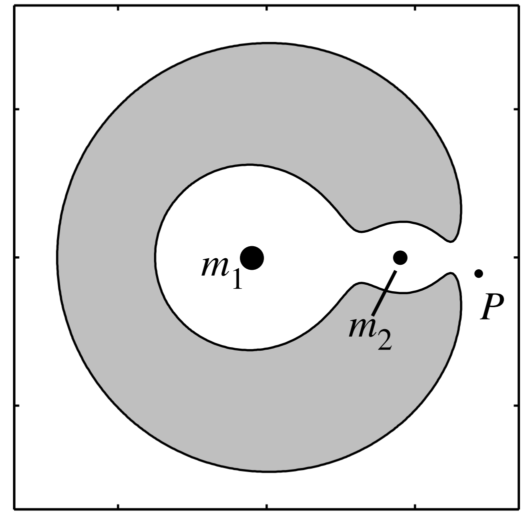
Realms of possible motion



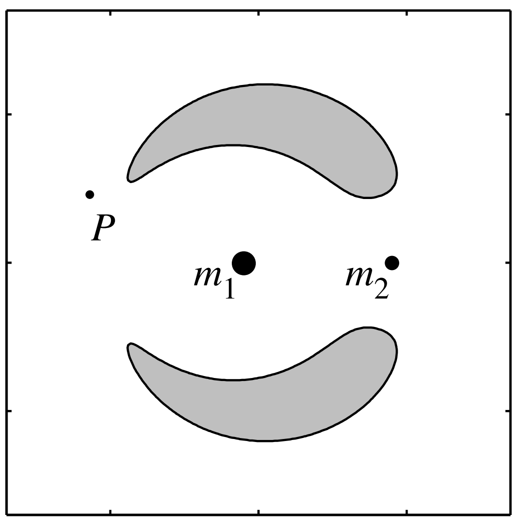
Case 1 : $E<E_1$



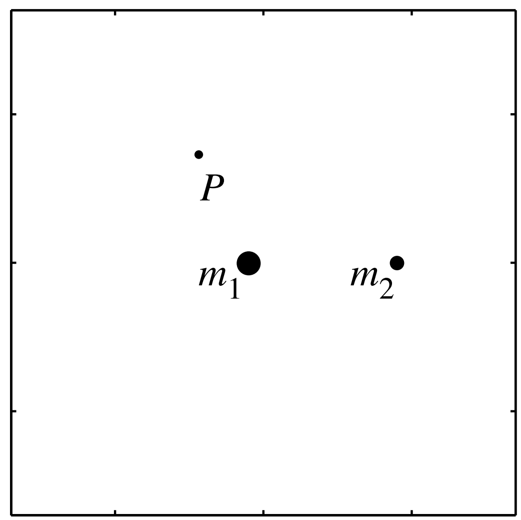
Case 2 : $E_1<E<E_2$



Case 3 : $E_2<E<E_3$



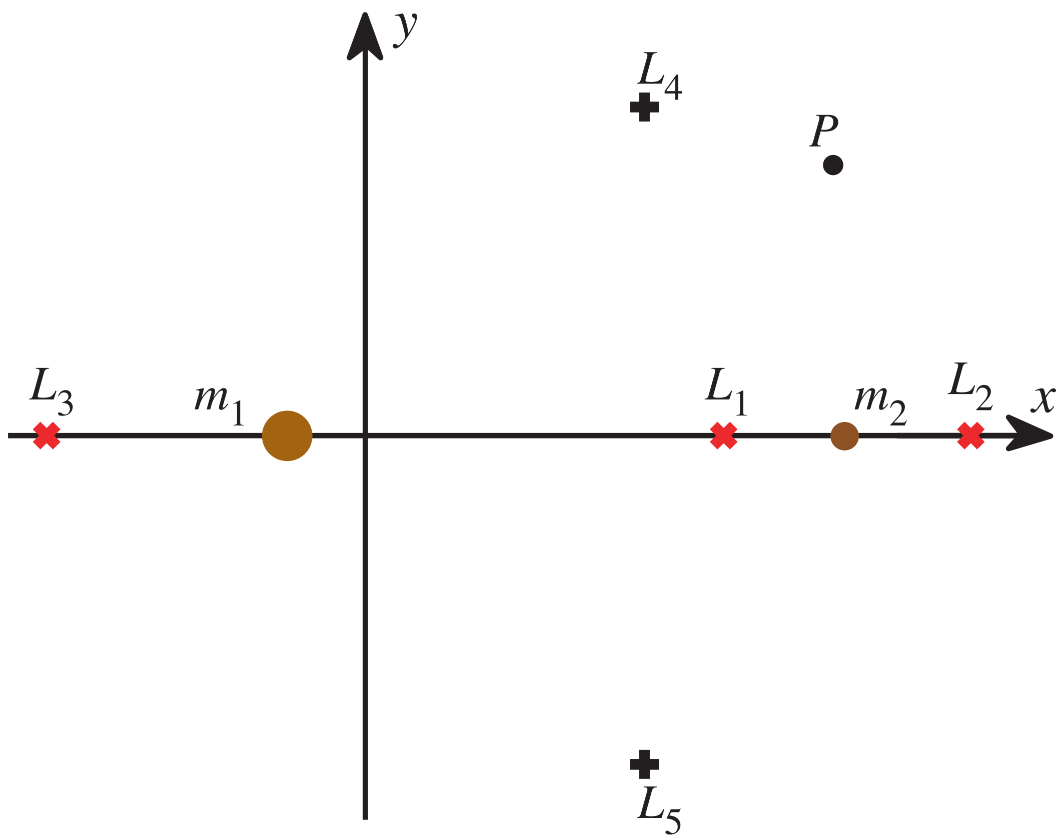
Case 4 : $E_3<E<E_4$



Case 5 : $E>E_4$

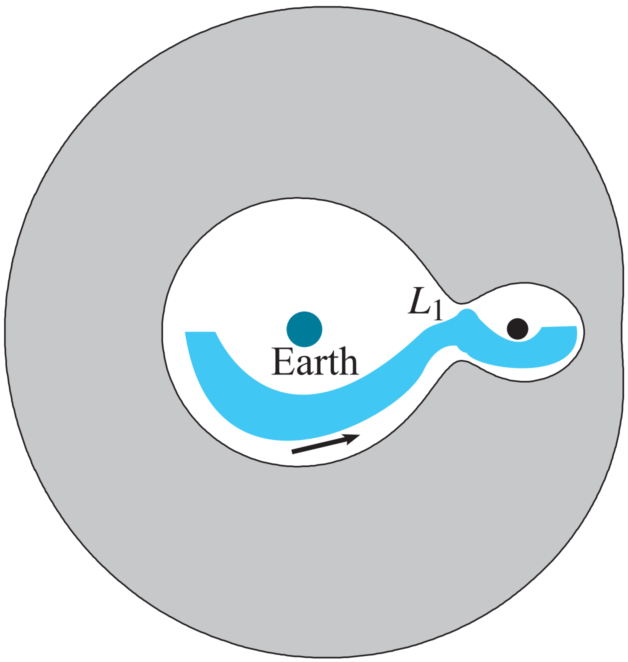
Orbits in neck regions between realms

- Orbits exist around L_1 & L_2; periodic & quasi-periodic
  - Unstable bound orbits: Lyapunov, halo and Lissajous orbits
  - their stable/unstable invariant manifolds are tubes, play a key role

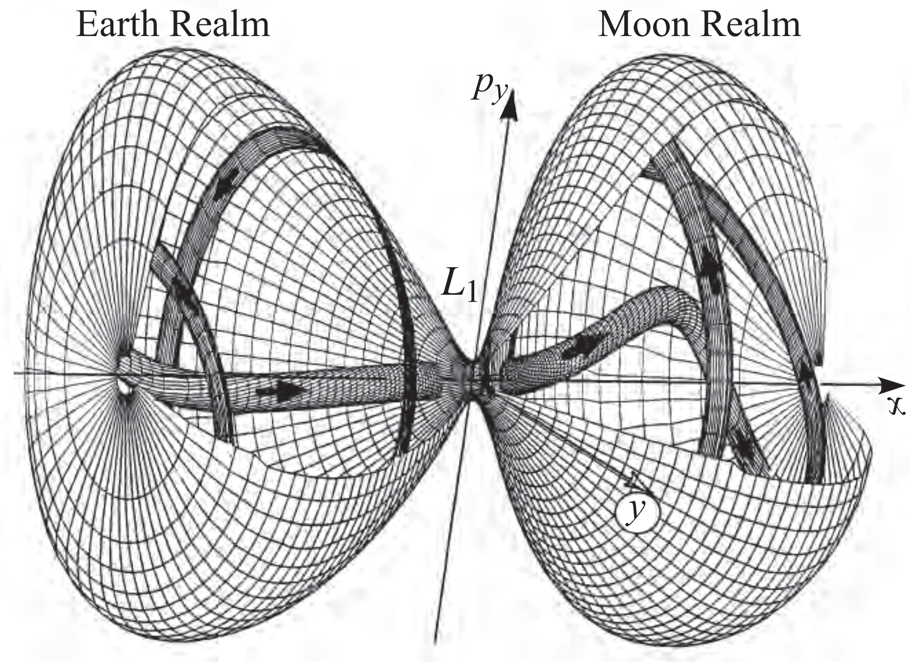


The location of all the equilibria for $\mu = 0.3$

Realms and tubes



Position Space



Phase Space (Position + Velocity)

- Realms connected by **tubes** in phase space $\simeq S^k \times \mathbb{R}$
  — Conley & McGehee, 1960s, found these locally for planar case, speculated on use for **"low energy transfers"**

Motion near saddles

- Near L_1 or L_2, linearized vector field has eigenvalues

$$\pm\lambda \text{ and } \pm i\omega_j,\ j = 2, \ldots, N$$

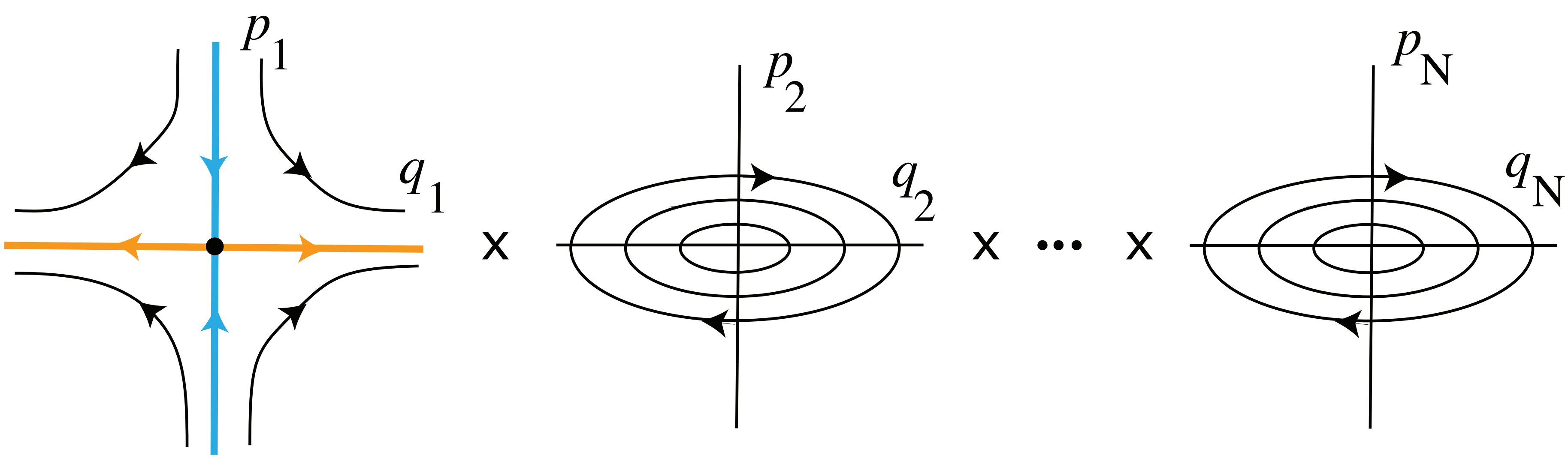
- Under local change of coordinates

$$H(q,p) = \lambda q_1 p_1 + \sum_{i=2}^{N} \frac{\omega_i}{2}\left(p_i^2 + q_i^2\right)$$

Motion near saddles

- Equilibrium point is of type
  saddle $\times$ center $\times \cdots \times$ center ($N-1$ centers)

  i.e., **rank 1 saddle**

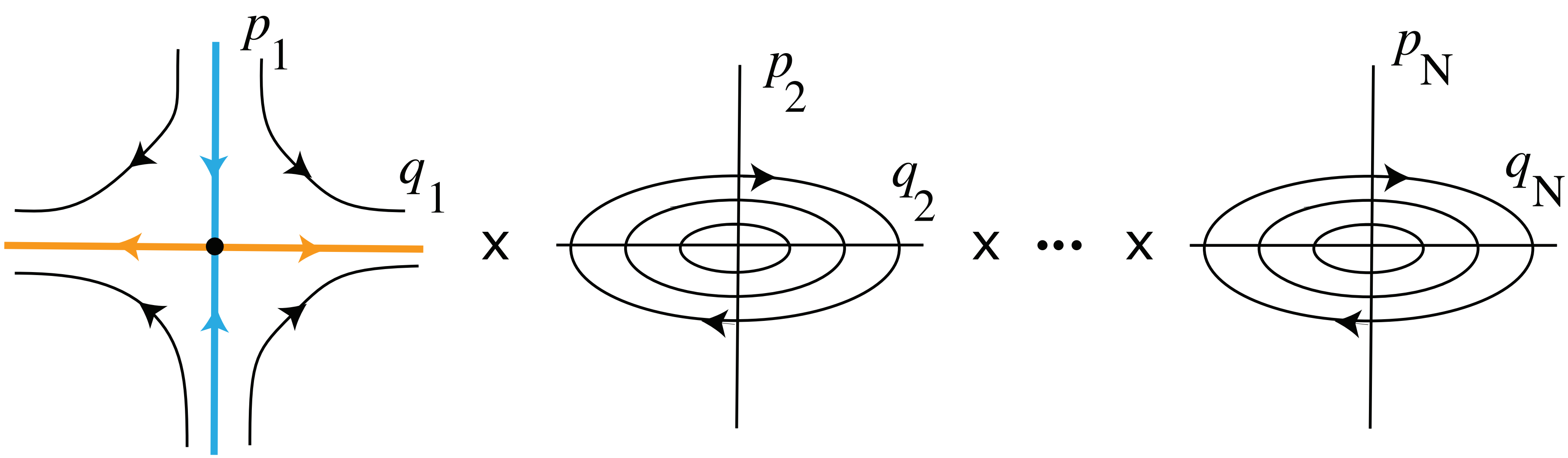


the N canonical planes

Motion near saddles

- For energy h just above saddle pt, $(q_1, p_1) = (0, 0)$ is normally hyperbolic invariant manifold of bound orbits

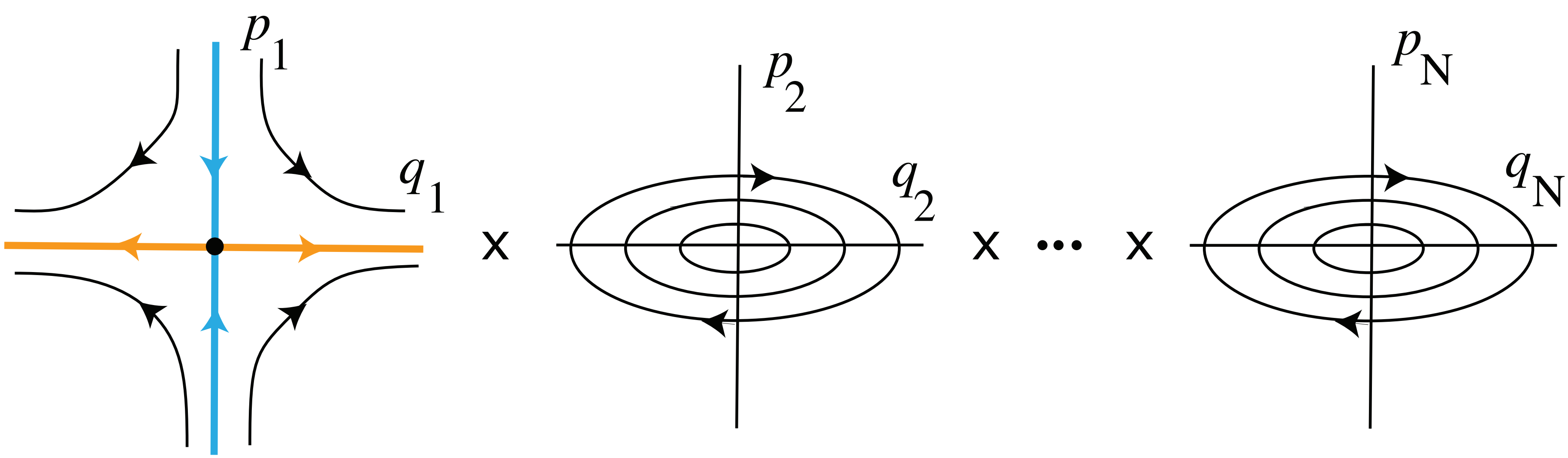
$$\mathcal{M}_h = \sum_{i=2}^{N} \frac{\omega_i}{2} \left(p_i^2 + q_i^2\right) = h > 0.$$



the N canonical planes

Motion near saddles

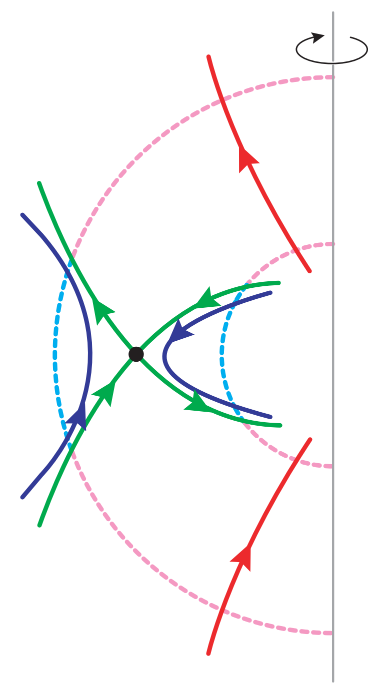
- Note that $\mathcal{M}_h \simeq S^{2N-3}$
  - $N=2$, the circle S^1, a single periodic orbit
  - $N=3$, the 3-sphere S^3, a set of periodic and quasi-periodic orbits



the N canonical planes

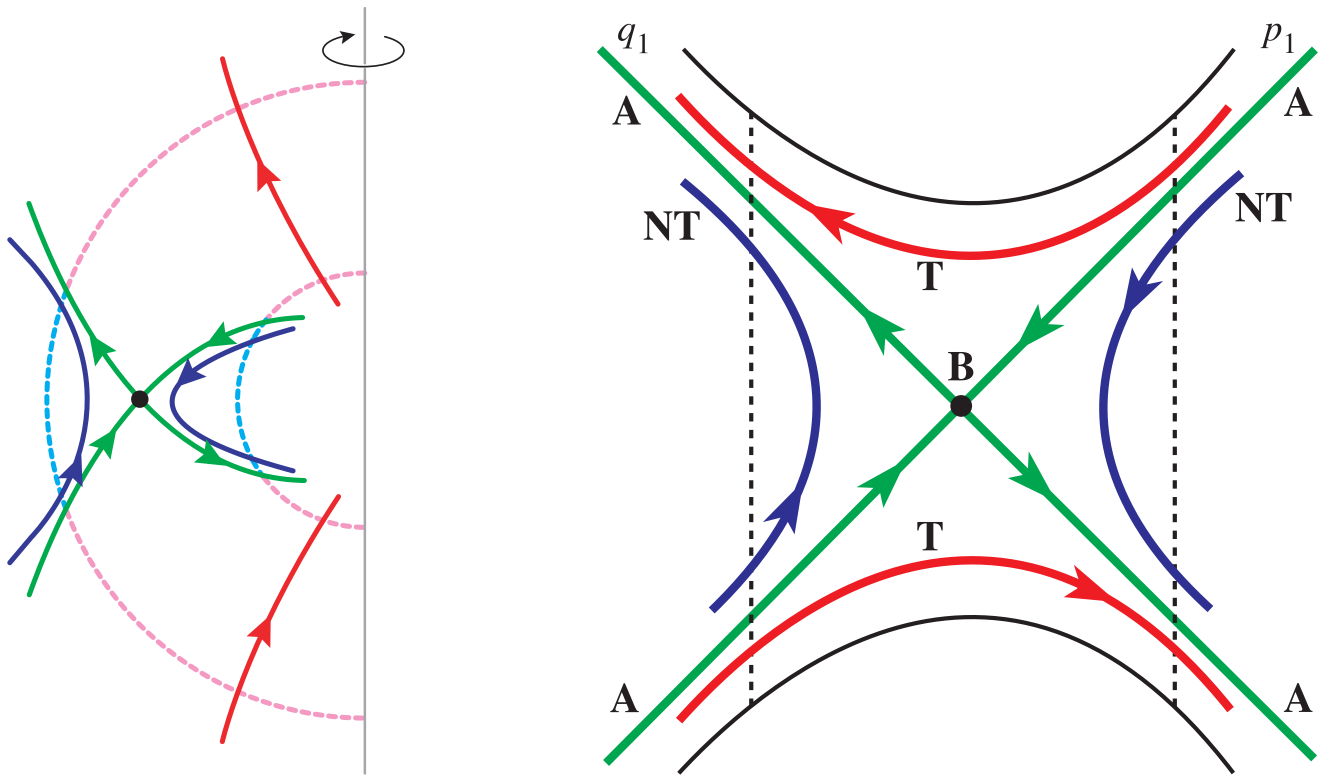
Motion near saddles

- Note that $\mathcal{M}_h \simeq S^{2N-3}$
  - $N = 2$, the circle S^1, a single periodic orbit
  - $N = 3$, the 3-sphere S^3, a set of periodic and quasi-periodic orbits
- Four "cylinders" or **tubes** of asymptotic orbits: stable, unstable manifolds, $W^s_{\pm}(\mathcal{M}_h), W^u_{\pm}(\mathcal{M}_h)$, $\simeq S^1 \times \mathbb{R}$ for $N = 2$



