Escape from potential wells in multi-dimensional systems: experiments and partial control

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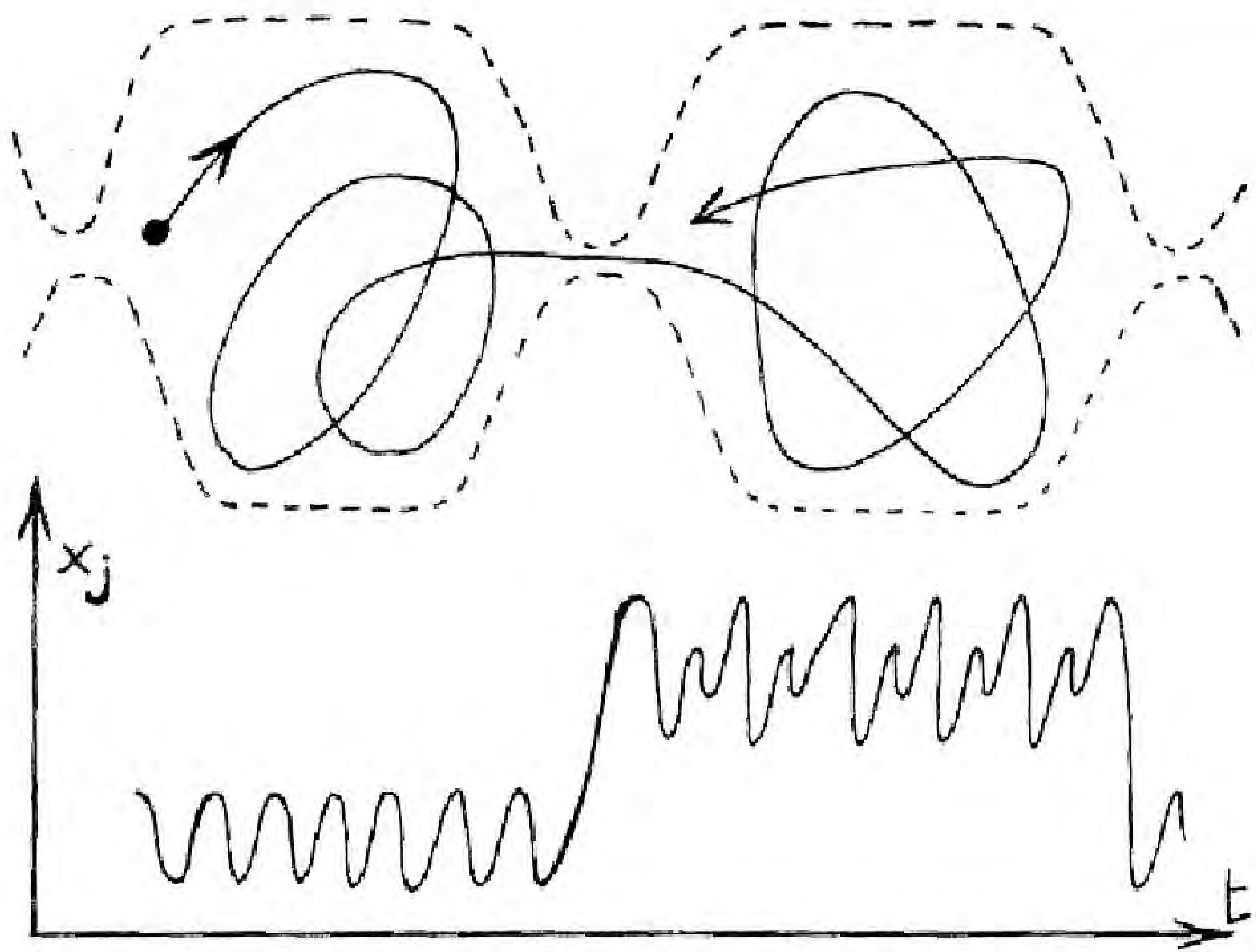
ESM 1908 NSF MultiSTEPS Virginia Tech 1872

Intermittency and chaotic transitions

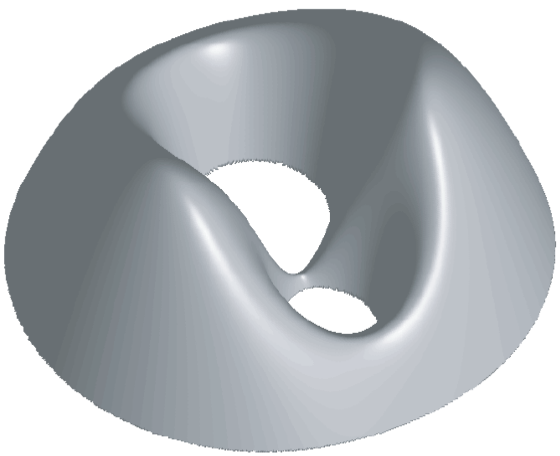
e.g., transitioning across “bottlenecks” in phase space; ‘metastability’