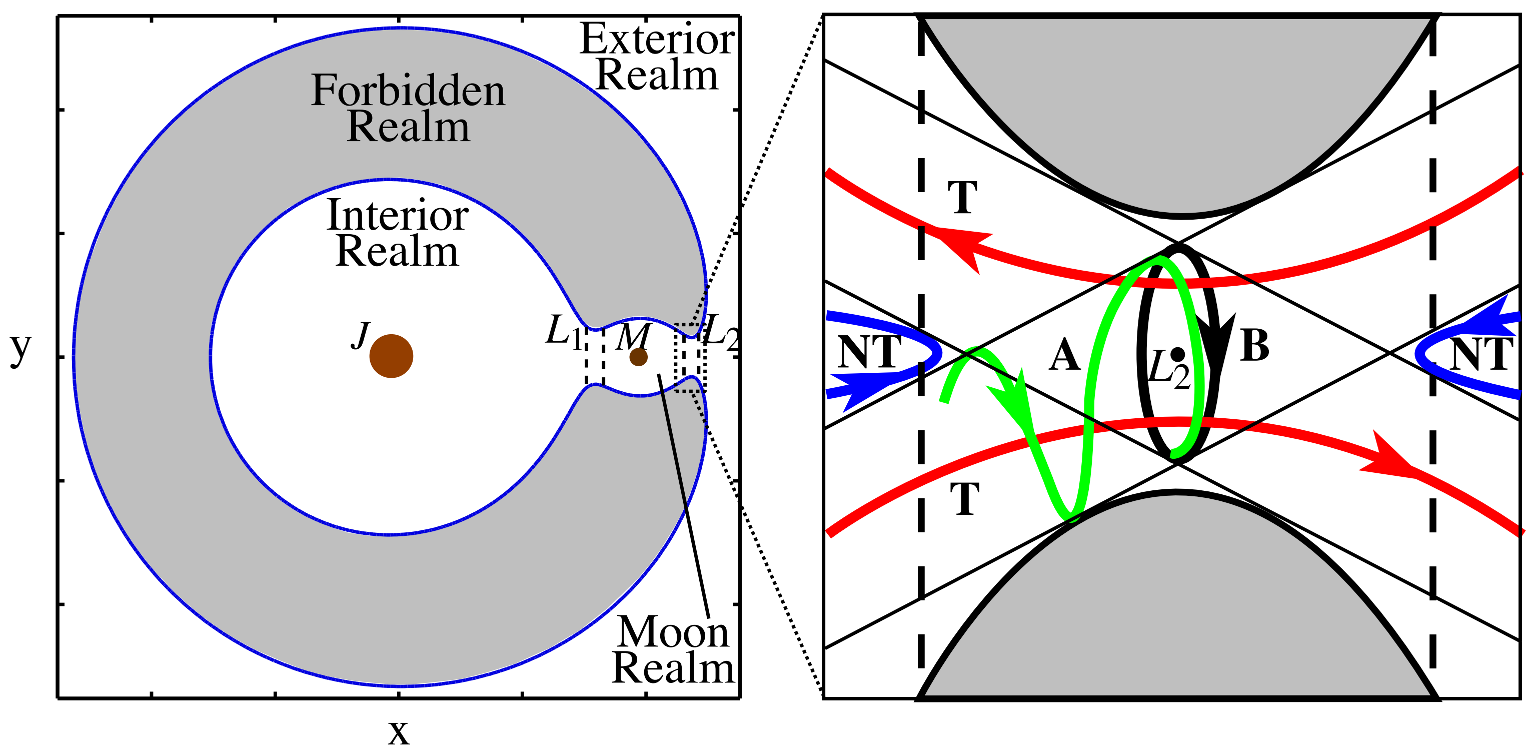
Motion near saddles

- **B** : **bounded orbits** (periodic/quasi-periodic): S^3
- **A** : **asymptotic orbits** to 3-sphere: $S^3 \times \mathbb{R}$ **(tubes)**
- **T** : **transit** and **NT** : **non-transit** orbits.



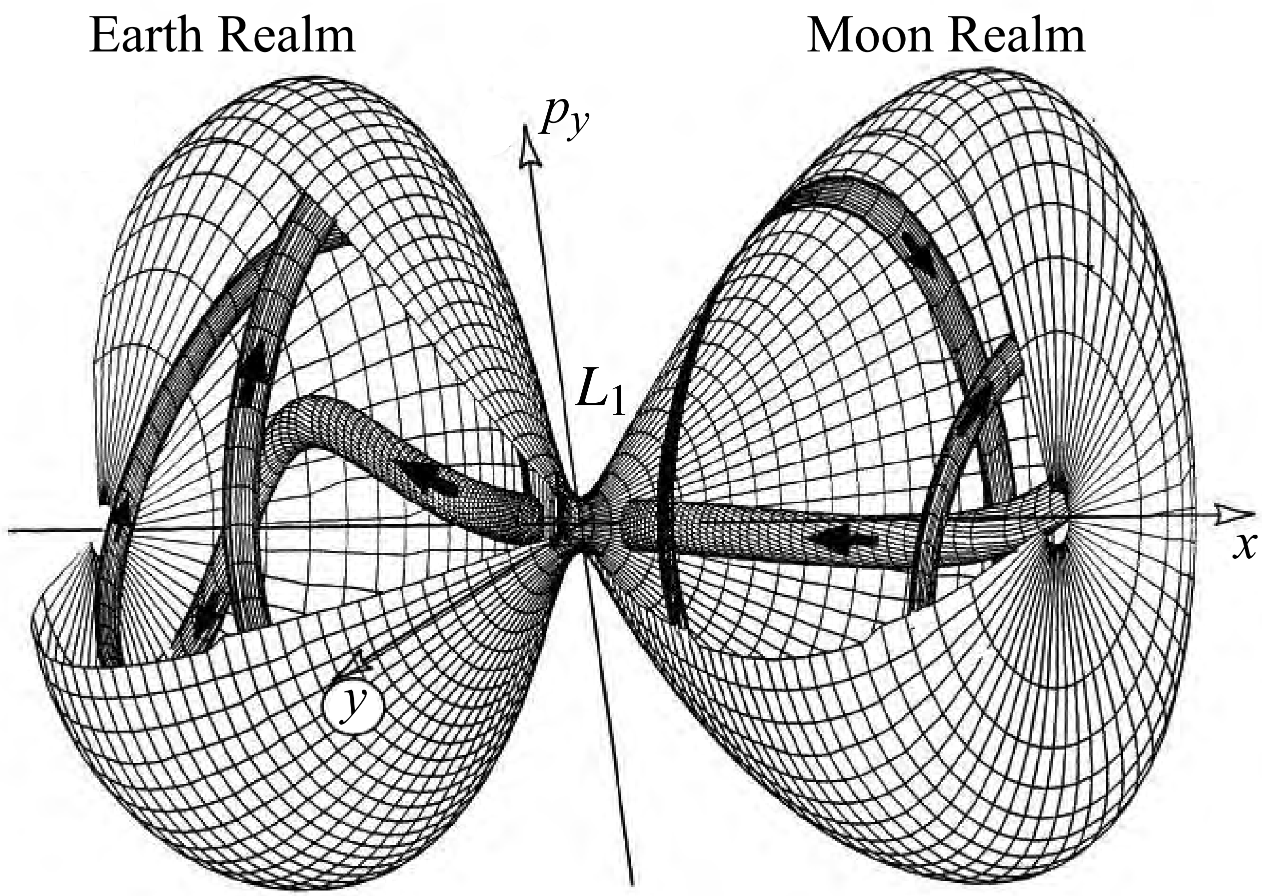
Motion near saddles: 3-body problem

- **B** : **bounded orbits** (periodic/quasi-periodic): S^3
- **A** : **asymptotic orbits** to 3-sphere: $S^3 \times \mathbb{R}$ **(tubes)**
- **T** : **transit** and **NT** : **non-transit** orbits.



Projection to configuration space.

Tube dynamics: inter-realm transport



- **Tube dynamics:** All motion between realms connected by necks around saddles must occur through the interior of tubes[1]

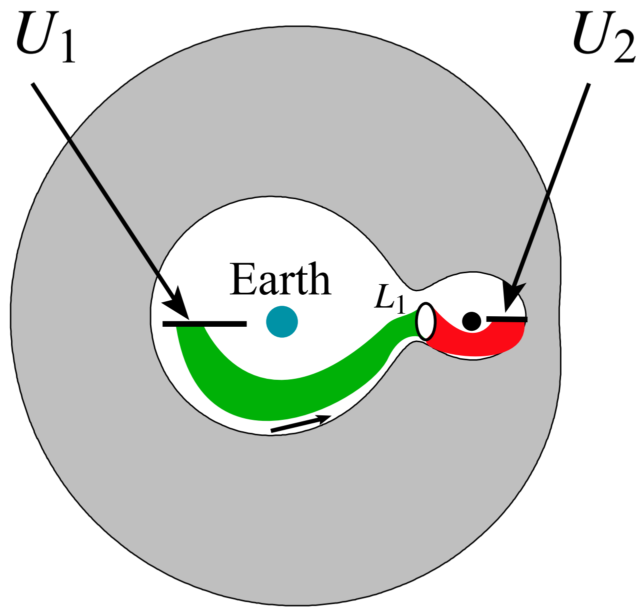
[1]Koon, Lo, Marsden, Ross [2000,2001,2002], Gómez, Koon, Lo, Marsden, Masdemont, Ross [2004]

Some remarks on tube dynamics

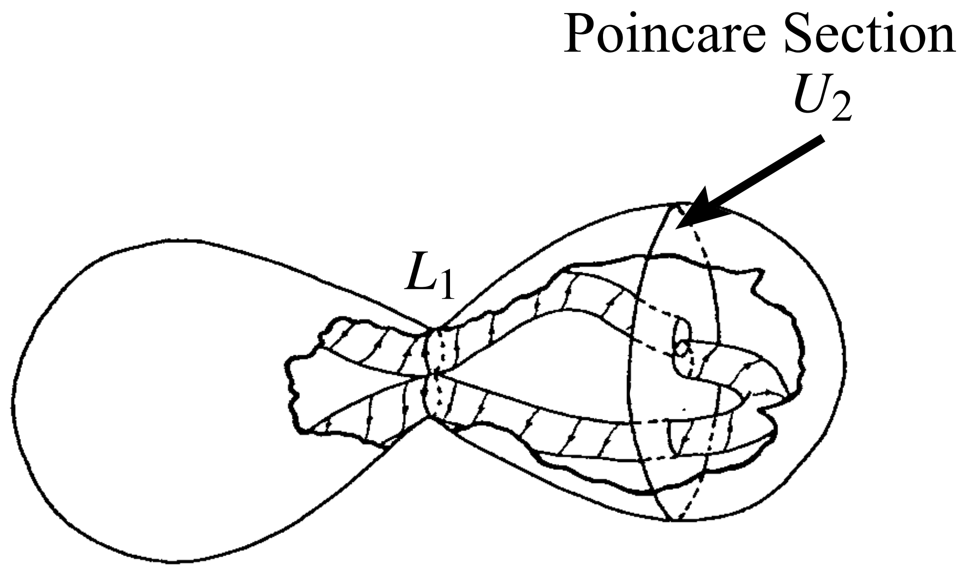
- Tubes are general; consequence of rank 1 saddle – e.g., ubiquitous in chemistry
- Tubes persist
  – in presence of additional massive body
  – when primary bodies' orbit is eccentric
- Observed in the solar system (e.g., Oterma)
- Even on galactic and atomic scales!

Koon, Lo, Marsden, & Ross [2000], Gómez, Koon, Lo, Marsden, Masdemont, & Ross [2004], Gabern, Koon, Marsden, & Ross [2005], Ross & Marsden [2006], Gawlik, Marsden, Du Toit, Campagnola [2008],Combes, Leon, Meylan [1999], Heggie [2000], Romero-Gómez, et al. [2006,2007,2008]

Tube dynamics



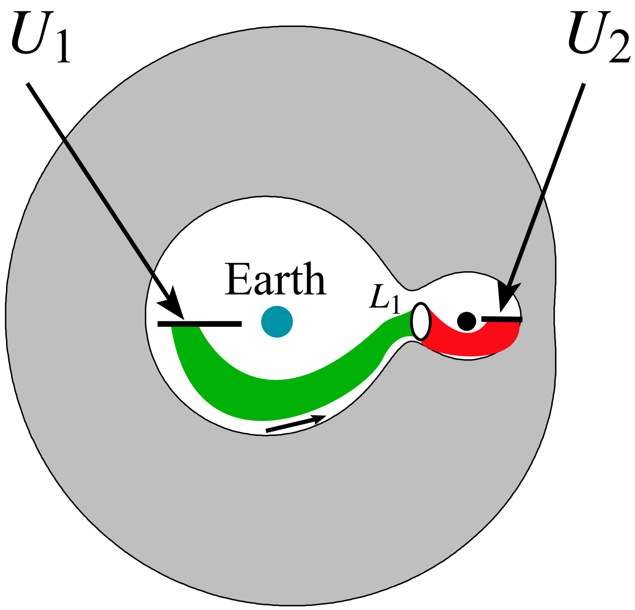
Position Space



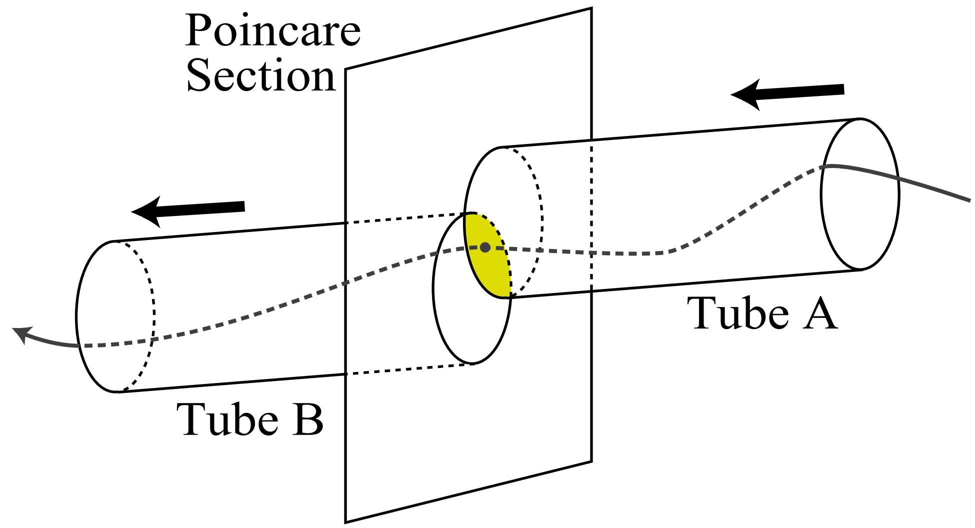
Phase Space

- Motion between Poincaré sections on $\mathcal{M}(E)$
- System reduced to k-map dynamics between the k U_i

Tube dynamics



Position Space

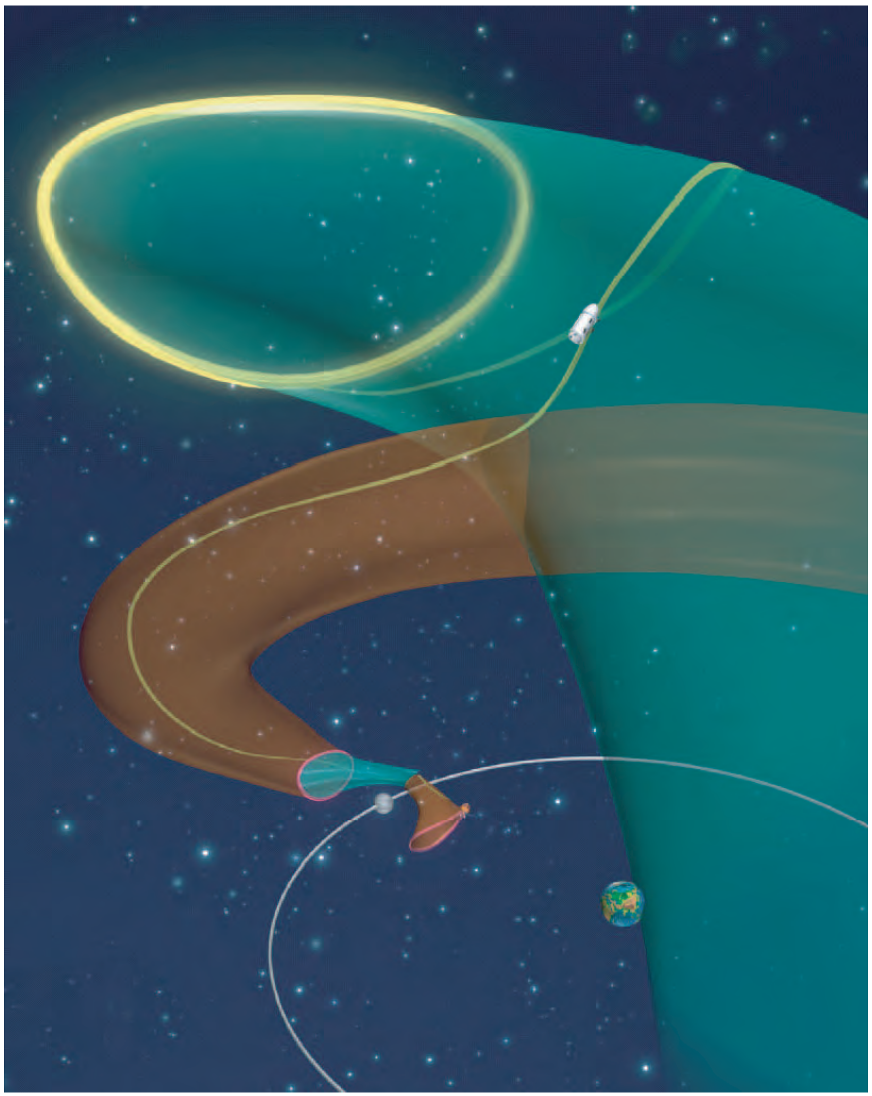


Phase Space

- Motion between Poincaré sections on $\mathcal{M}(E)$
- System reduced to k-map dynamics between the k U_i

Identifying orbits by itinerary

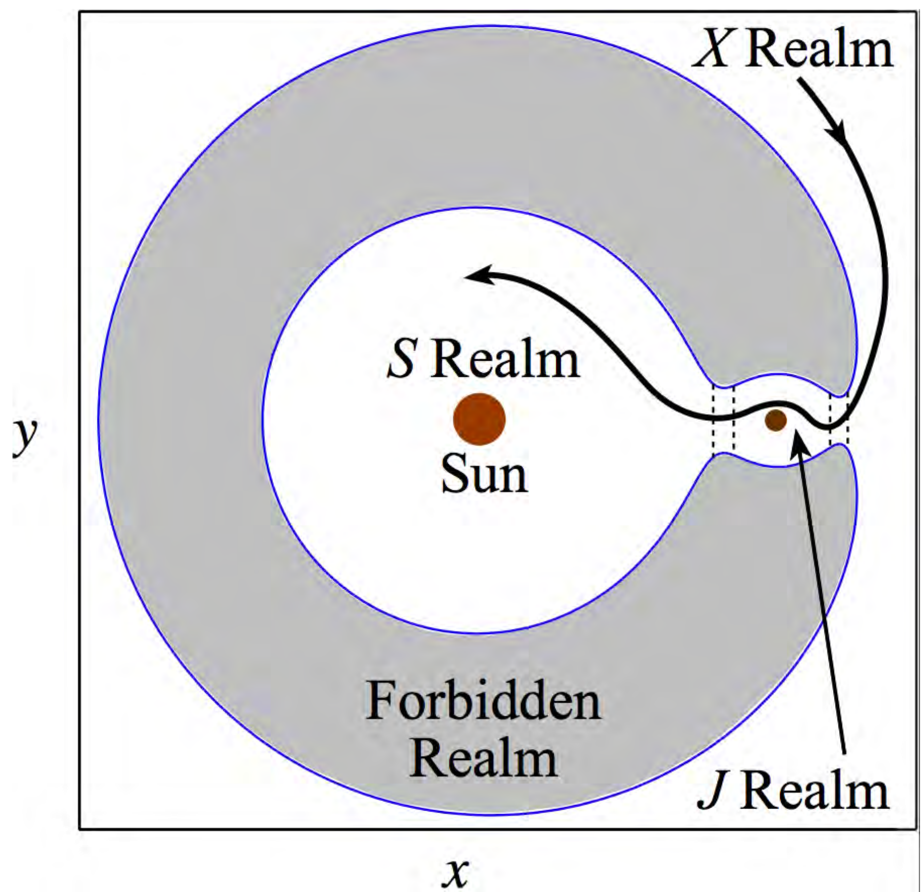
- Regions of common orbits labeled using itineraries
  - by looking at intersections of labeled tubes $\longrightarrow$ **tube hopping**



Itineraries for multiple 3-body systems possible too.

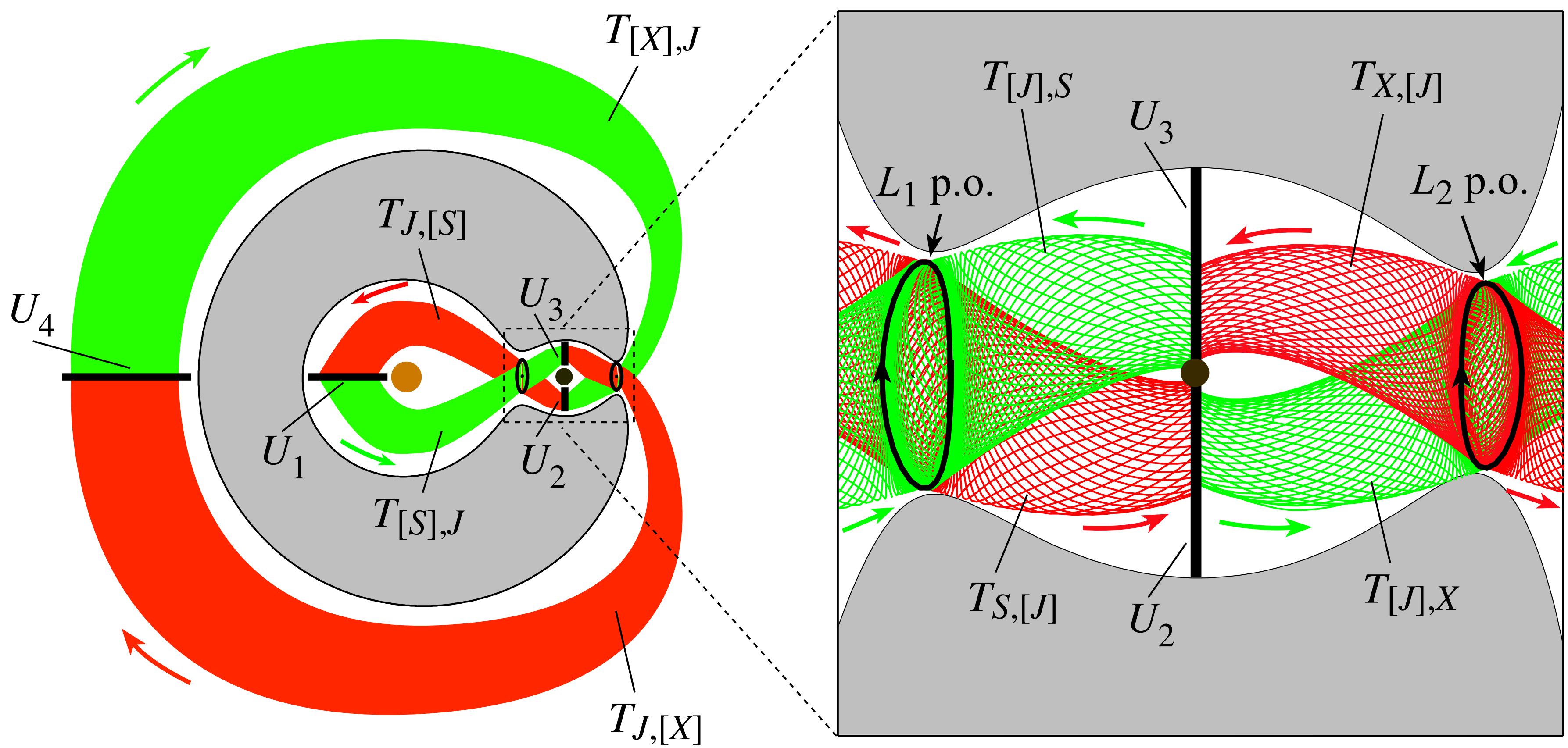
Identifying orbits by itinerary

- *itinerary* (X, J, S), *same as Oterma*
  - search for an initial condition with this itinerary
  - first in 2 d.o.f., then in 3 d.o.f.

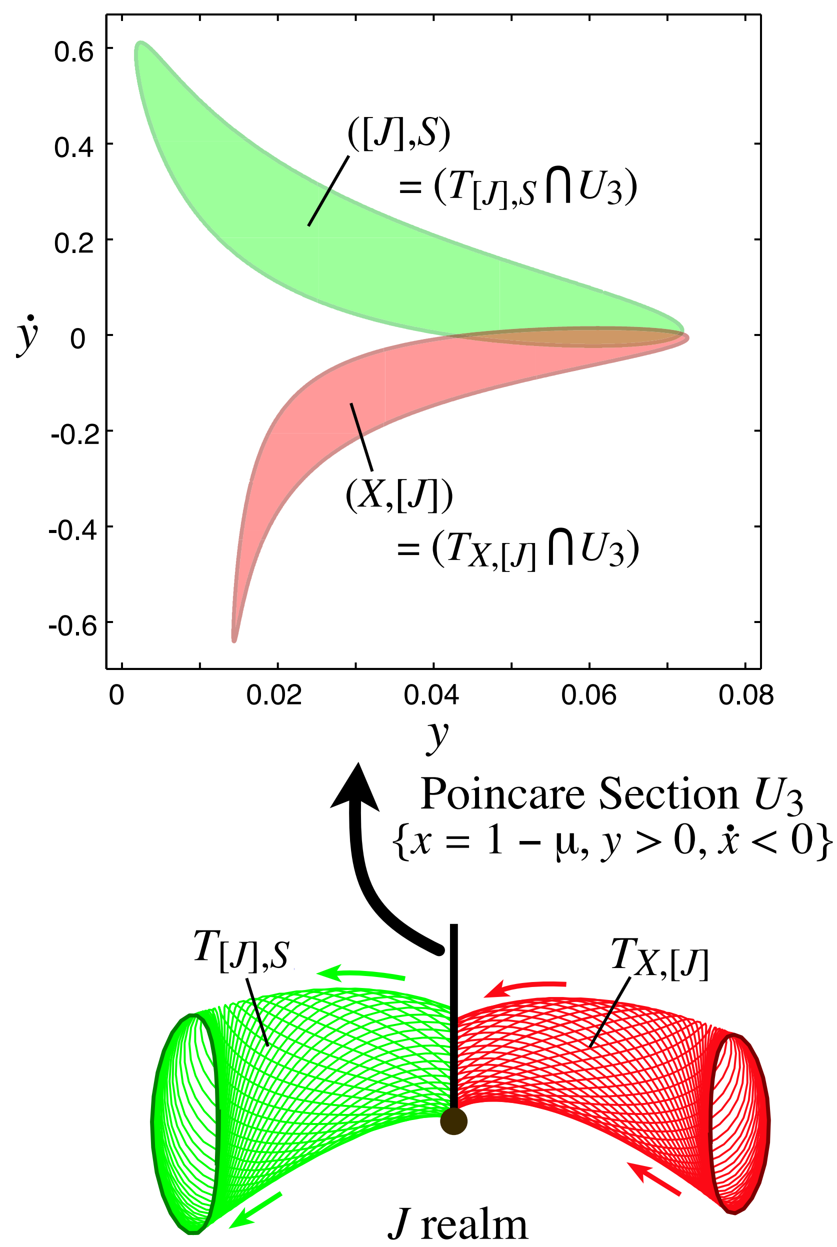


Identifying orbits by itinerary — 2 d.o.f.

- Consider how tubes connect Poincaré sections U_i

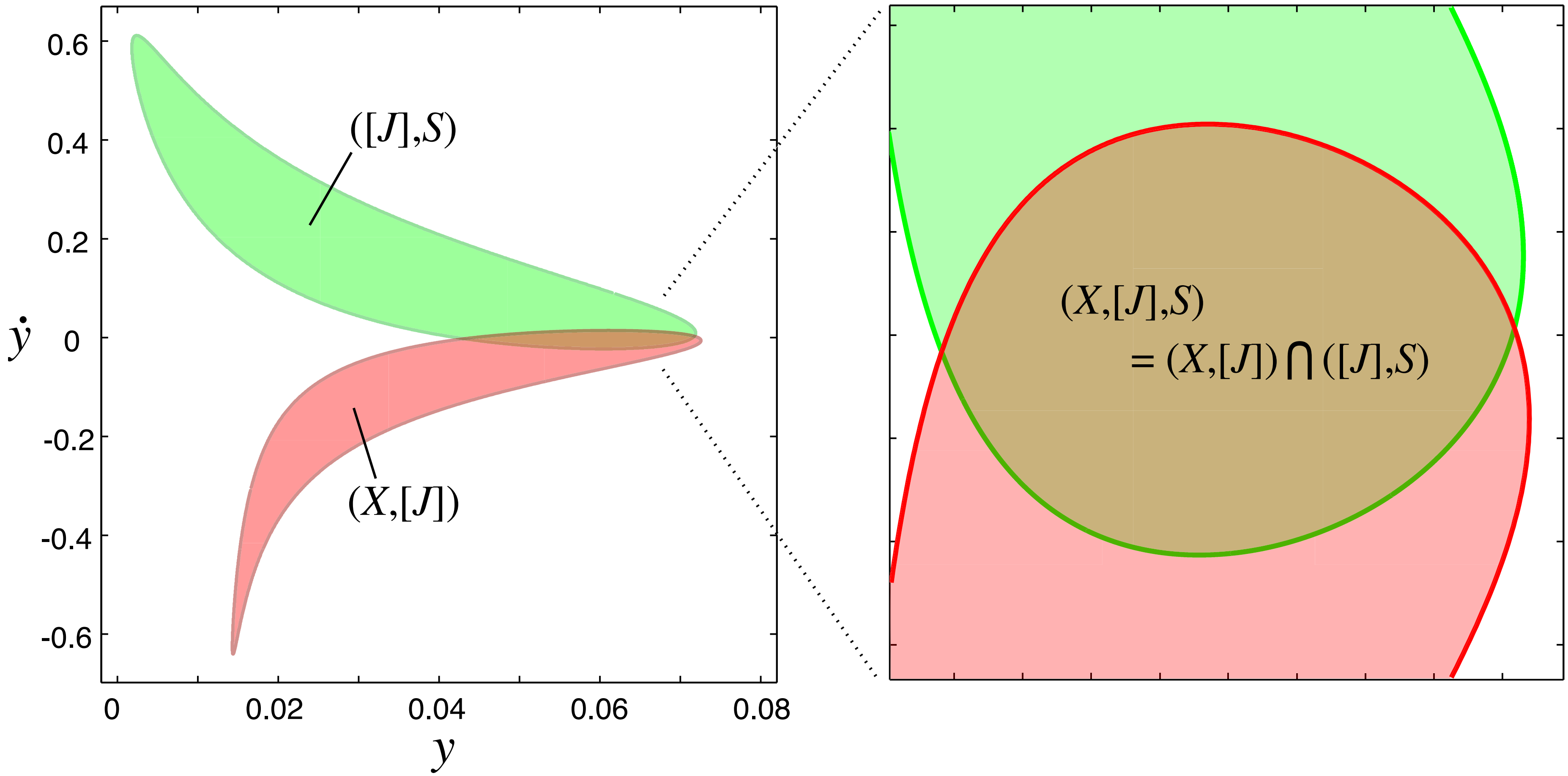


Identifying orbits by itinerary — 2 d.o.f.



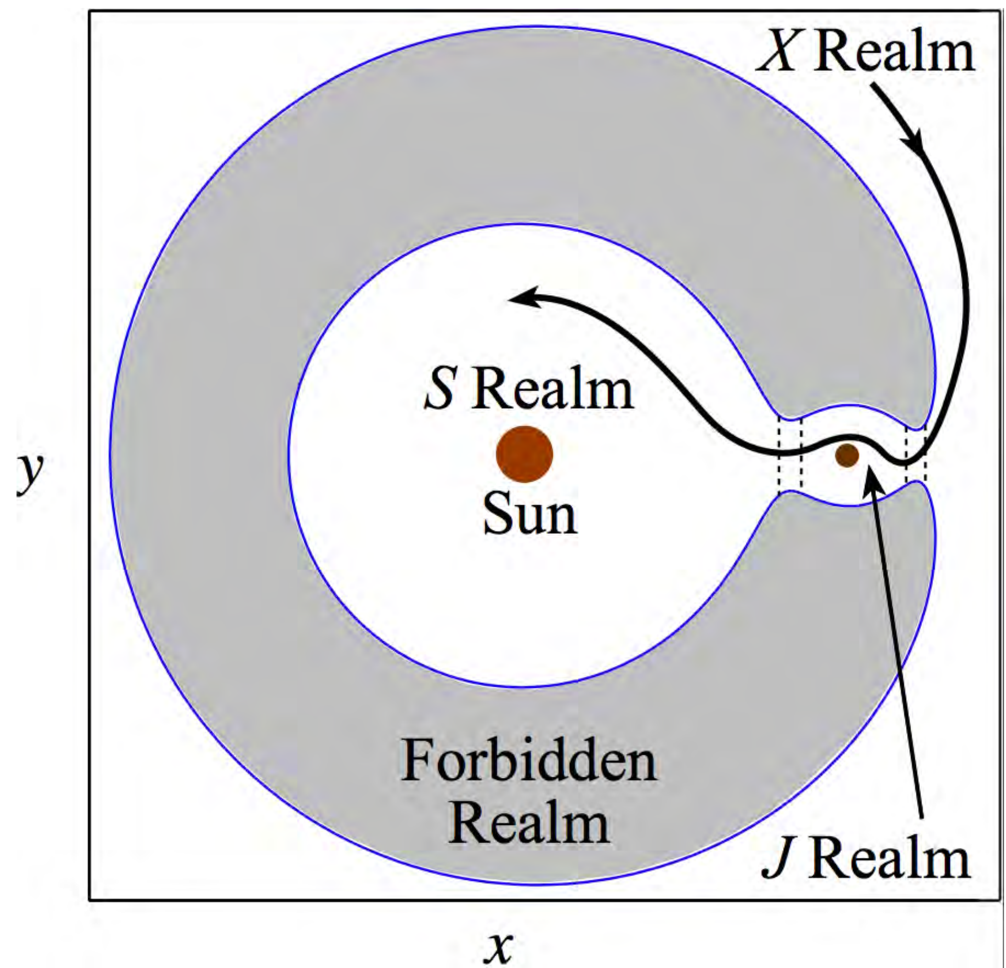
Identifying orbits by itinerary — 2 d.o.f.

- ***Tile with label (X,[J],S)***
  - Denote the intersection $(X,[J]) \bigcap ([J],S)$ by $(X,[J],S)$

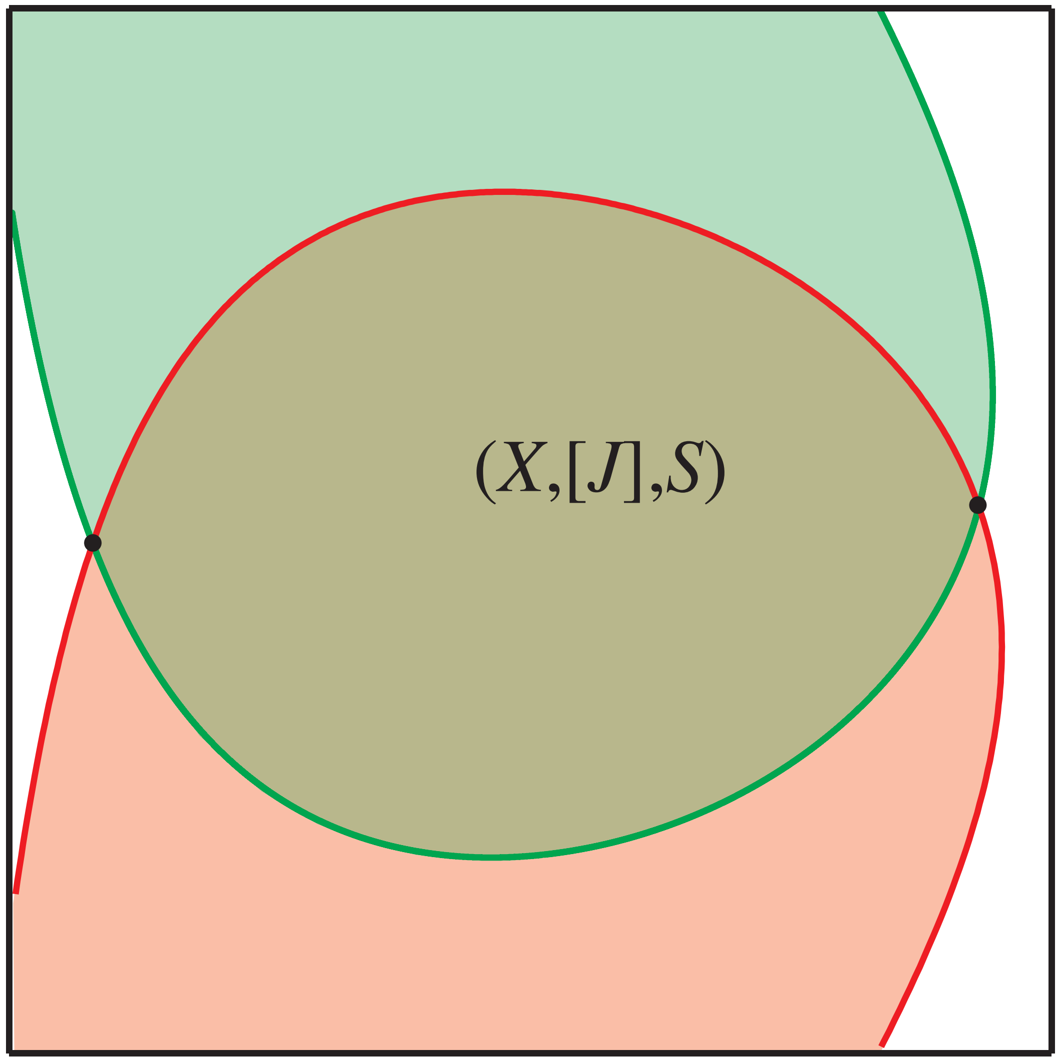


Identifying orbits by itinerary — 2 d.o.f.