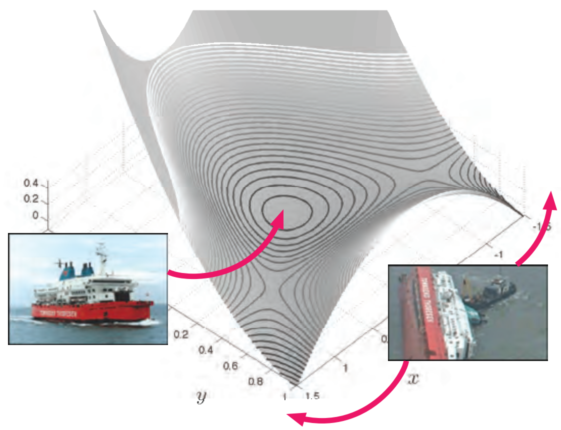


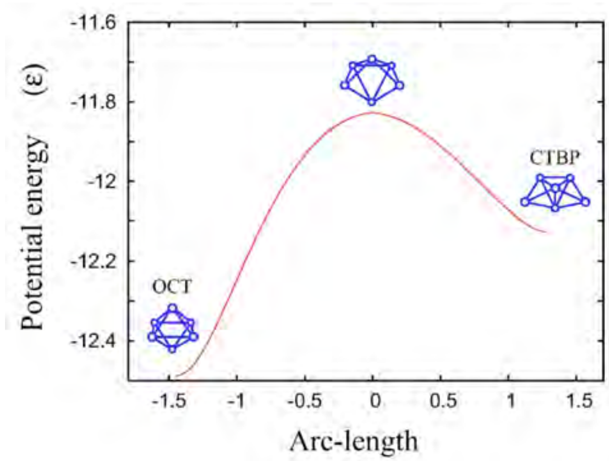
Marchal [1990]

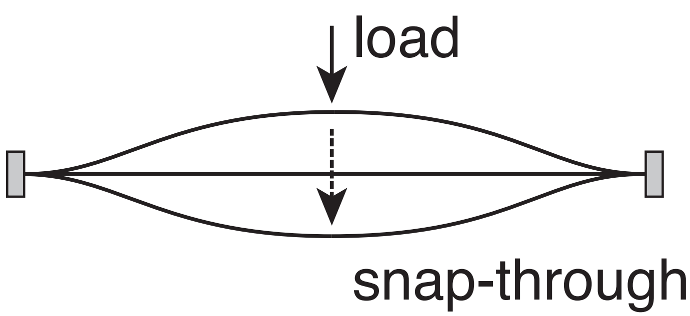
Multi-well multi-degree of freedom systems

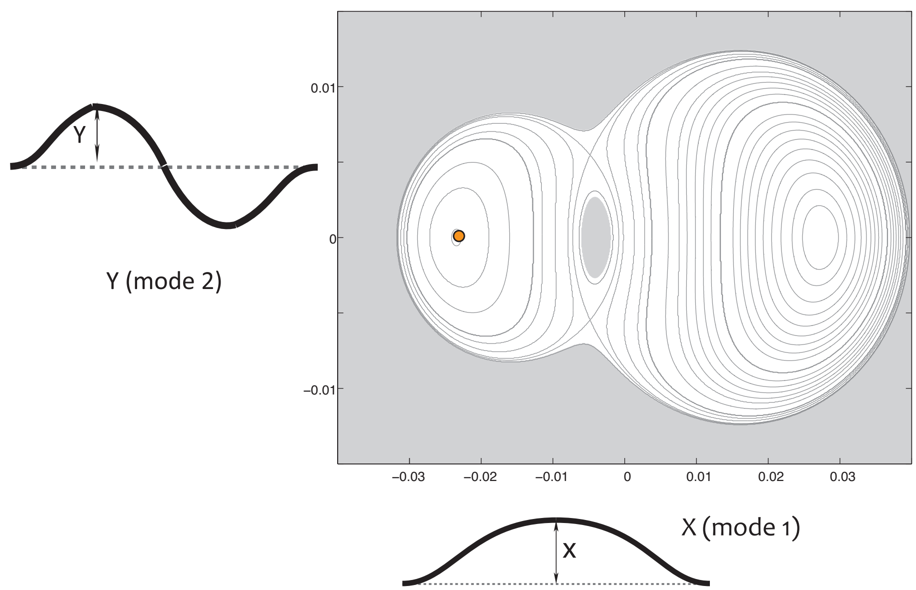
- Examples: chemistry, vehicle dynamics, structural mechanics



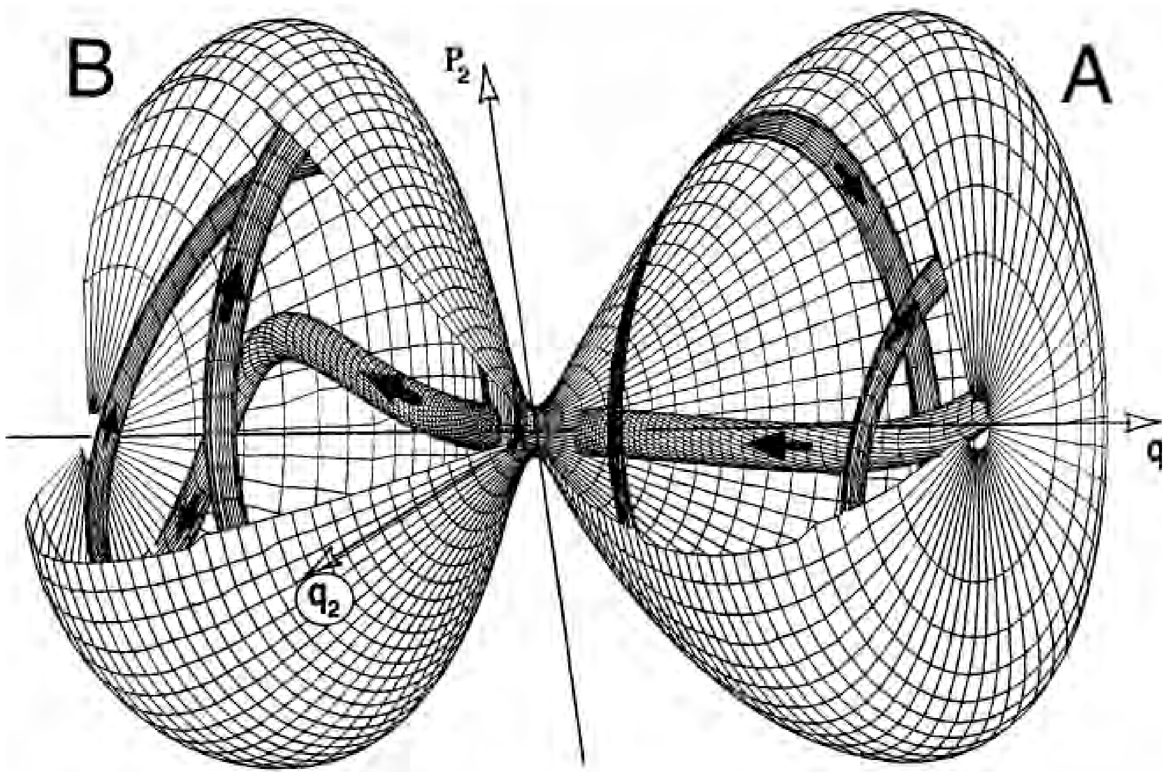








Transitions through bottlenecks via tubes



Topper [1997]

- Wells connected by phase space **transition tubes** $\simeq S^1 \times \mathbb{R}$ for 2 DOF
  — Conley, McGehee, 1960s
  — Llibre, Martínez, Simó, Pollack, Child, 1980s
  — De Leon, Mehta, Topper, Jaffé, Farrelly, Uzer, MacKay, 1990s
  — Gómez, Koon, Lo, Marsden, Masdemont, Ross, Yanao, 2000s

Motion near saddles

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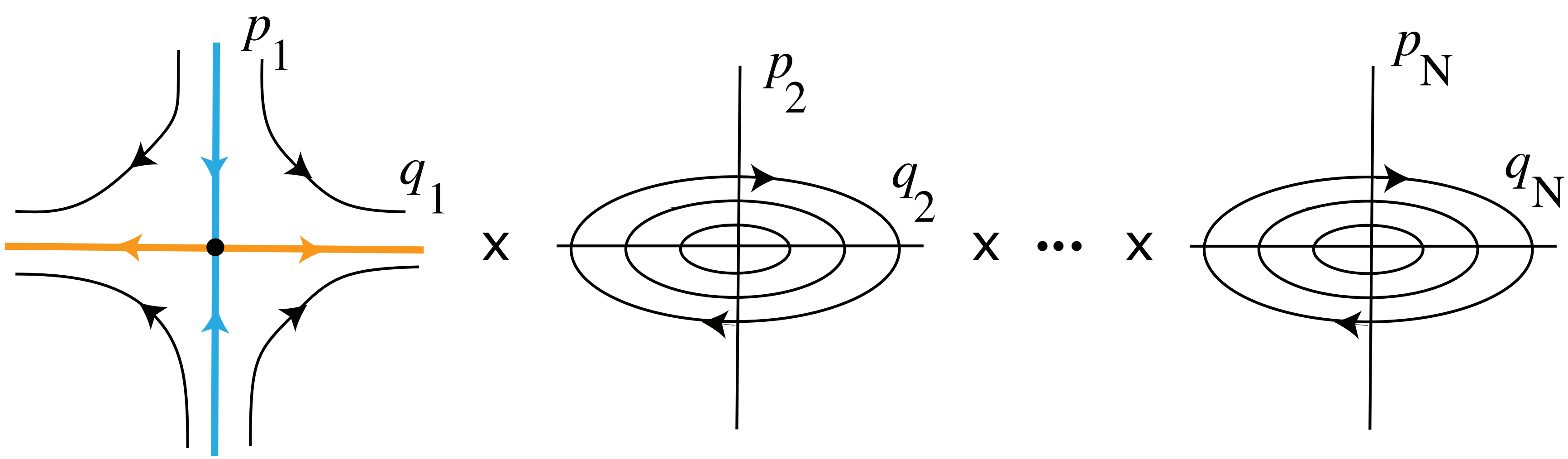
- Equilibrium point is of type saddle $\times$ center $\times \cdots \times$ center ($N-1$ centers).



the saddle-space projection and $N-1$ center projections — the N canonical planes