- Forward and backward numerical integration

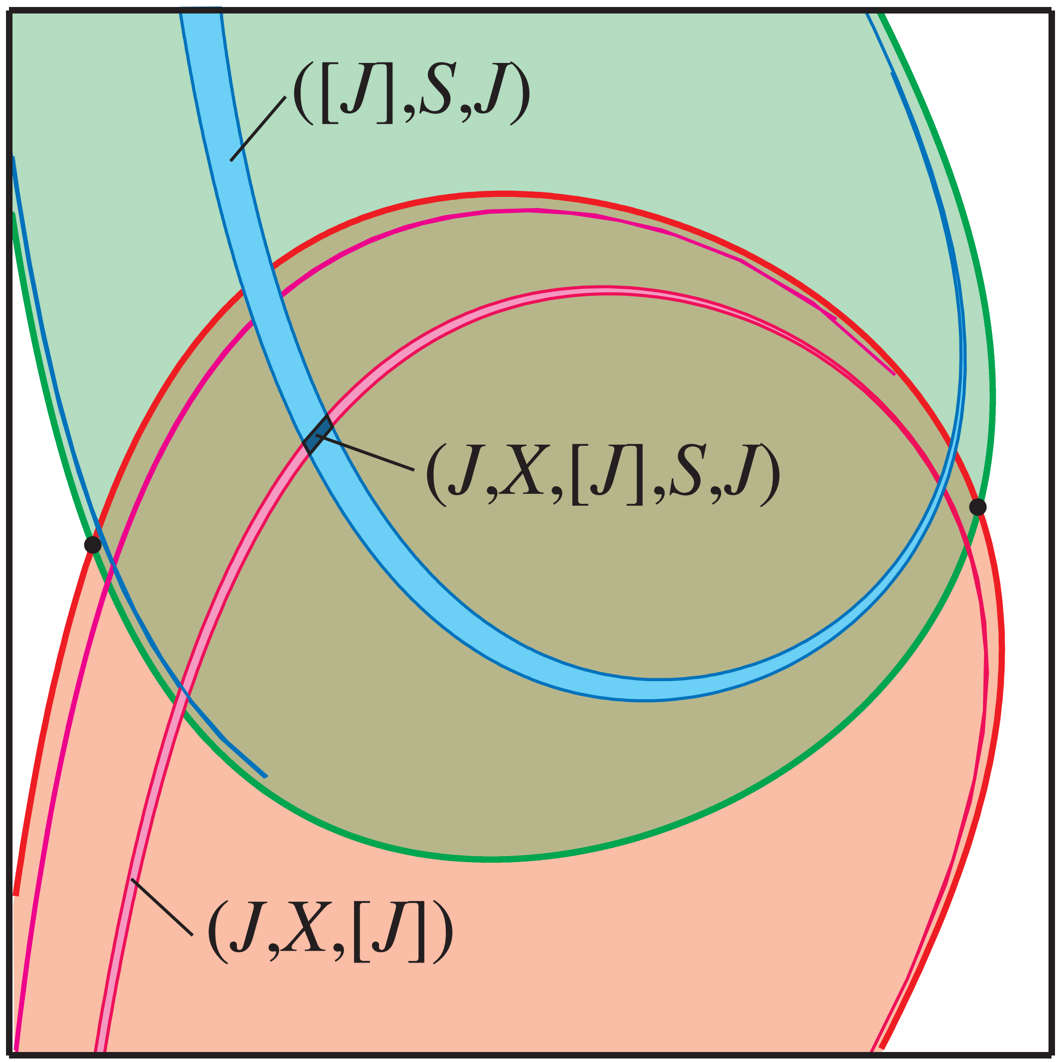


Identifying orbits by itinerary — 2 d.o.f.



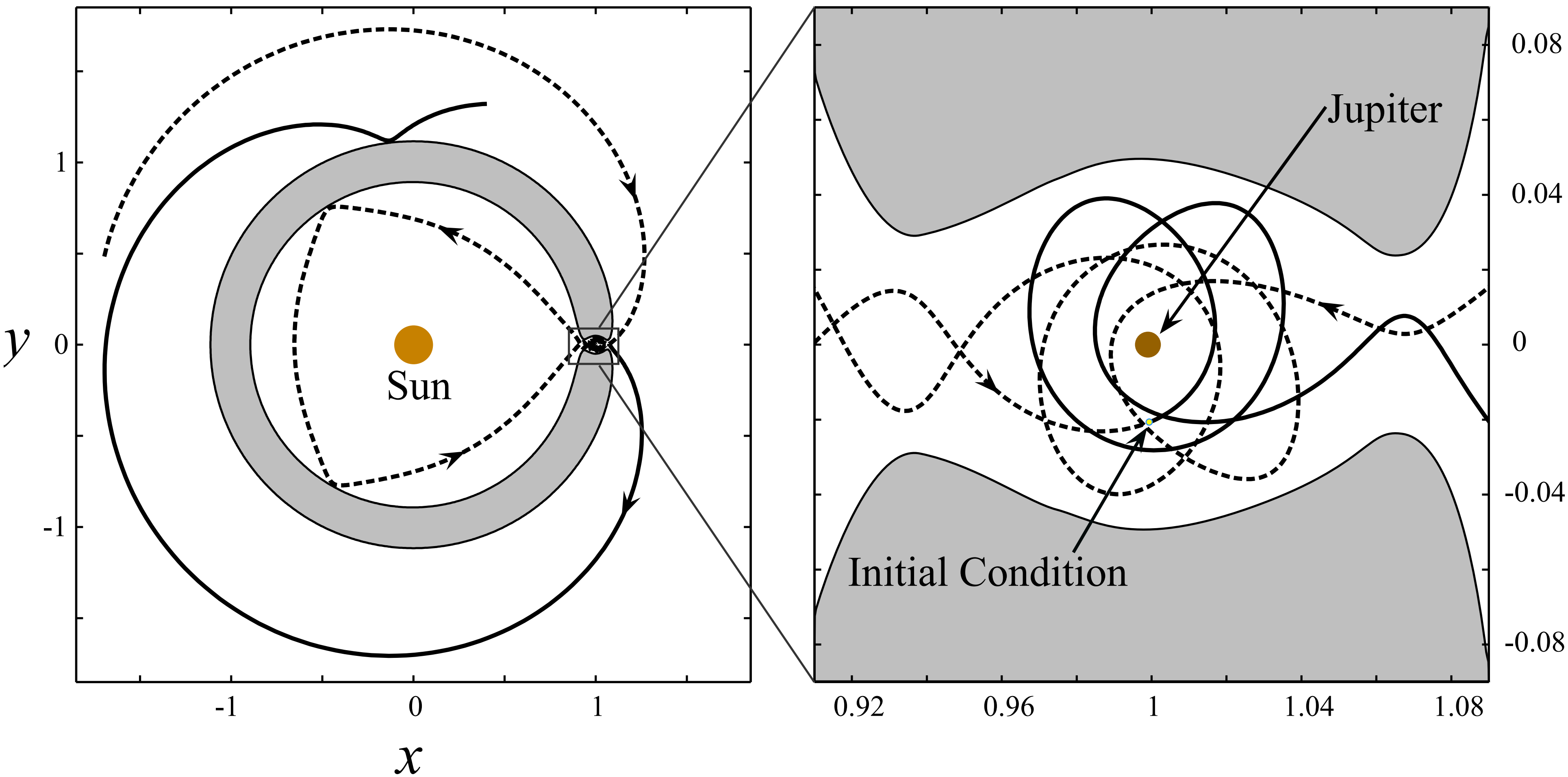
Longer itineraries...

Identifying orbits by itinerary — 2 d.o.f.



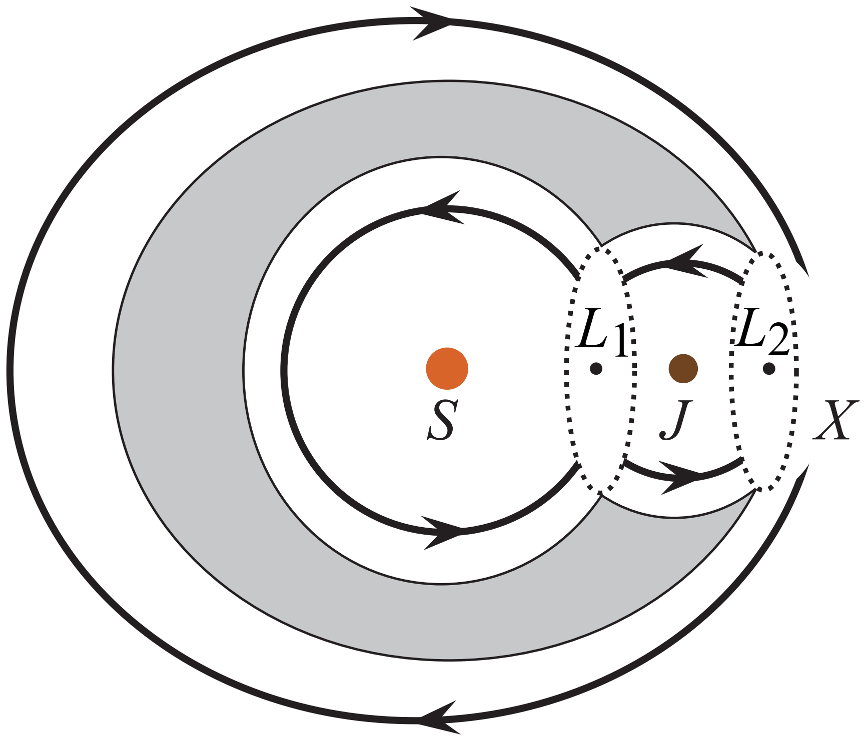
… correspond to smaller pieces of phase space

Identifying orbits by itinerary — 2 d.o.f.



Orbit with (X, J, S, J, X)

Tube dynamics: theorem

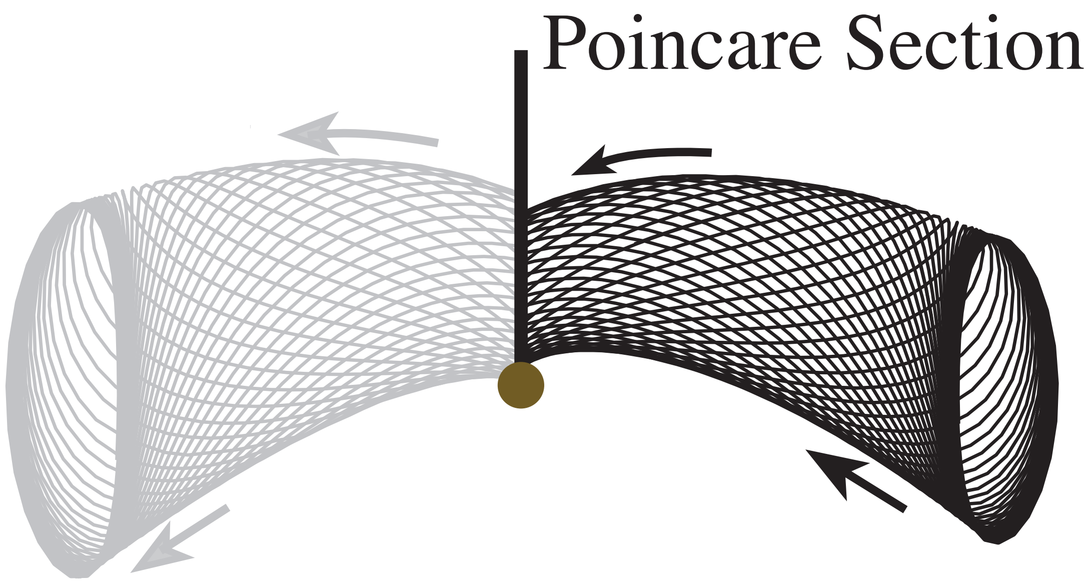


- ***Theorem of global orbit structure***
  - says we can construct an orbit with any **itinerary**, eg $(\ldots, J, X, J, S, J, S, \ldots)$, where X, J and S denote the different realms (symbolic dynamics)[2]

[2]Main theorem of Koon, Lo, Marsden, and Ross [2000] *Chaos*

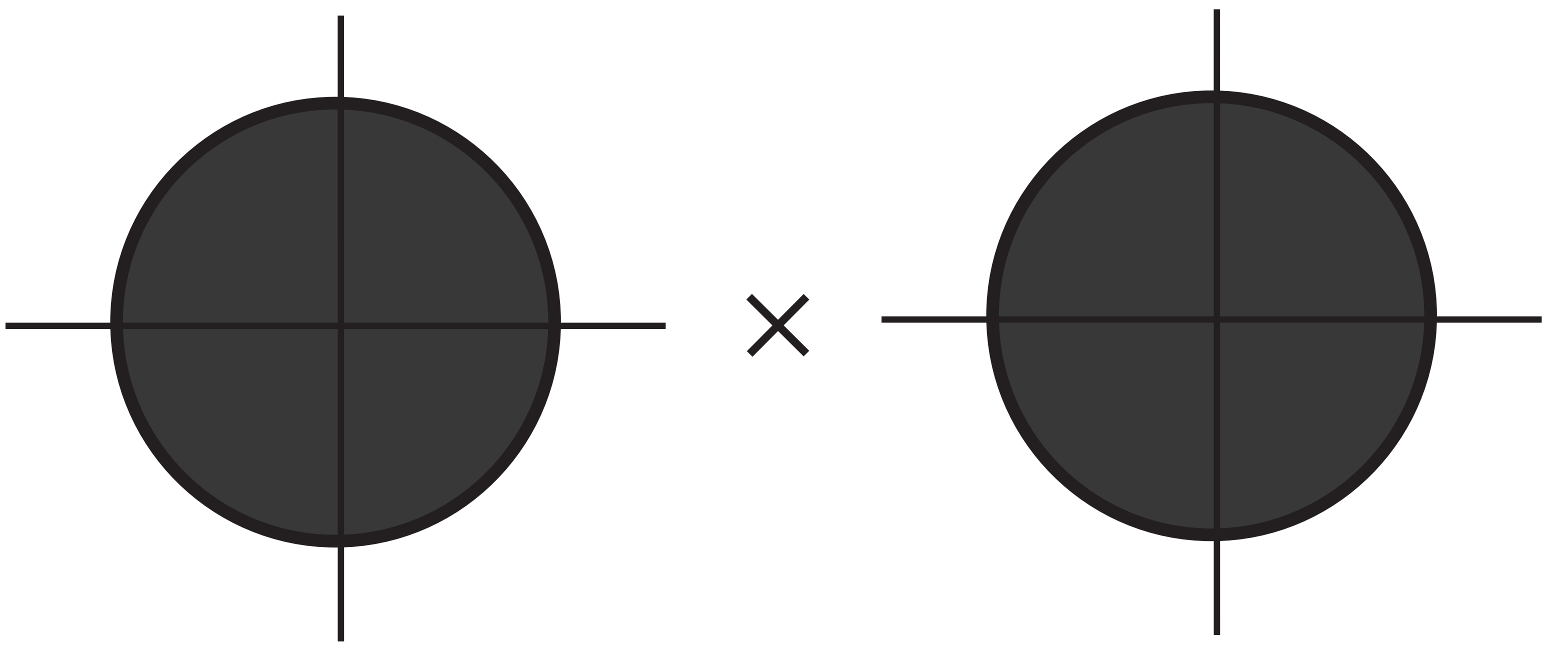
Identifying orbits by itinerary — 3 d.o.f.

- **Similar for 3 d.o.f.**: Invariant manifold tubes $S^3 \times \mathbb{R}$
- Poincaré section of energy surface
  - at $x =$ constant, $(y, \dot{y}, z, \dot{z}) \subset \mathbb{R}^4$



Identifying orbits by itinerary — 3 d.o.f.

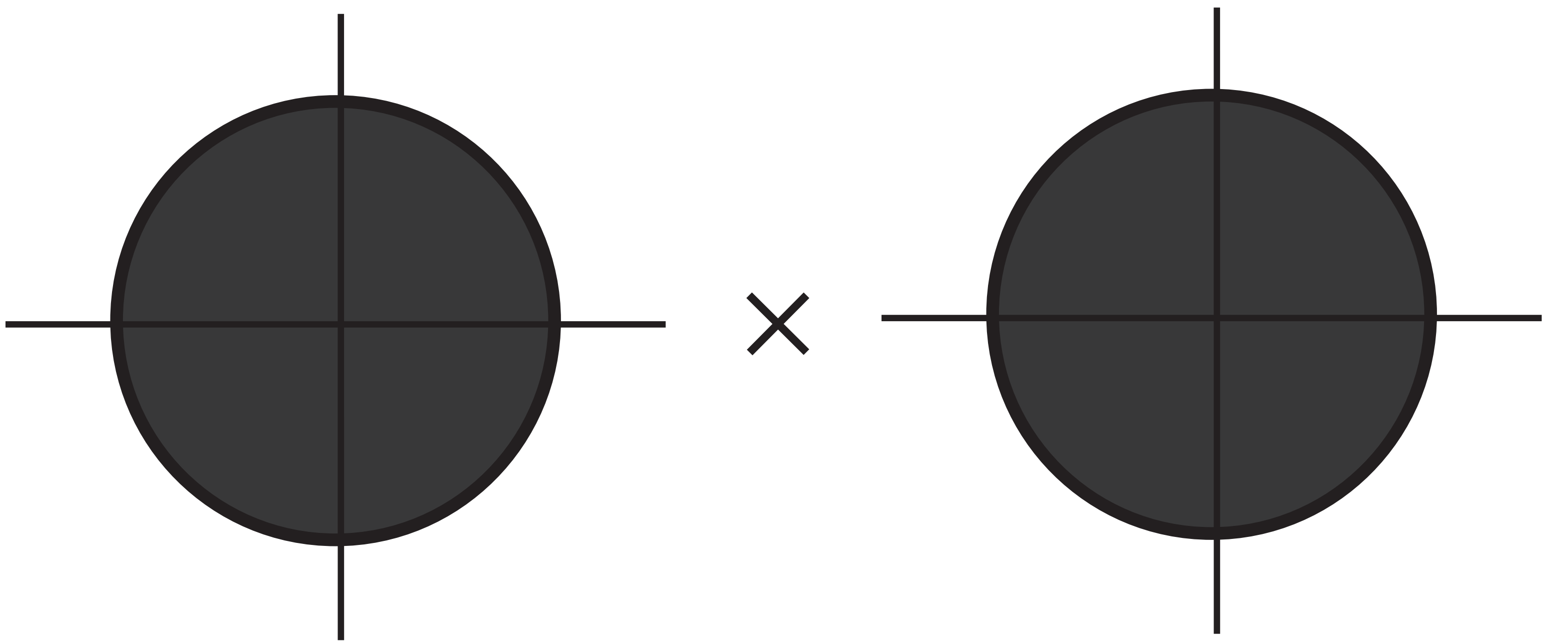
- **Similar for 3 d.o.f.**: Invariant manifold tubes $S^3 \times \mathbb{R}$
- Poincaré section of energy surface
  - at $x =$ constant, $(y, \dot{y}, z, \dot{z}) \subset \mathbb{R}^4$
- Tube **cross-section** is a topological **3-sphere** S^3 of radius r
  - S^3 projection: **disk** $\times$ **disk**



Determining interior of S^3

- S^3 projection: **disk** $\times$ **disk**

$$y^2 + \dot{y}^2 + z^2 + \dot{z}^2 = r^2$$
$$r_y^2 + r_z^2 = r^2$$

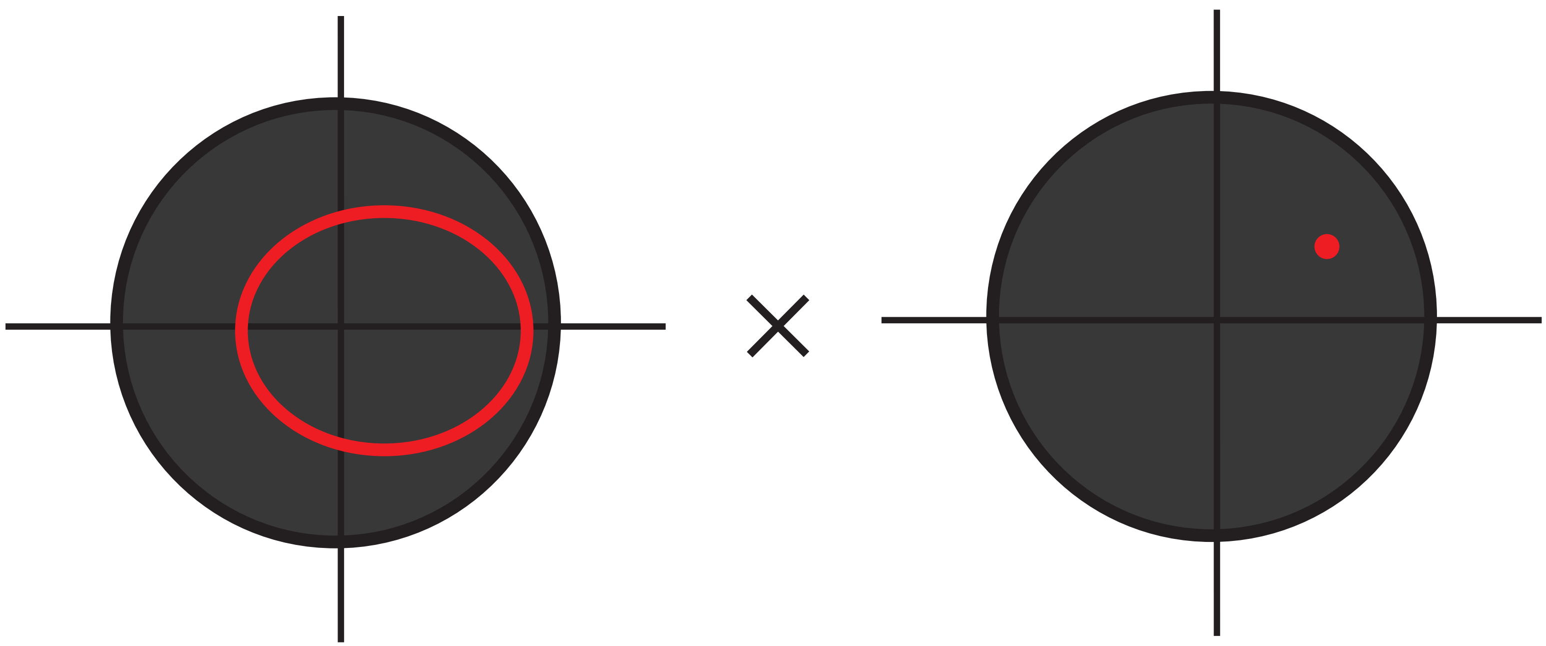


$(y, \dot{y})$ Plane $(z, \dot{z})$ Plane

Determining interior of S^3

- For fixed $(z, \dot{z})$, projection onto $(y, \dot{y})$ is a **closed curve**

$$
\begin{aligned}
y^2 + \dot{y}^2 &= r^2 - (z^2 + \dot{z}^2) \\
r_y^2 &= r^2 - r_z^2
\end{aligned}
$$