Motion near saddles

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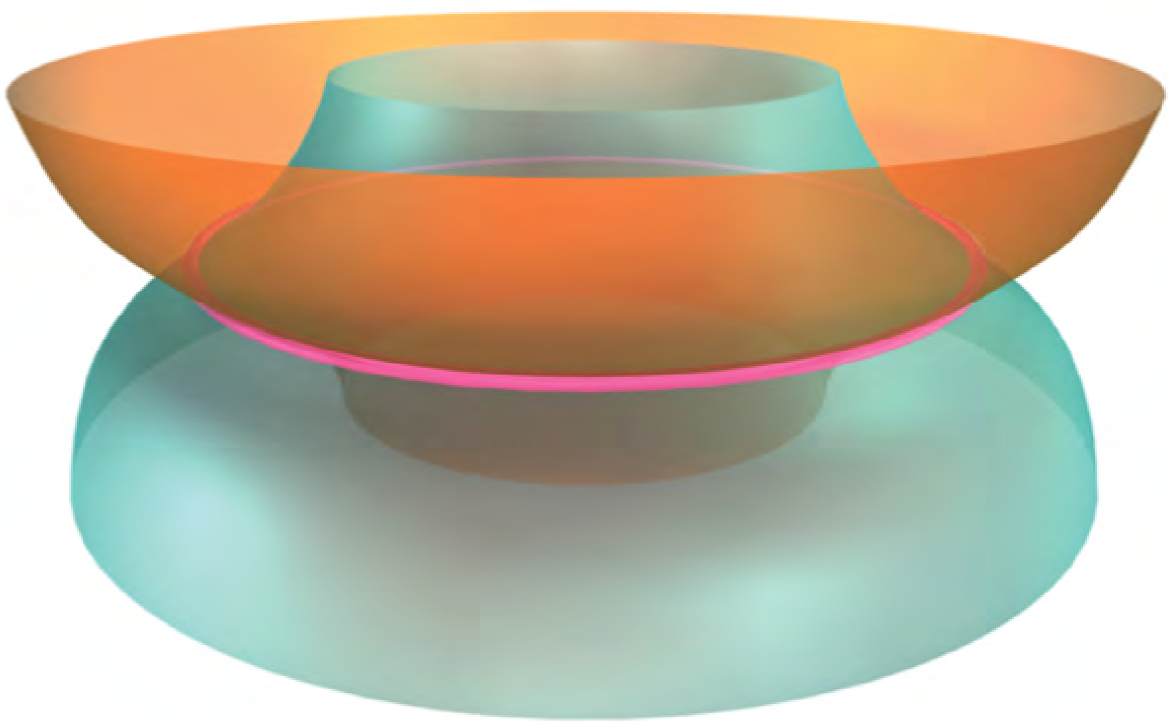
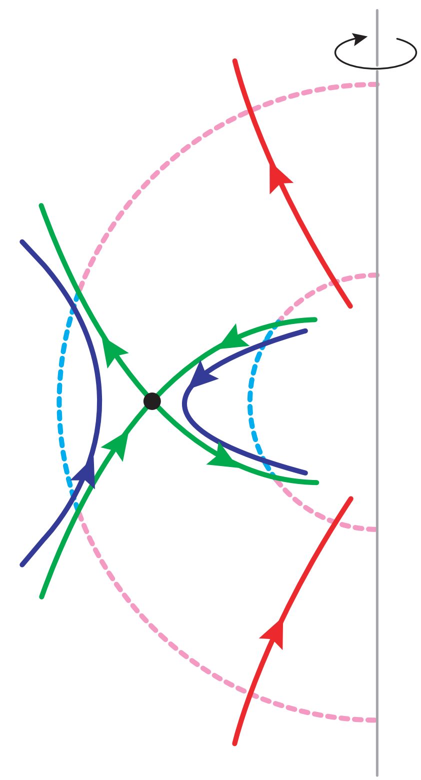
$\mathcal{M}_{\Delta E} \simeq S^1$, a periodic orbit of period $T_{\text{po}} = 2\pi/\omega$

Motion near saddles: 2 DOF

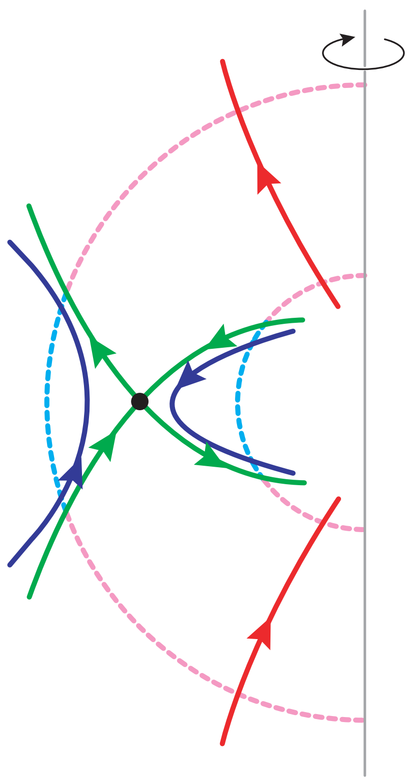
- Cylindrical **tubes** of orbits asymptotic to $\mathcal{M}_{\Delta E}$: stable and unstable invariant manifolds, $W^s_{\pm}(\mathcal{M}_{\Delta E}), W^u_{\pm}(\mathcal{M}_{\Delta E})$, $\simeq S^1 \times \mathbb{R}$
- Enclose transitioning trajectories

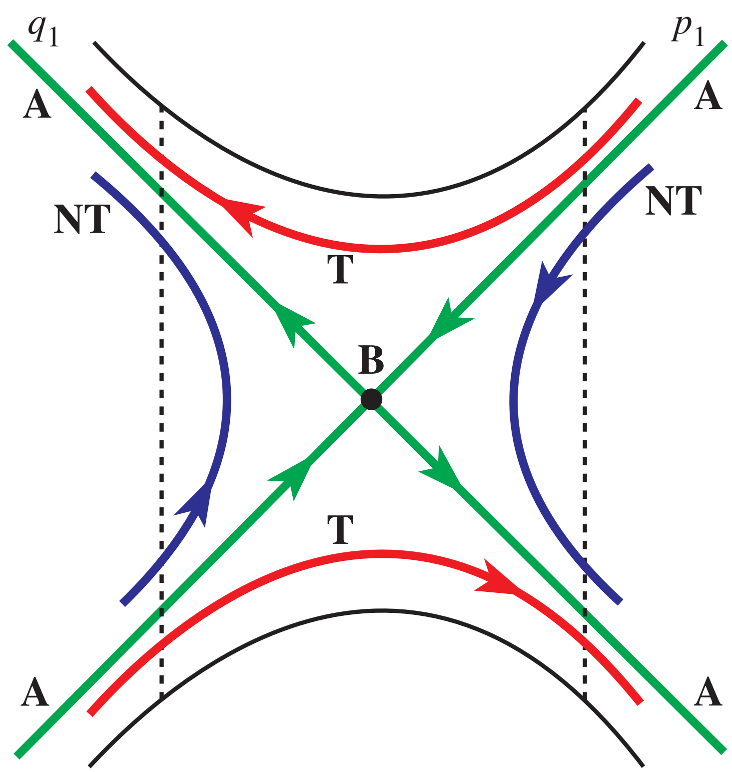




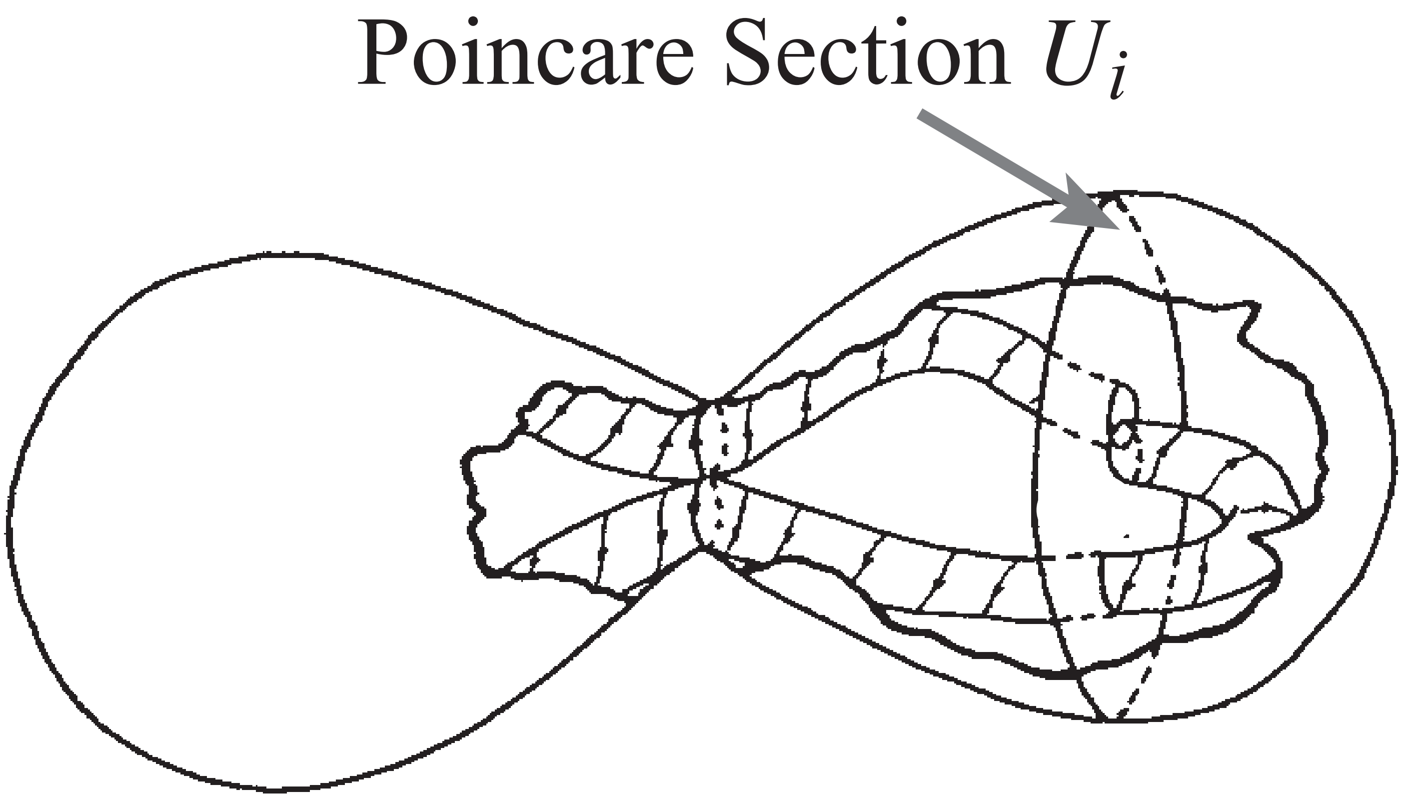
Motion near saddles: 2 DOF

- **B** : **bounded orbits** (periodic): S^1
- **A** : **asymptotic orbits** to 1-sphere: $S^1 \times \mathbb{R}$ **(tubes)**
- **T** : transitioning and **NT** : non-transitioning orbits.





Tube dynamics — global picture



De Leon [1992]

- **Tube dynamics:** All transitioning motion between wells connected by bottlenecks must occur through tube
  - Imminent transition regions, transitioning fractions
  - Consider k Poincaré sections U_i, various excess energies ΔE

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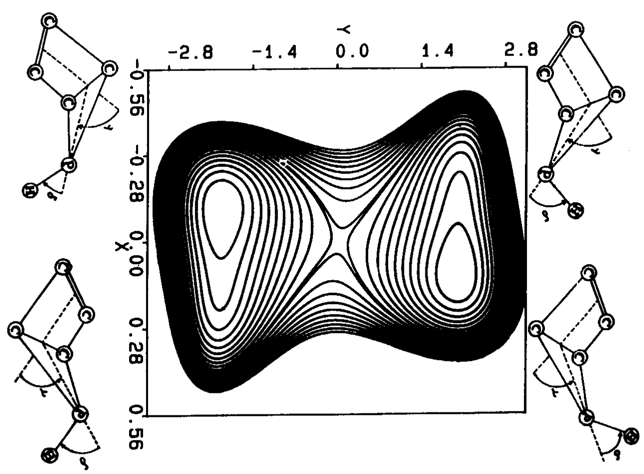