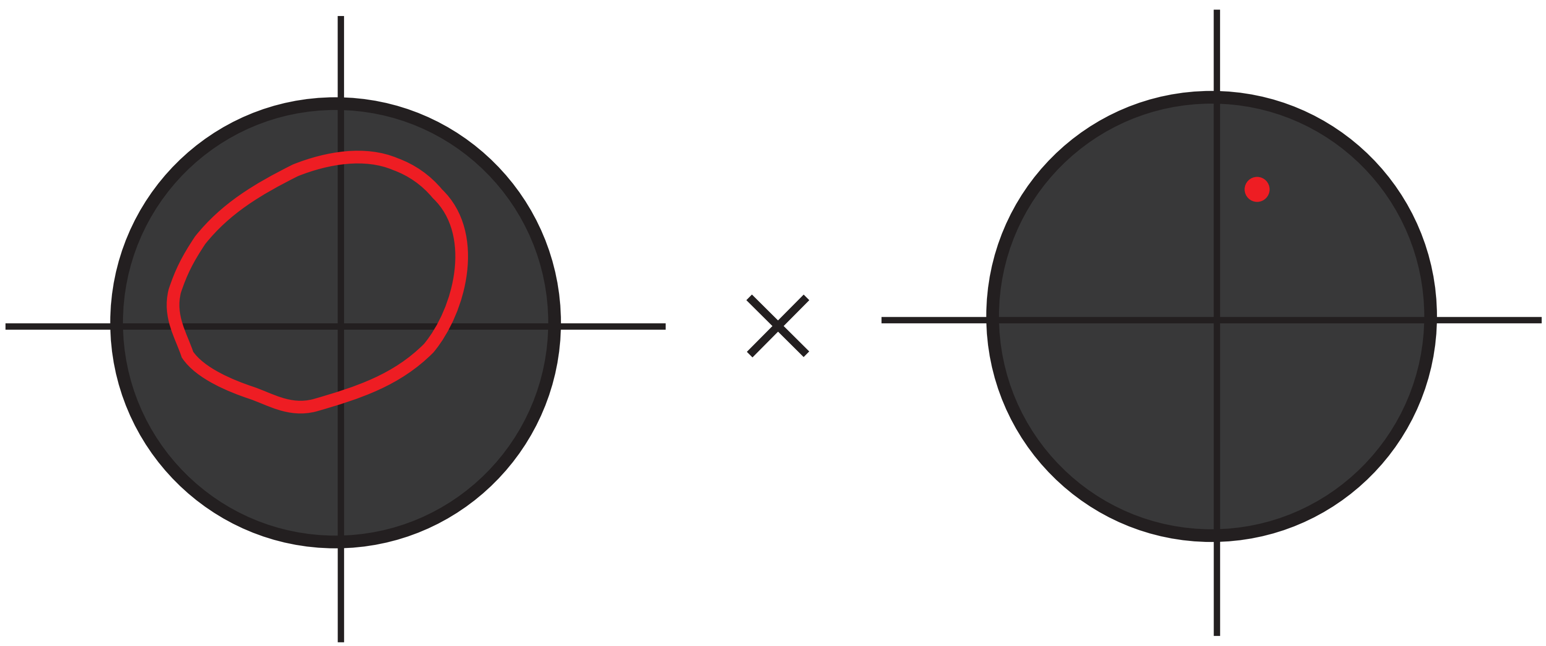


$(y, \dot{y})$ Plane $\qquad$ $(z, \dot{z})$ Plane

Determining interior of S^3

- For different $(z, \dot{z})$, a different **closed curve** in $(y, \dot{y})$

$$y^2 + \dot{y}^2 = r^2 - (z^2 + \dot{z}^2)$$
$$r_y^2 = r^2 - r_z^2$$



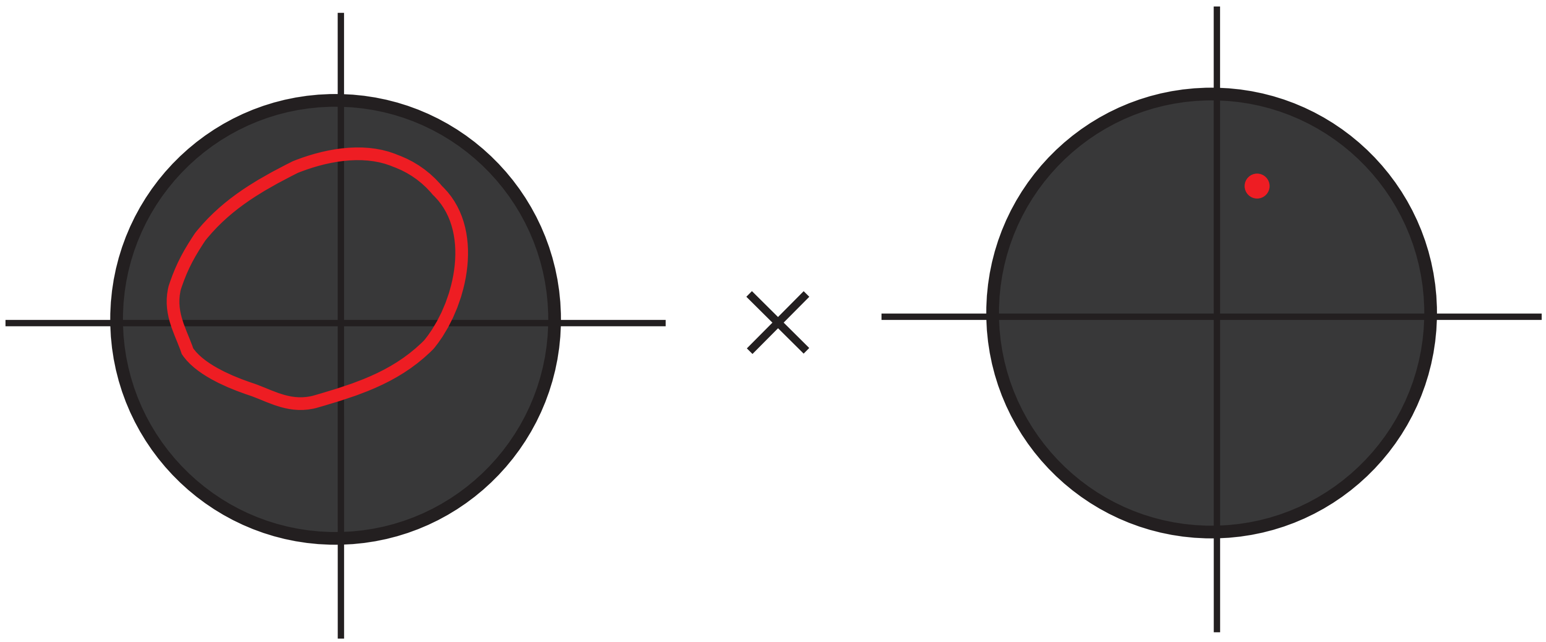
$(y, \dot{y})$ Plane

$(z, \dot{z})$ Plane

Determining interior of S^3

- Cross-section of tube effectively reduced to a **two-parameter family of closed curves**

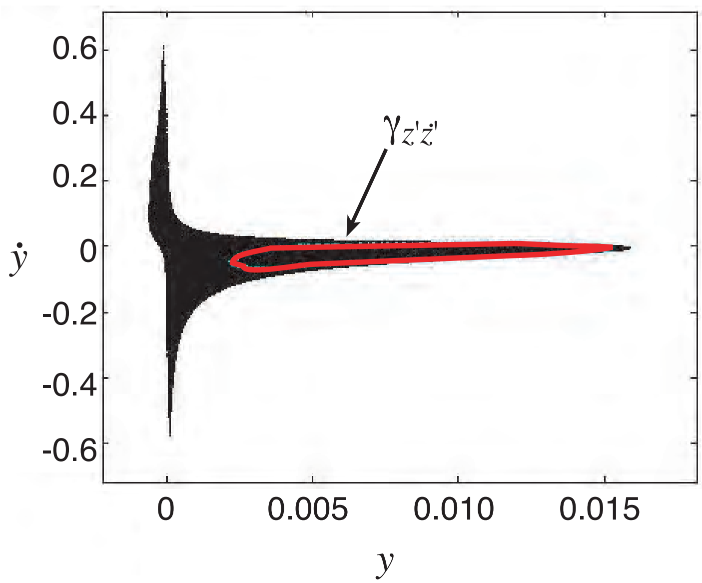
$$y^2 + \dot{y}^2 = r^2 - (z^2 + \dot{z}^2)$$



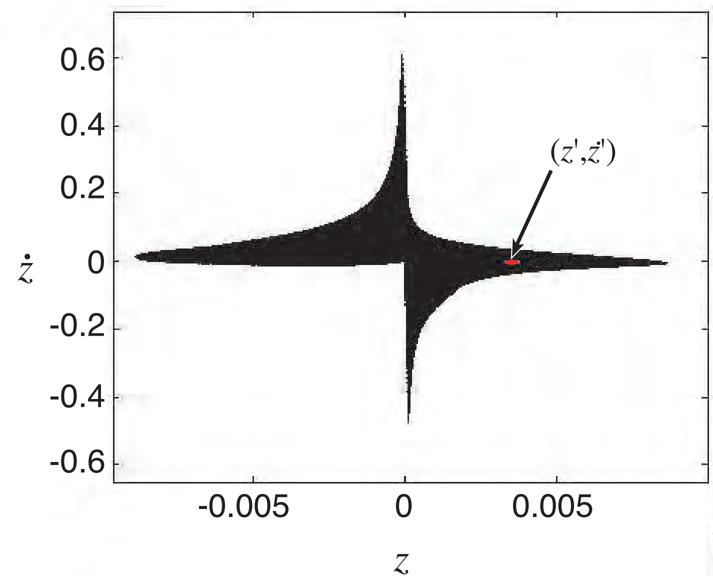
$(y, \dot{y})$ Plane $\qquad$ $(z, \dot{z})$ Plane

Determining interior of S^3

- Can be demonstrated numerically: $\{\mathrm{int}(\gamma_{z\dot{z}})\}_{(z,\dot{z})}$



$(y, \dot{y})$ Plane

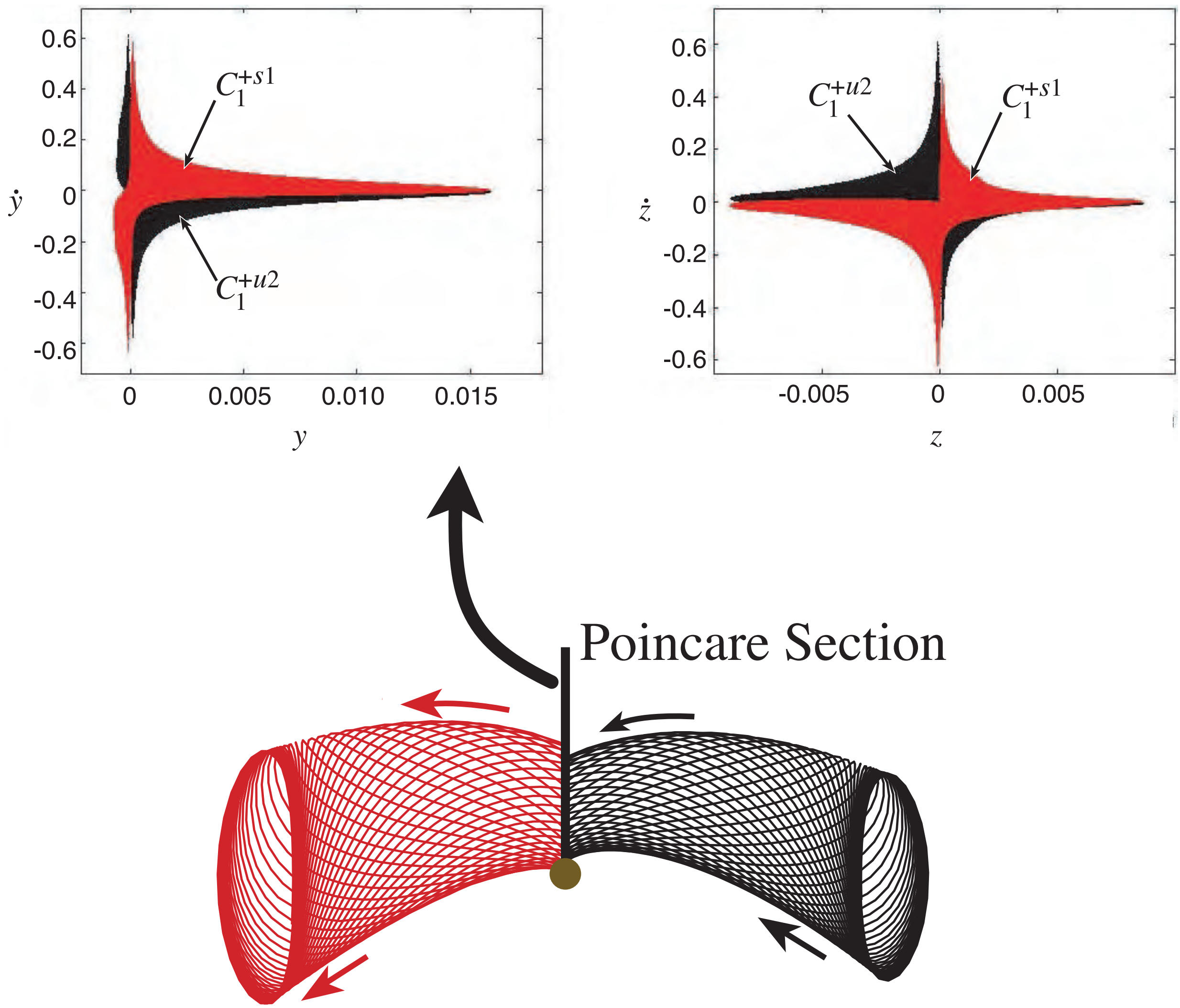


$(z, \dot{z})$ Plane

- Provides nice way to calculate interior of tube, intersections of tubes, etc.

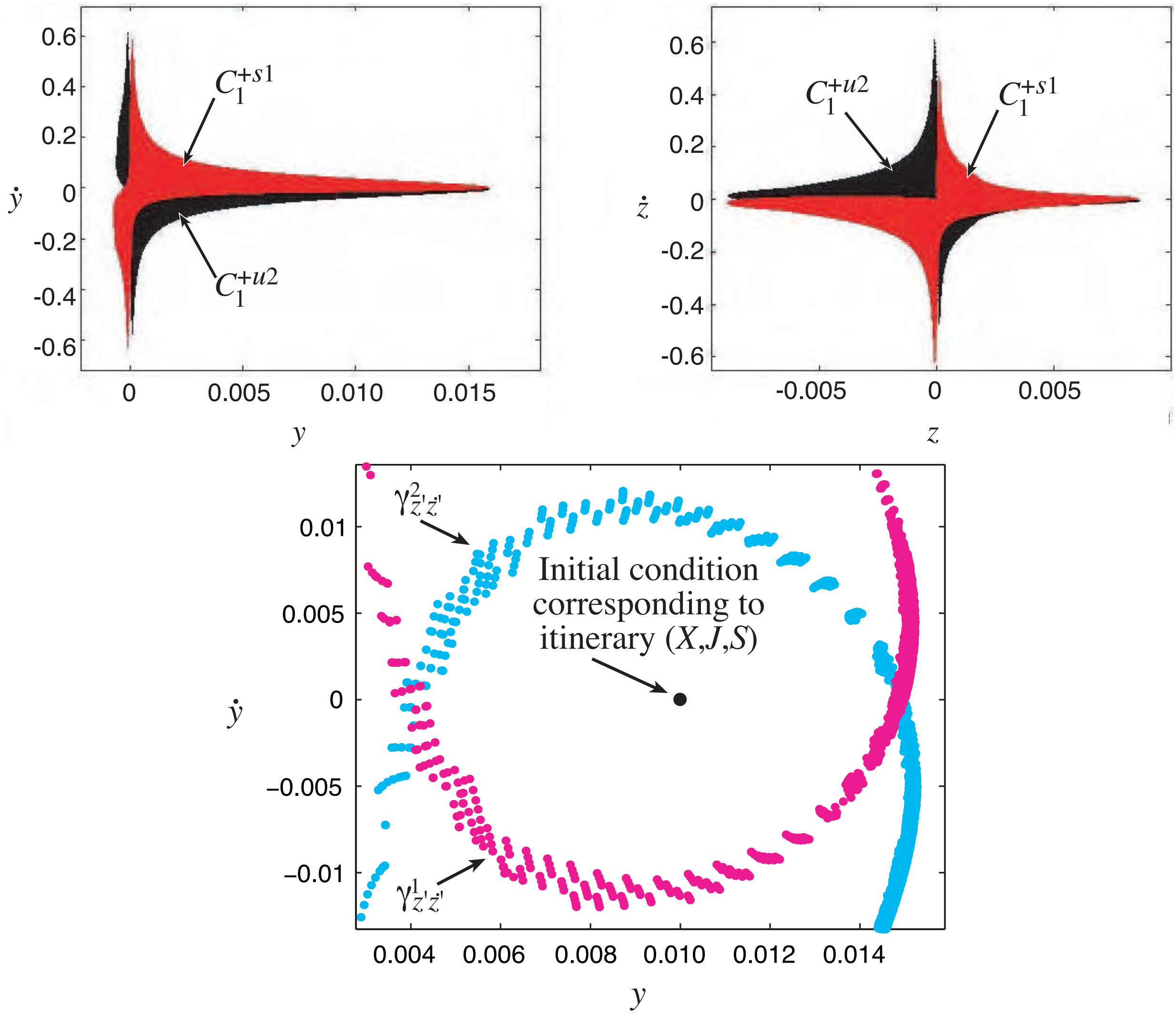
Intersection of phase volumes

- Find (X,J,S) orbit via tube intersection

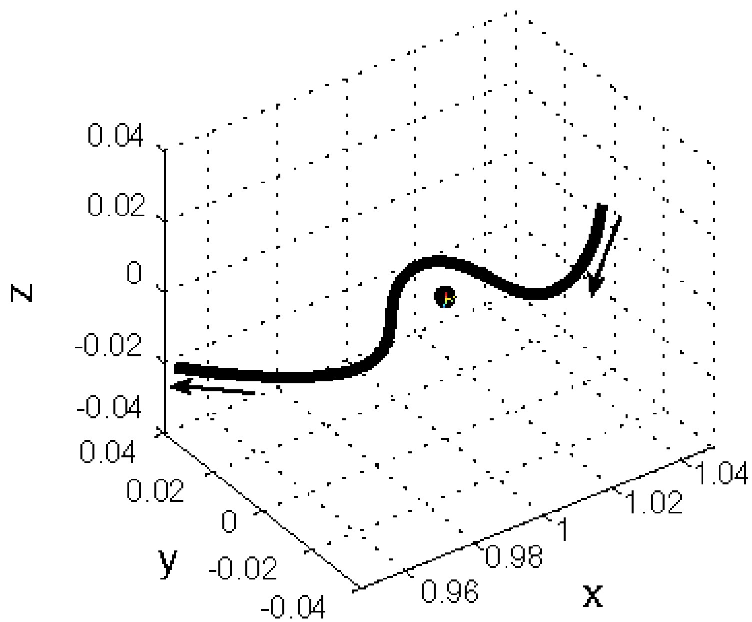


Intersection of phase volumes

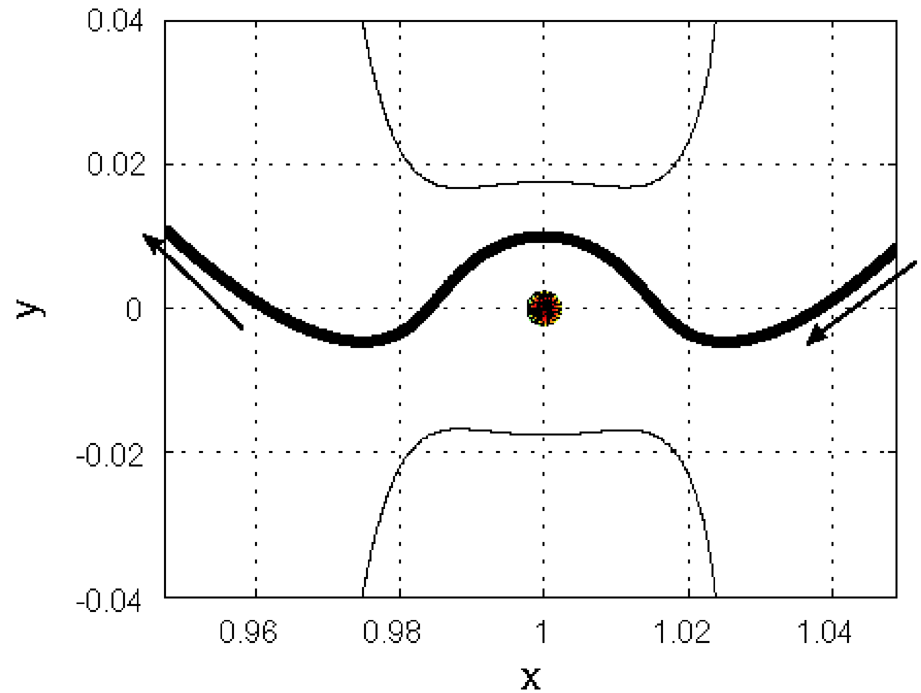
- Find (X,J,S) orbit via tube intersection