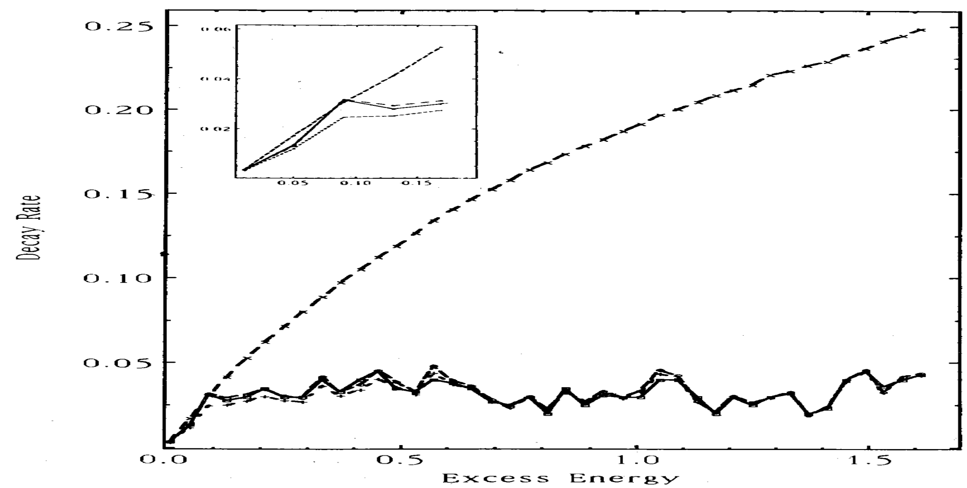

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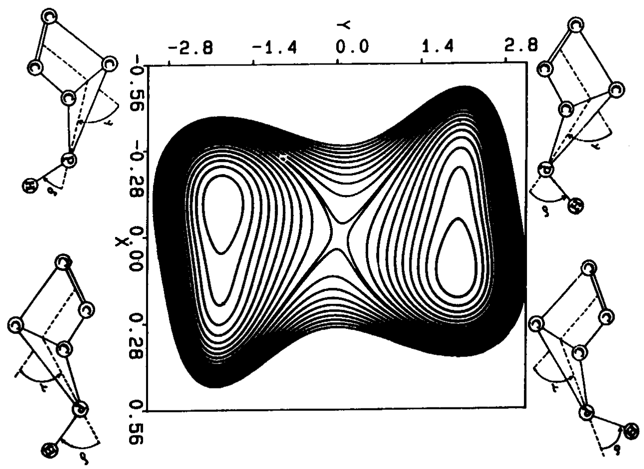
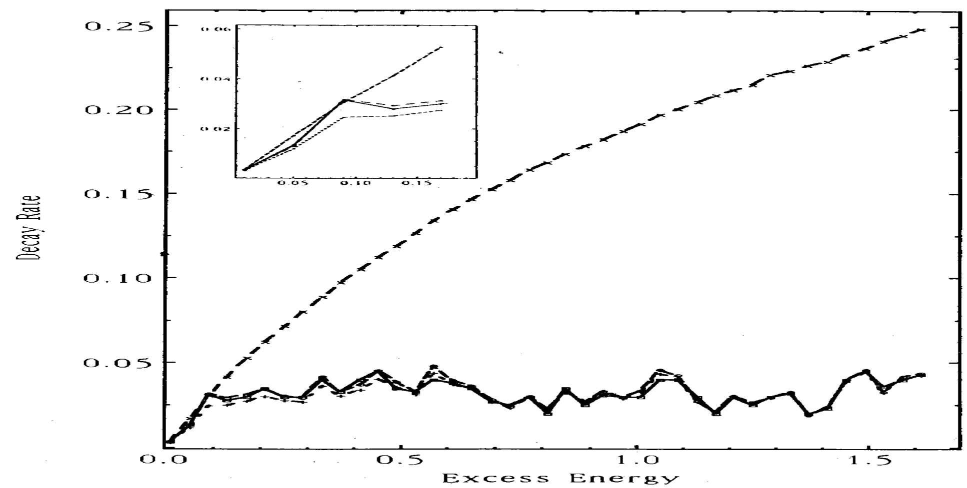
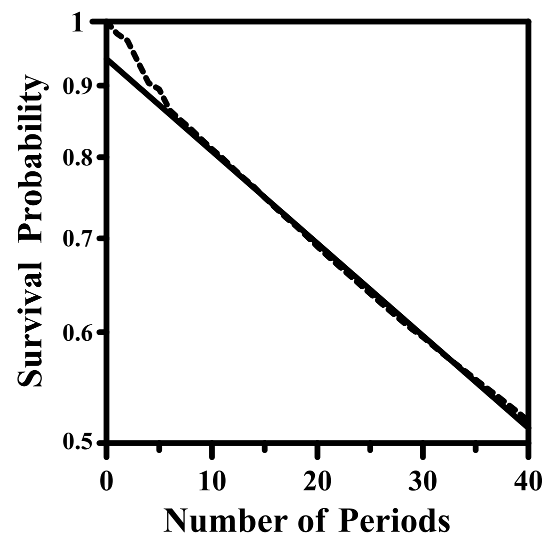




  - — celestial mechanics, asteroid escape rates e.g., Jaffé, Ross, Lo, Marsden, Farrelly, Uzer [2002]



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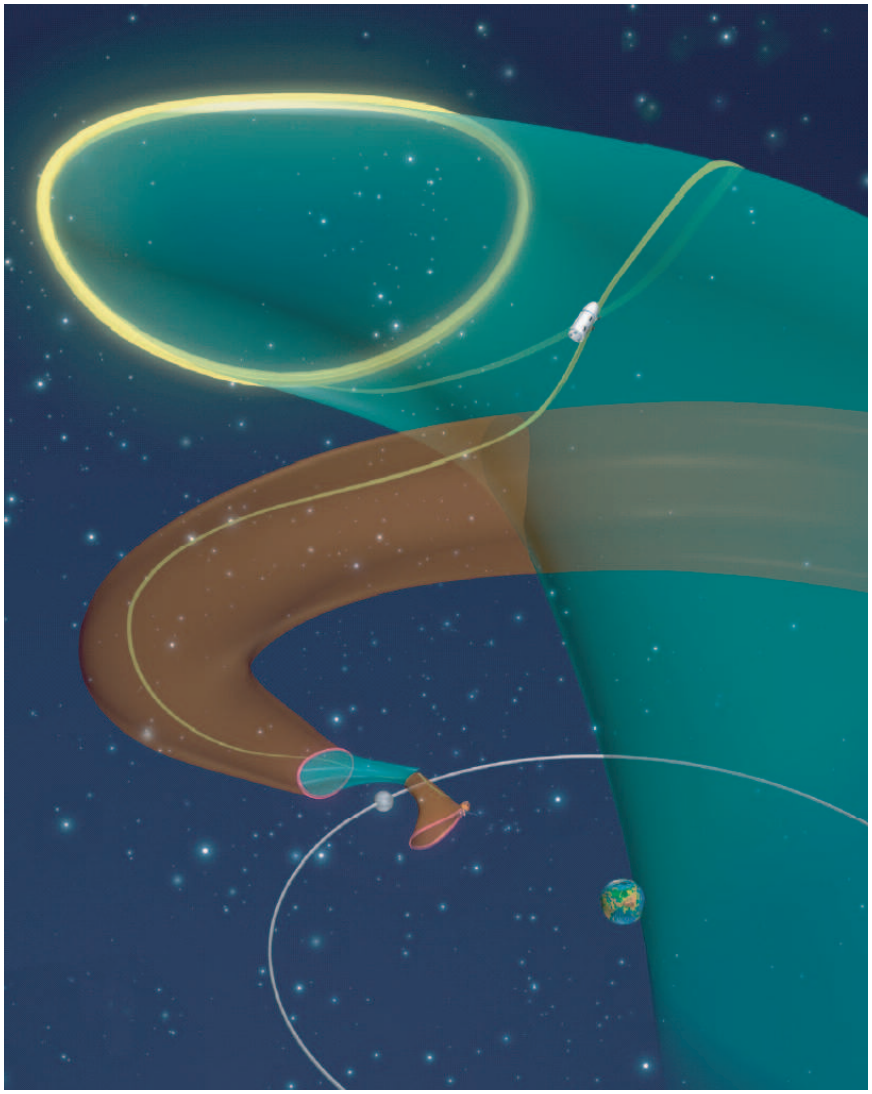

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- **Structural mechanics**
  — re-configurable deformation of flexible objects

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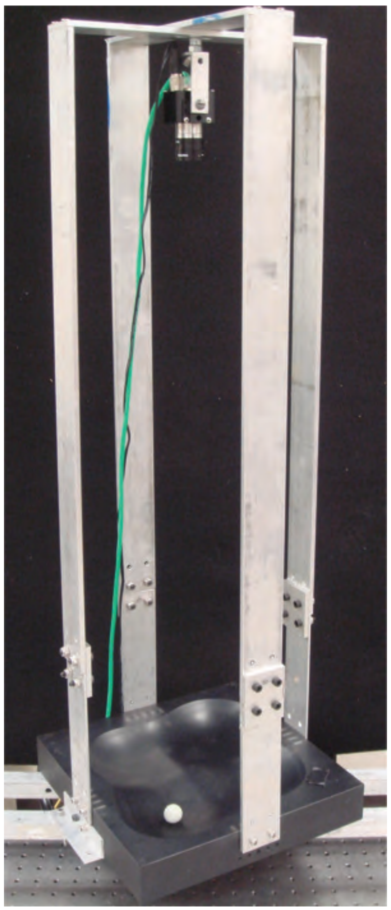
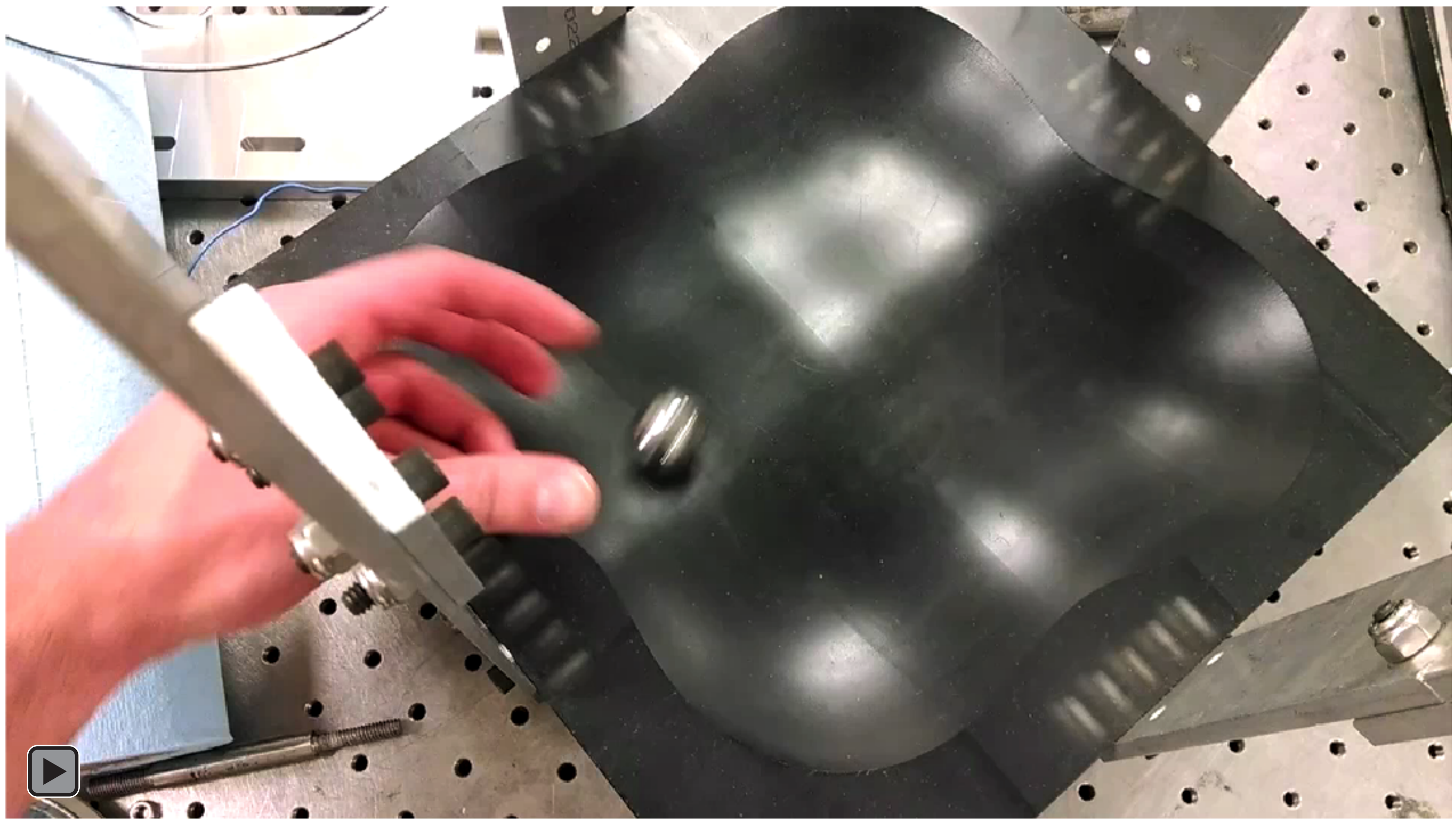