All orbits in intersection correspond to transition



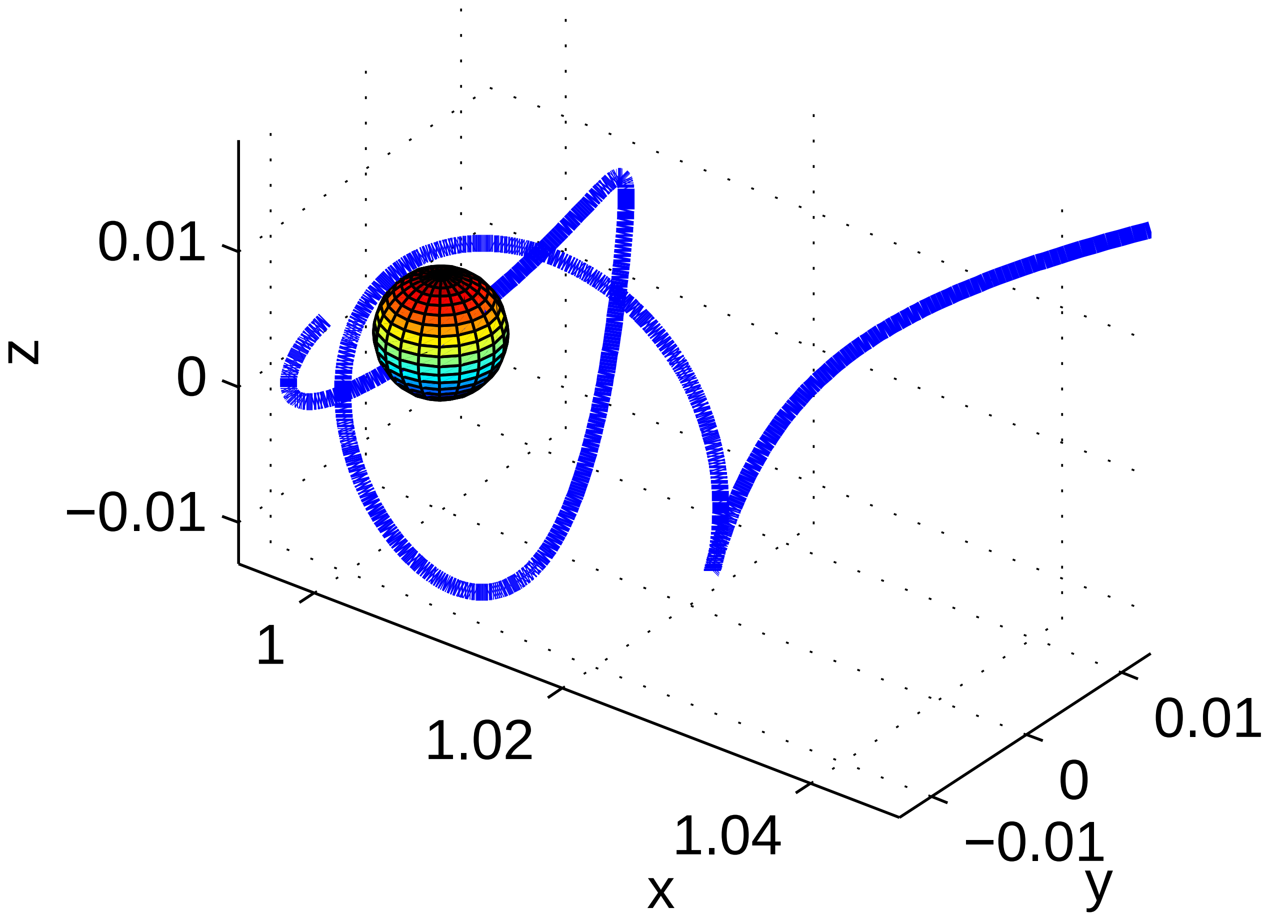
3D view



xy-plane projection

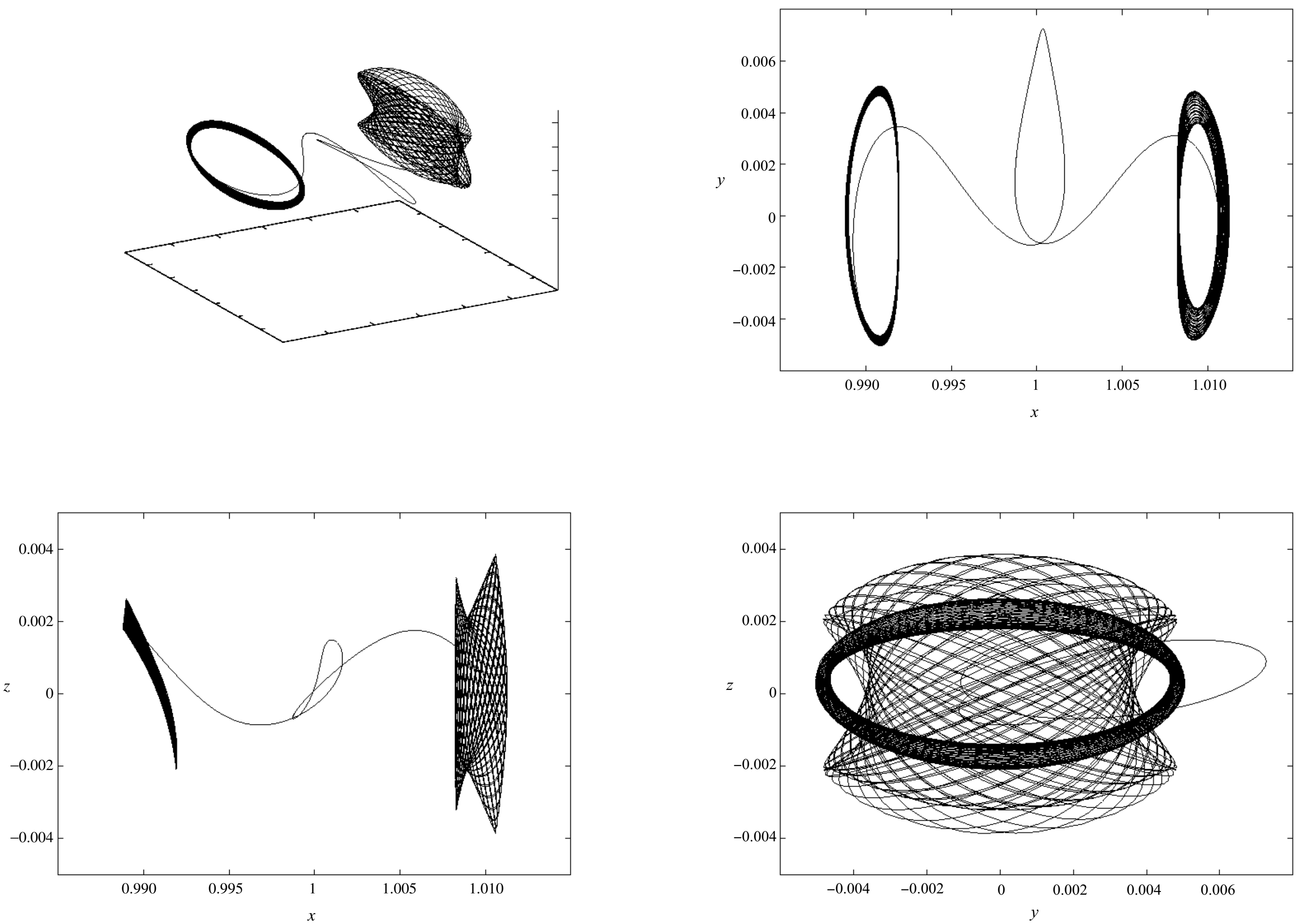
Gómez, Koon, Lo, Marsden, Masdemont, Ross, Nonlinearity [2004]

Other orbits obtained this way



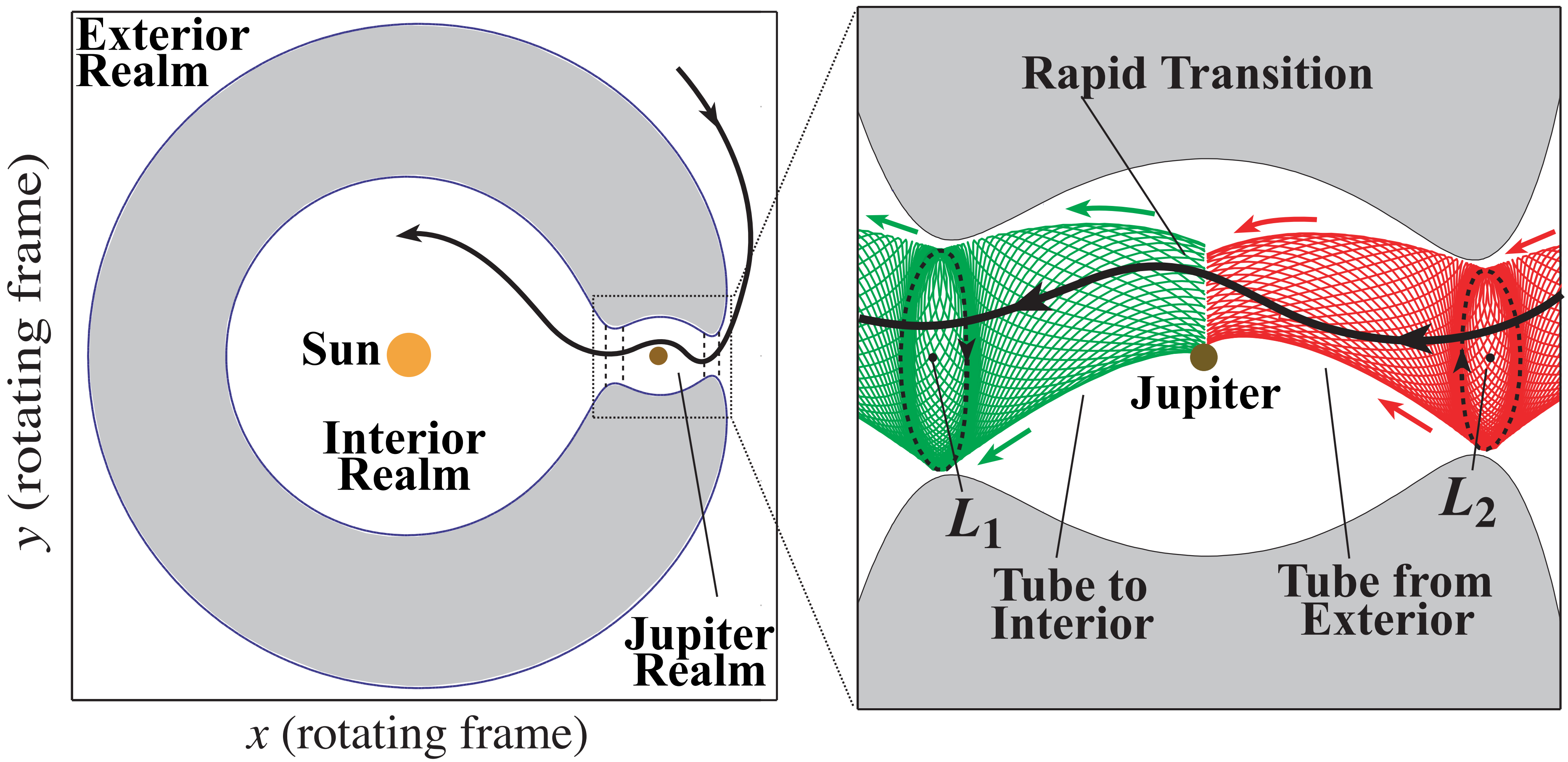
Another example

On the tubes, rather than in the tubes



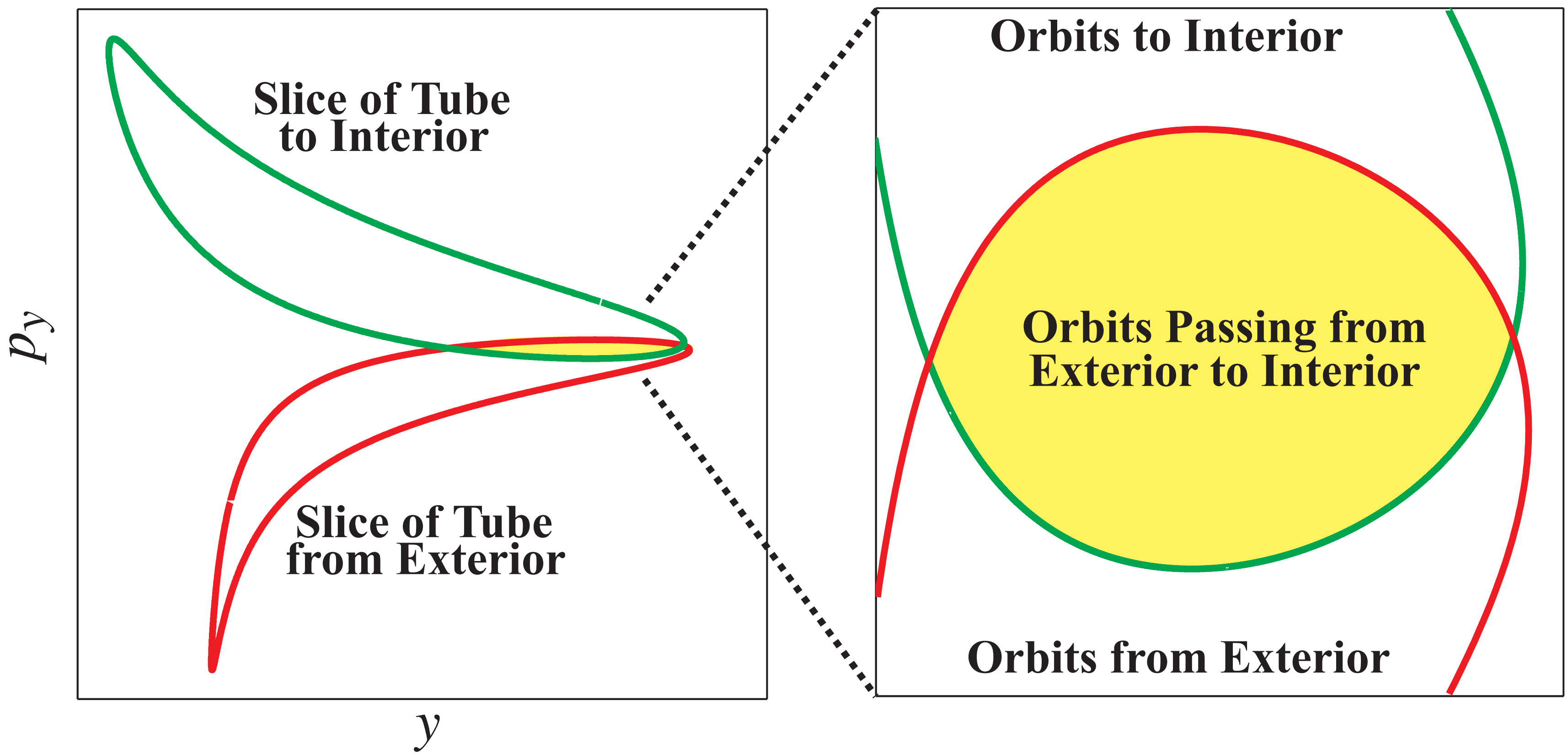
An L_1-L_2 heteroclinic connection

Transition probabilities



- Example: Comet transport between outside and inside of Jupiter (i.e., **Oterma**-like transitions)

Transition probabilities

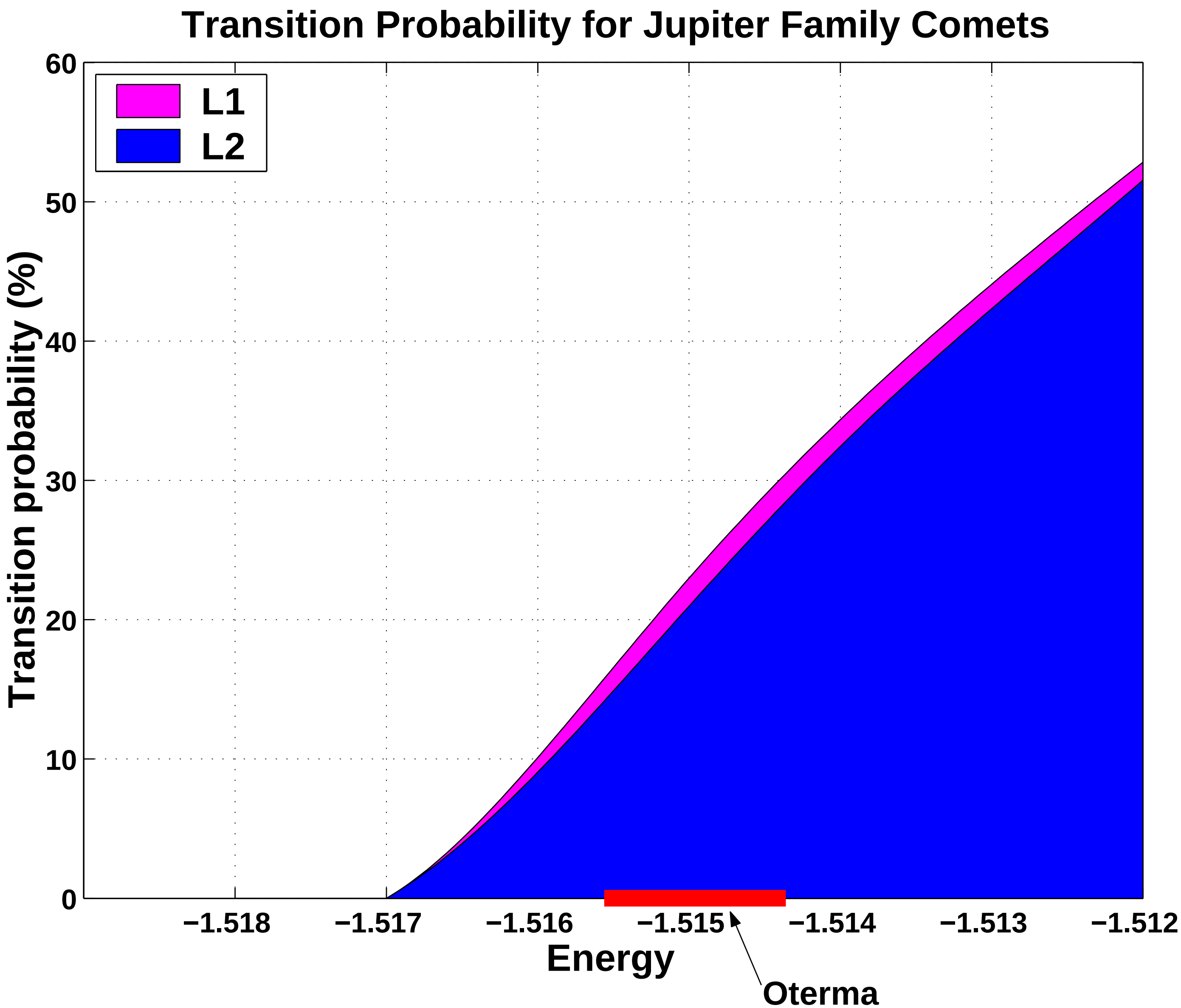


Poincaré Section

- Phase volume ratio gives the **relative probability** to pass from **outside** to **inside** Jupiter's orbit.

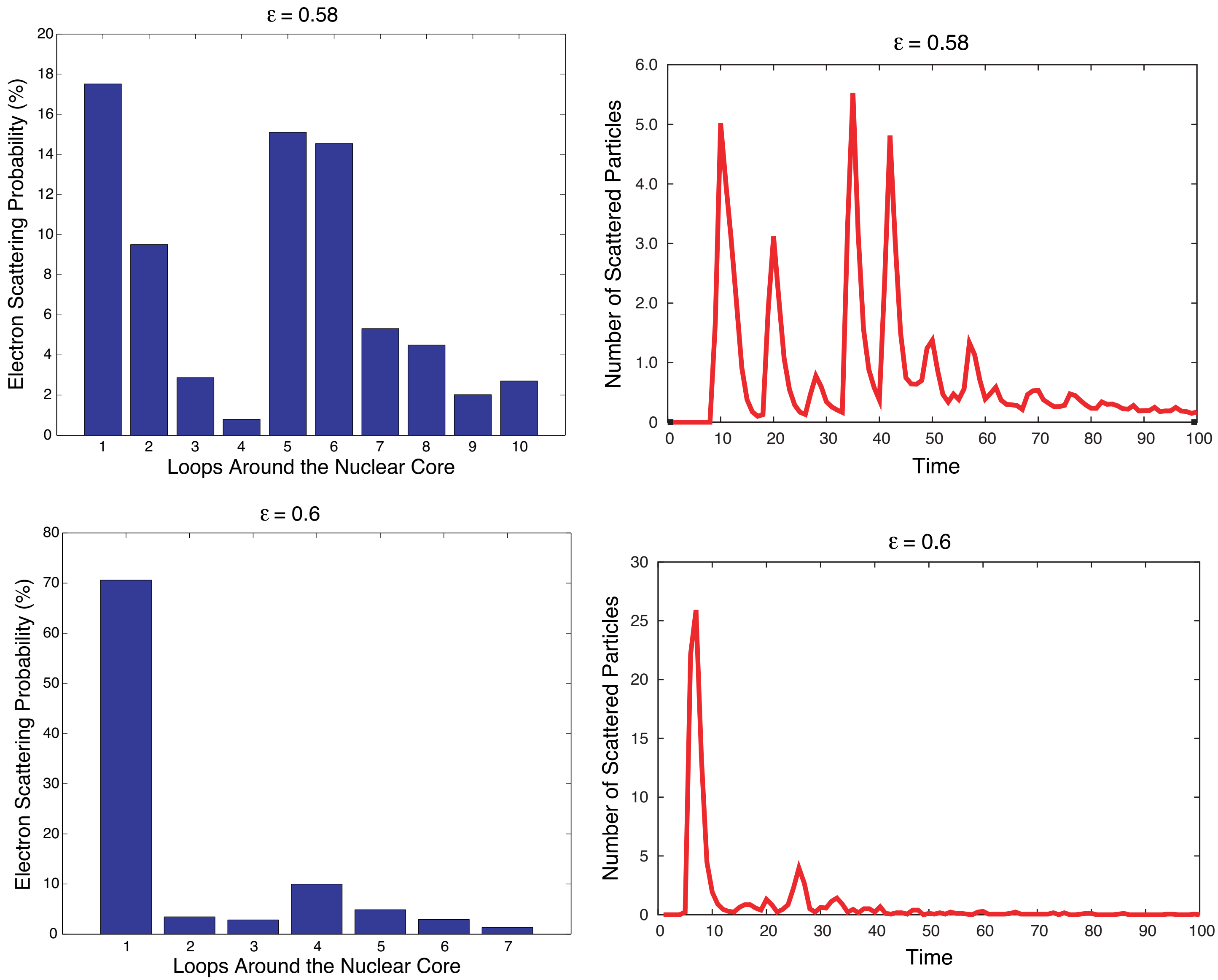
Transition probabilities

- Jupiter family comet transitions: **X** $\rightarrow$ **S**, **S** $\rightarrow$ **X**



Capture time determined by tube dynamics

- Temporary capture time profiles are structured



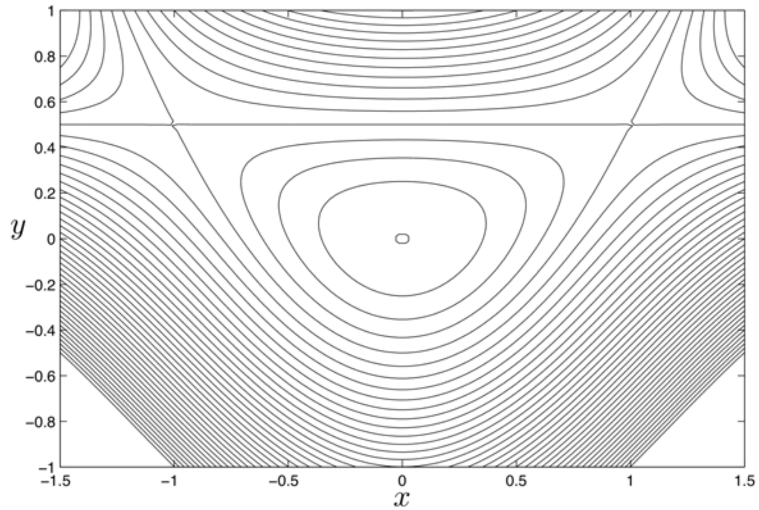
Related systems

- Results apply to similar problems in chemistry, biomechanics, **ship capsize**

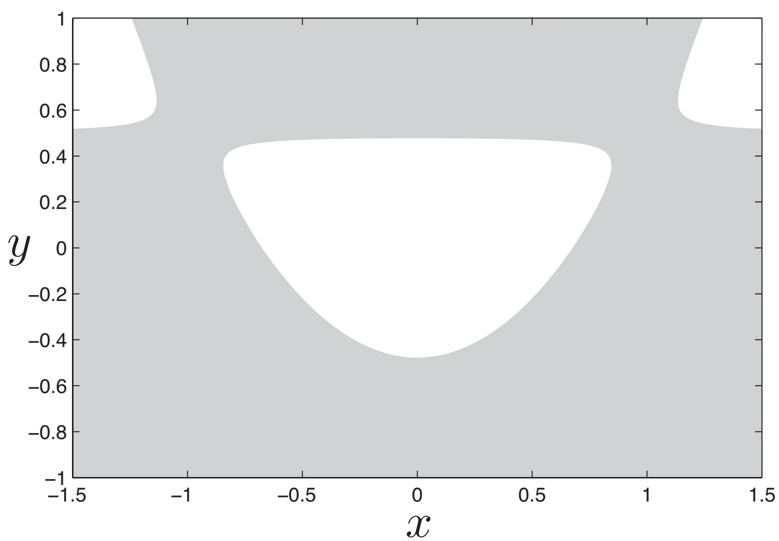
Tubes leading to capsize

- Ship motion is Hamiltonian,

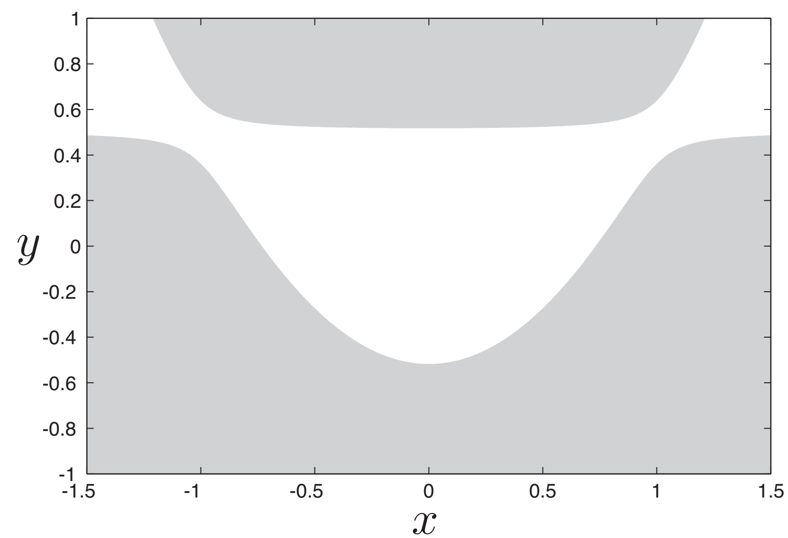
  $$H = p_x^2/2 + R^2 p_y^2/4 + V(x,y),$$



$V(x,y)$

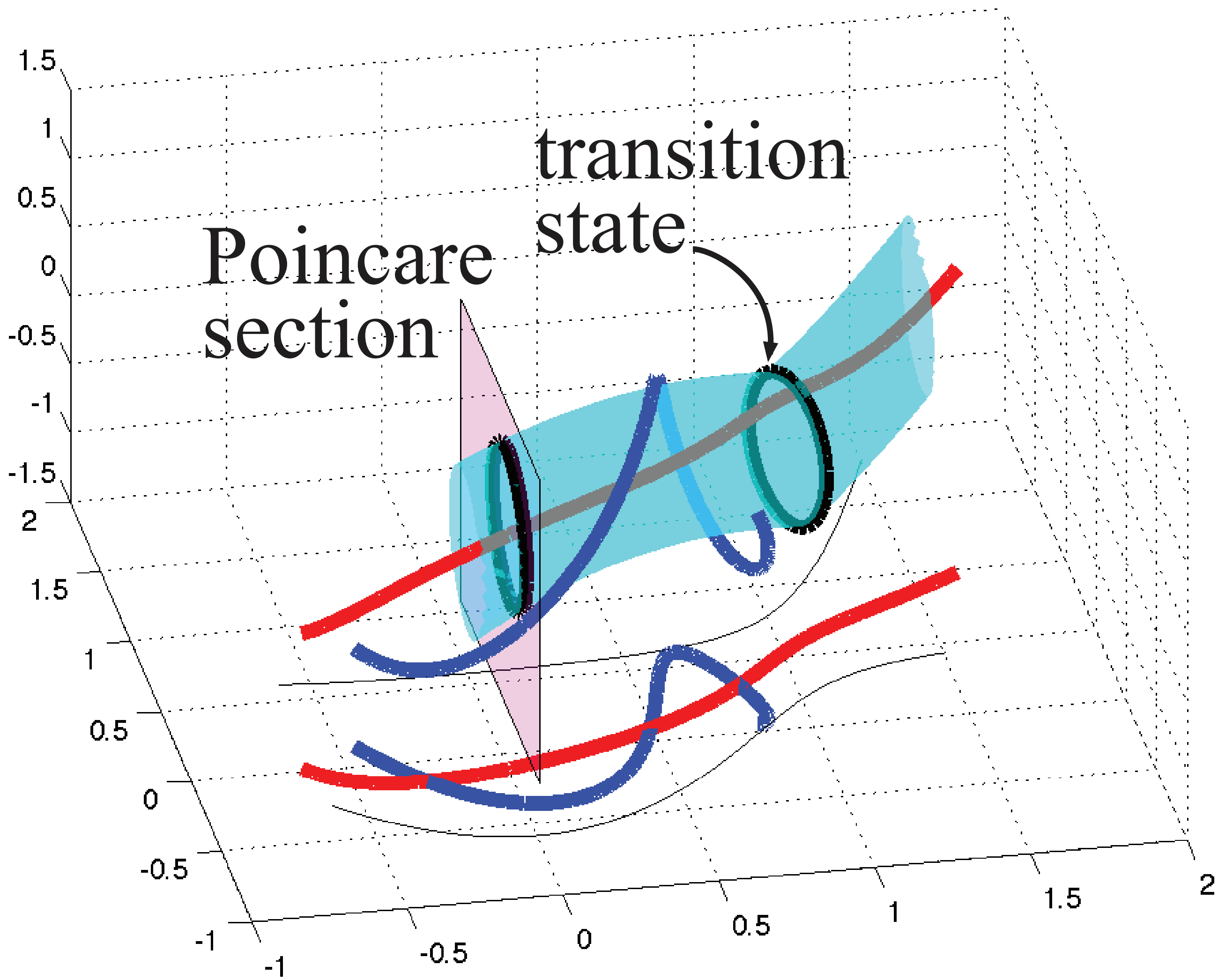


$E < E_c$



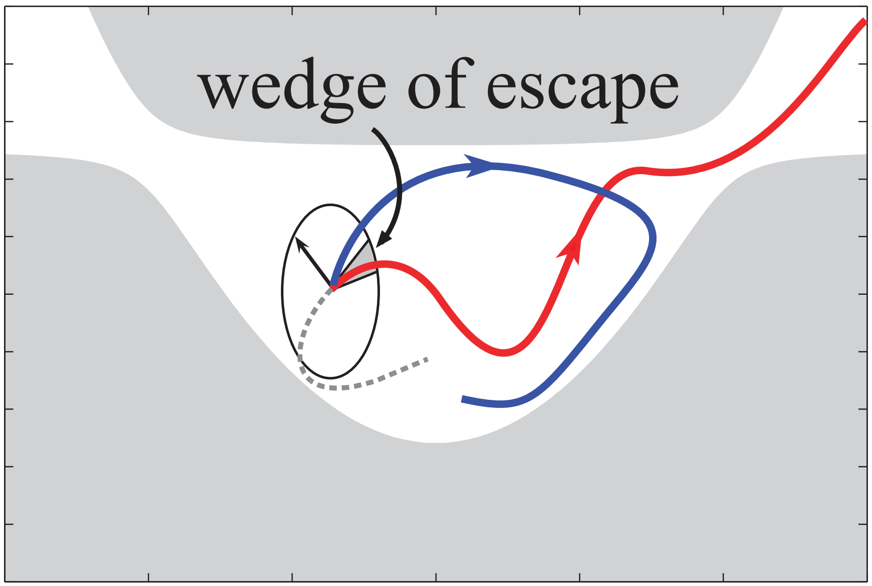
$E > E_c$

Tubes leading to capsize



Tubes leading to capsize

- Wedge of trajectories leading to imminent capsize



- Tubes are a useful paradigm for predicting capsize even in the presence of random forcing, e.g., random waves
- Could inform control schemes to avoid capsize in rough seas

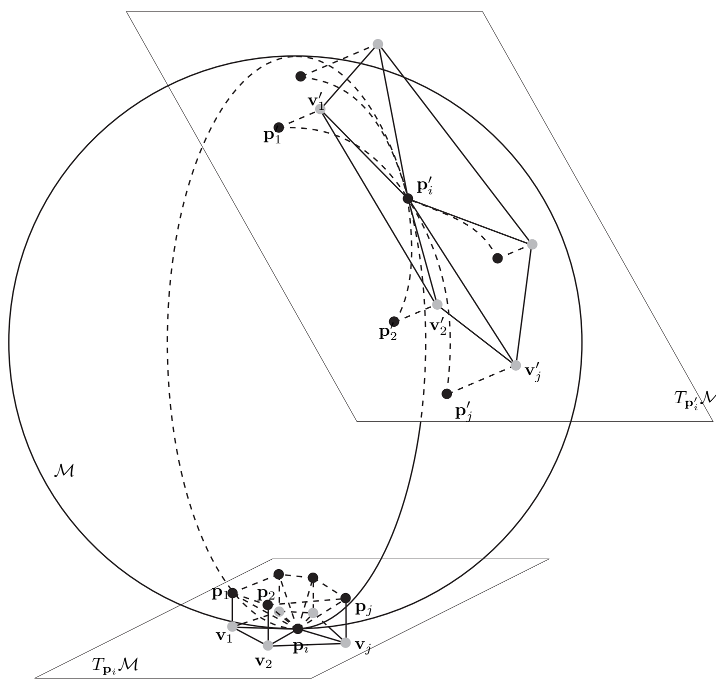
Some other transport activities inspired by Jerry

FTLE for Riemannian manifolds

- We can define the FTLE for Riemannian manifolds[3]

$$\sigma_t^T(x) = \frac{1}{|T|} \ln \left\| \mathrm{D}\phi_t^{t+T} \right\| \doteq \frac{1}{|T|} \log \left(\max_{\mathrm{y} \neq 0} \frac{\left\| \mathrm{D}\phi_t^{t+T}(\mathrm{y}) \right\|}{\|\mathrm{y}\|} \right)$$

with y a small perturbation in the tangent space at x.



[3]Lekien & Ross [2010] Chaos

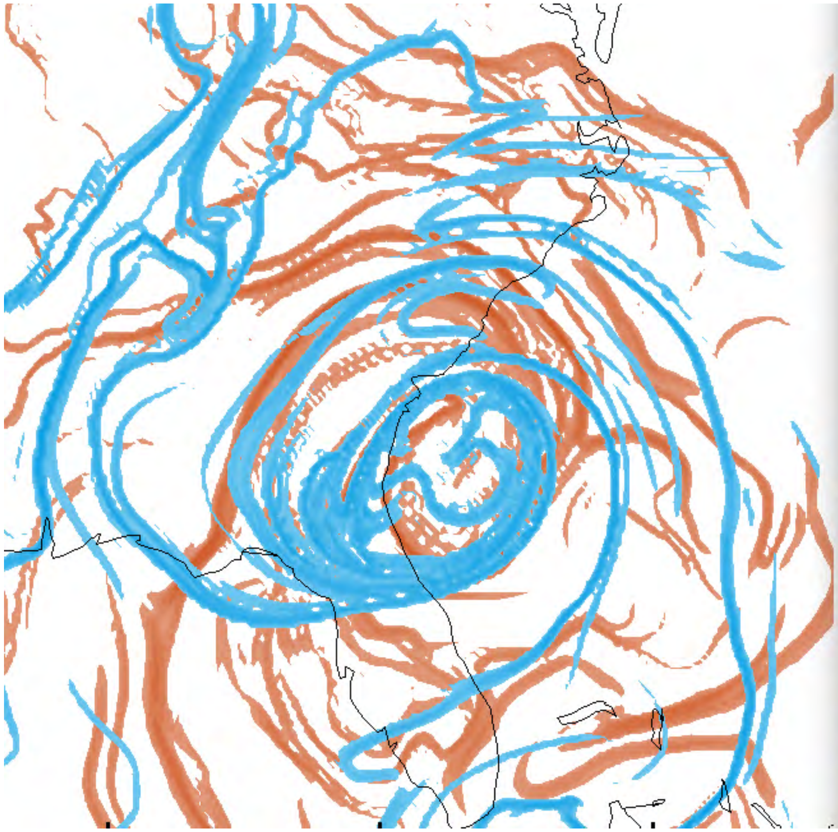
Atmospheric flows: Antarctic polar vortex

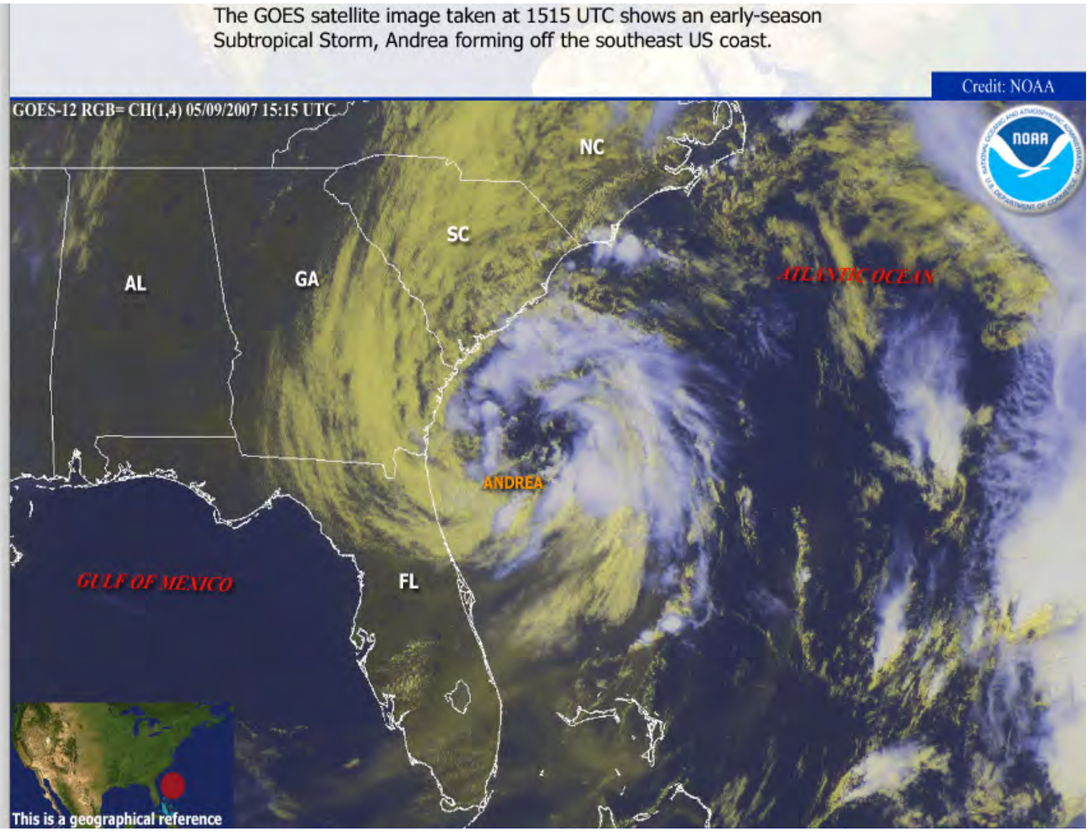
ozone data

Atmospheric flows: Antarctic polar vortex

ozone data + LCSs (red = repelling, blue = attracting)

Atmospheric flows and lobe dynamics





orange = repelling LCSs, blue = attracting LCSs satellite

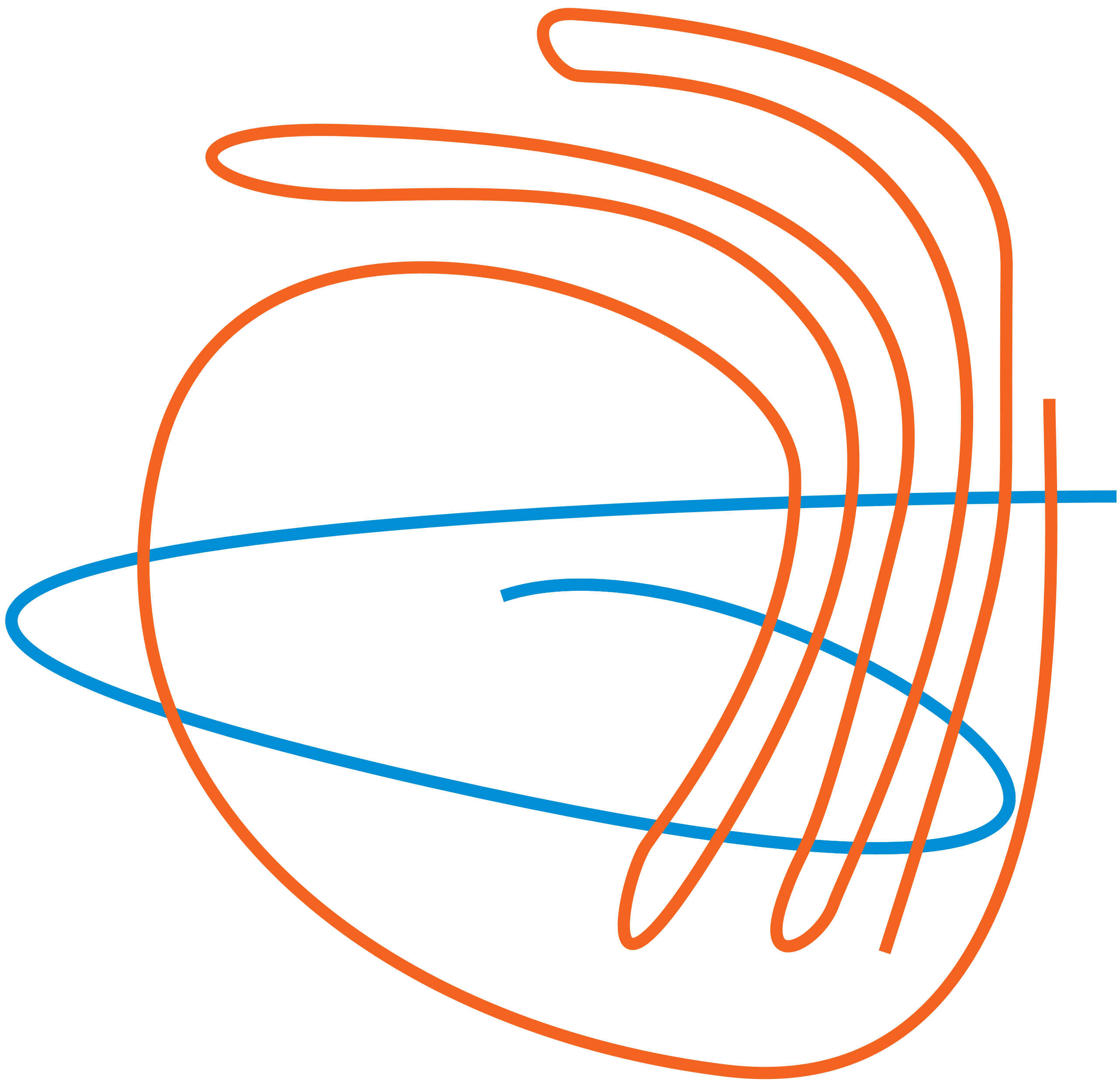
Hurricane Andrea, 2007

cf. Sapsis & Haller [2009], Du Toit & Marsden [2010], Lekien & Ross [2010], Tallapragada & Ross [2011]

Atmospheric flows and lobe dynamics

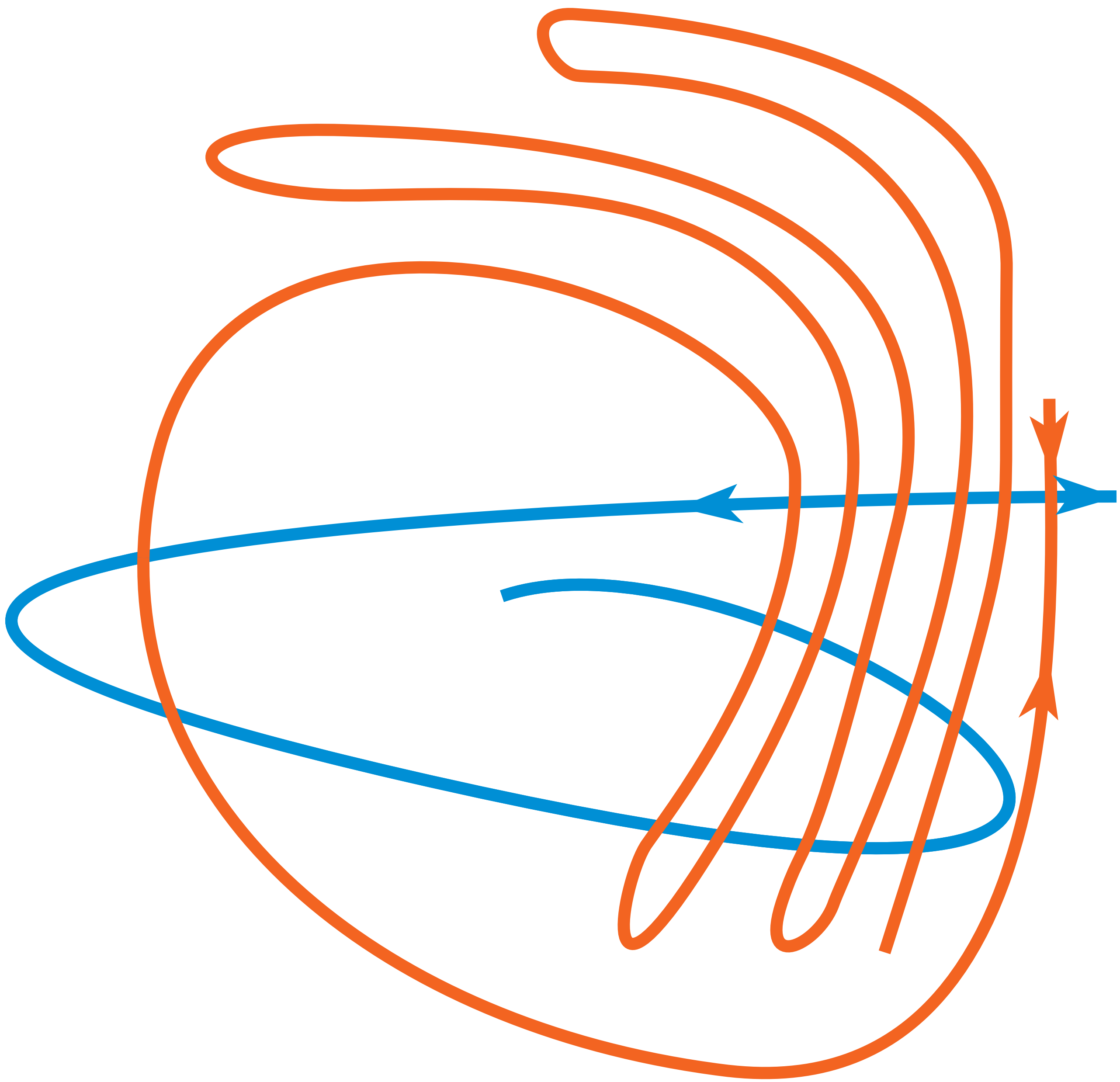
Hurricane Andrea at one snapshot; LCS shown (orange = repelling, blue = attracting)

Atmospheric flows and lobe dynamics



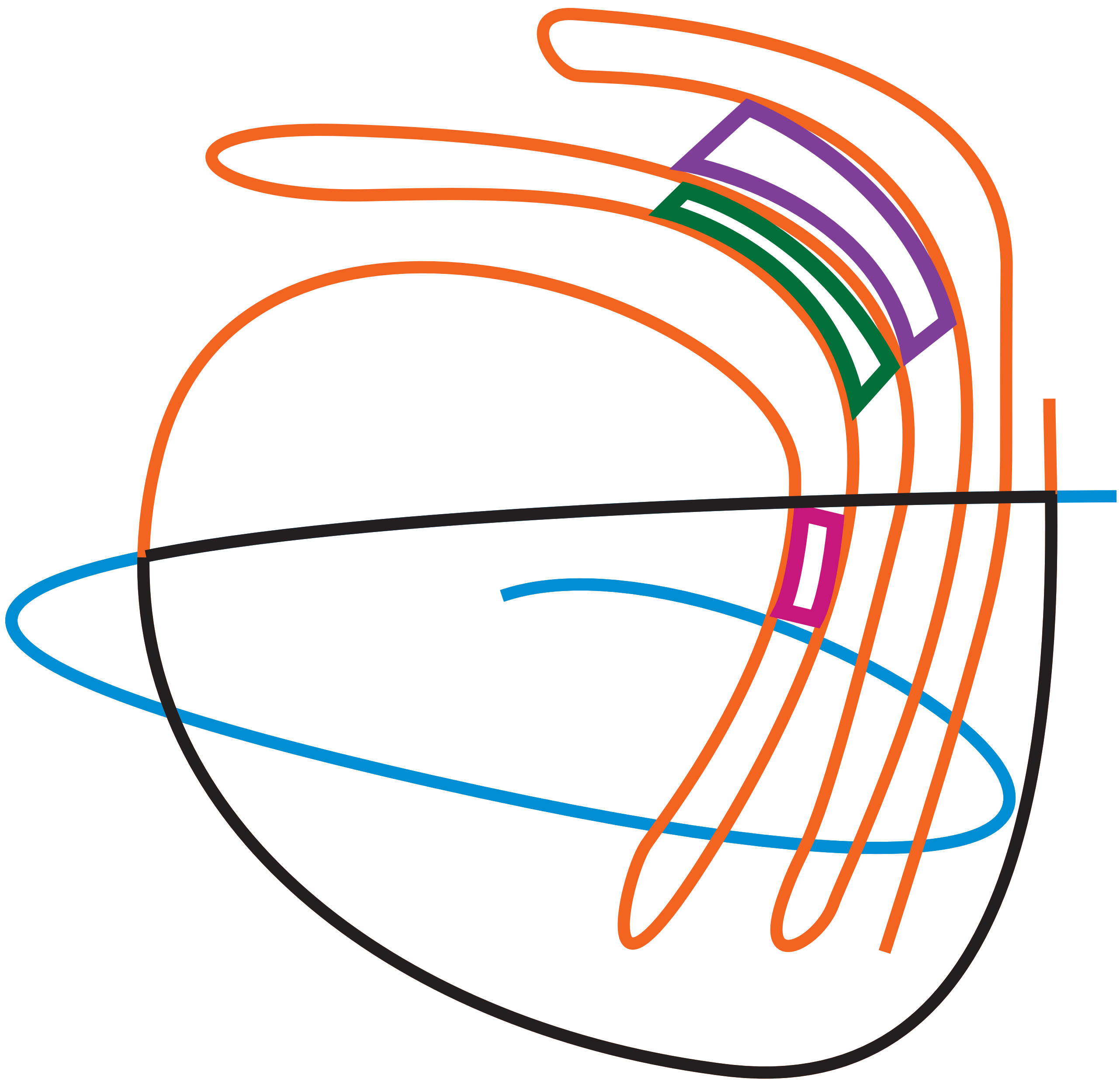
orange = repelling (stable manifold), blue = attracting (unstable manifold)

Atmospheric flows and lobe dynamics



orange = repelling (stable manifold), blue = attracting (unstable manifold)

Atmospheric flows and lobe dynamics



Portions of lobes colored; magenta = outgoing, green = incoming, purple = stays out

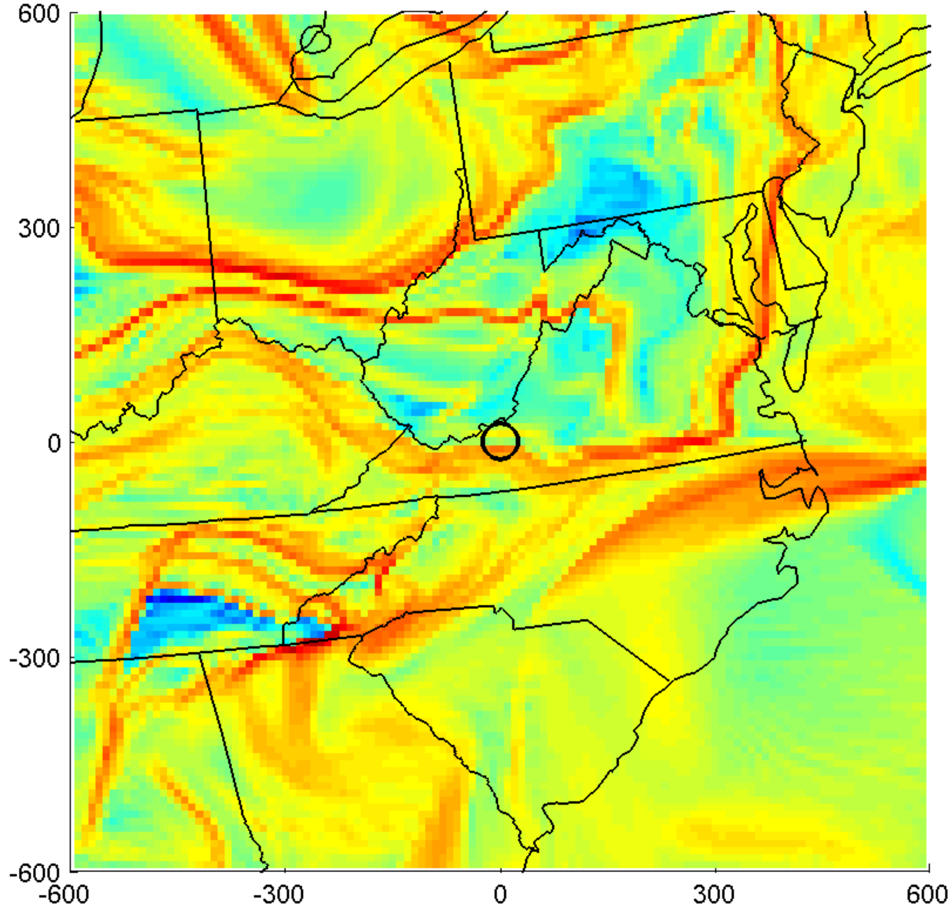
Atmospheric flows and lobe dynamics

Portions of lobes colored; magenta = outgoing, green = incoming, purple = stays out

Atmospheric flows and lobe dynamics

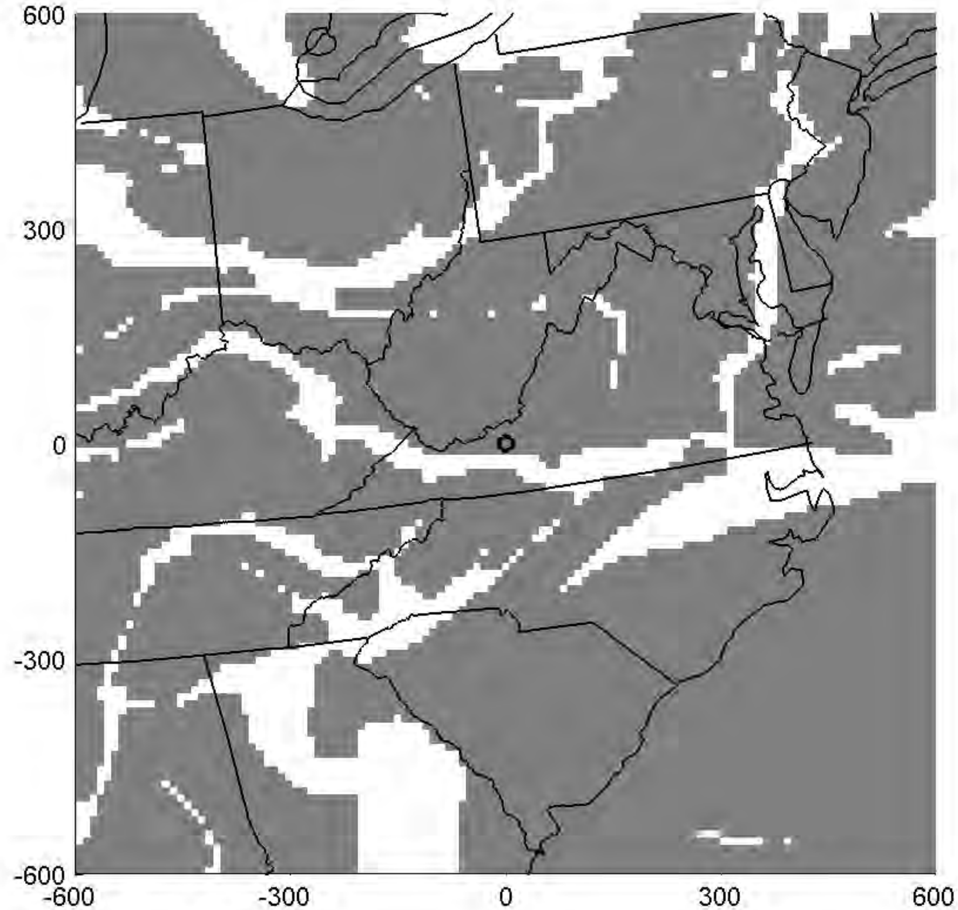
Sets behave as lobe dynamics dictates

Coherent sets and set-based definition of FTLE



- FTLE from covariance during 24 hours starting 09:00 1 May 2007

Coherent sets and set-based definition of FTLE

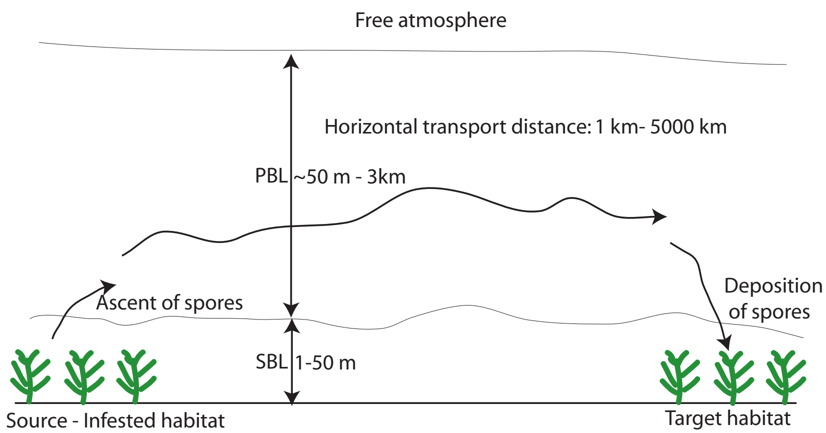


- Coherent sets during 24 hours starting 09:00 1 May 2007

Navigation in an aperiodic setting

- Selectively jumping between large air masses using control
- Moving between mobile subregions of different finite-time itineraries

Biological adaptation



Long range transport of plant pathogen spores

atmospheric superhighway

- Might organisms which travel via the atmosphere have adaptations to best take advantage of the "atmospheric superhighway"?

Final words on geometry of transport

- **Invariant manifold** and invariant manifold-like structures are related to transport; form template or skeleton

- In Hamiltonian systems with rank-1 saddles:
  - **Tube dynamics**: the interior of tube manifolds
    — related to capture, escape, transition, collision
    — applications to orbital mechanics, ship capsize, ...

- In the atmosphere:
  - **Lagrangian coherent structures**
    — the skeleton of air
    — boundaries between air masses
    — link with set-oriented and topological methods

The End — Thank you!

Thanks to: Phanindra Tallapragada, Carmine Senatore, Piyush Grover, David Schmale, Daniel Scheeres, Francois Lekien, Mark Stremler

For papers, movies, etc., visit:
www.shaneross.com

Some Papers:

- Schmale, Ross, Fetters, Tallapragada, Wood-Jones, Dingus [2011] Isolates of *Fusarium graminearum* collected 40-320 meters above ground level cause Fusarium head blight in wheat and produce trichothecene mycotoxins. *Aerobiologia*, published online.
- Tallapragada & Ross [2011] A geometric and probabilistic description of coherent sets. Submitted preprint.
- Lekien & Ross [2010] The computation of finite-time Lyapunov exponents on unstructured meshes and for non-Euclidean manifolds. *Chaos* 20, 017505.
- Marsden & Ross [2006] New methods in celestial mechanics and mission design. *Bulletin of the American Mathematical Society*, 43(1), 43.
- Koon, Lo, Marsden, Ross [2000] Heteroclinic connections between periodic orbits and resonance transitions in celestial mechanics. *Chaos* 10, 427.

FREE Book

Book available:

Dynamical systems, the three-body problem, and space mission design
Koon, Lo, Marsden, Ross

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