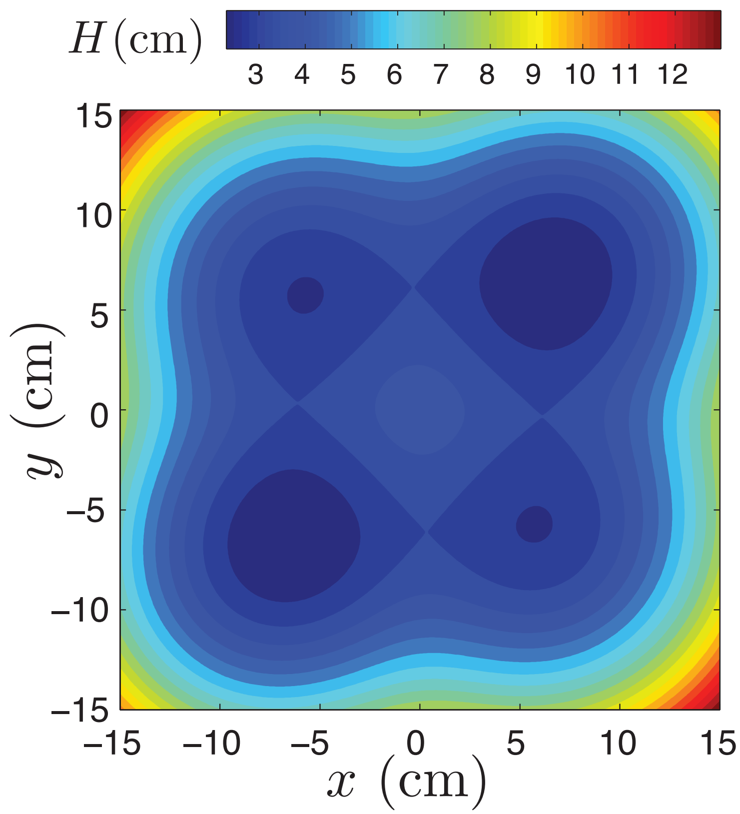


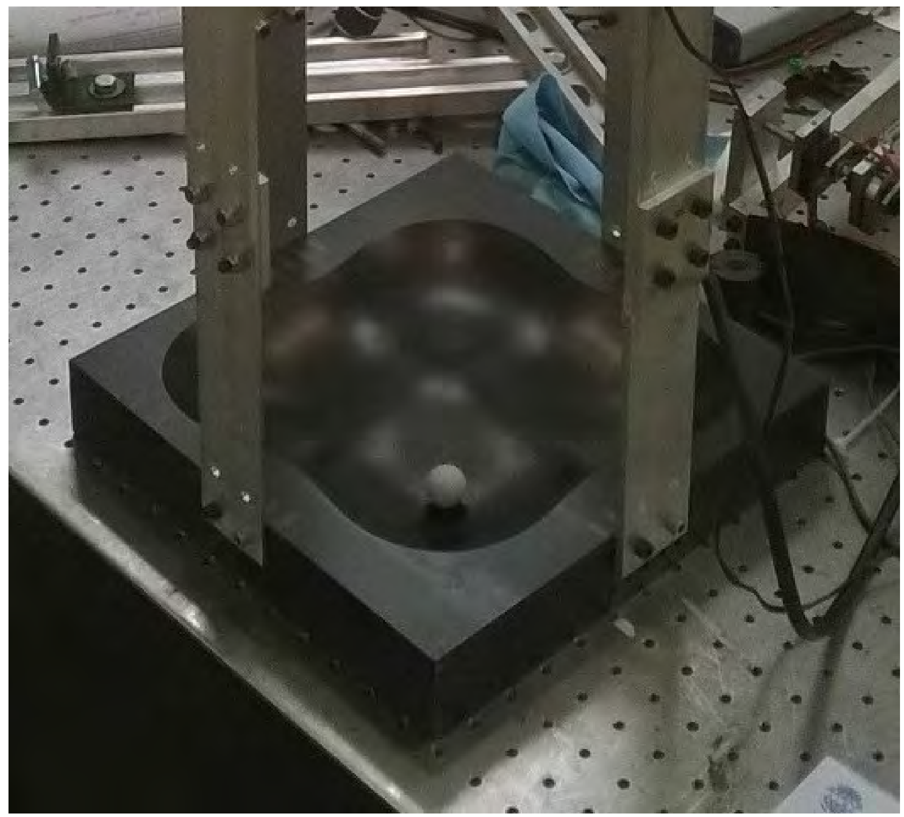
Virgin, Lyman, Davis [2010] Am. J. Phys.

Ball rolling on a surface — 2 DOF

- The potential energy is $V(x,y) = gH(x,y)$, where the surface is arbitrary, e.g., we chose

$$H(x,y) = \alpha(x^2+y^2) - \beta(\sqrt{x^2+\gamma} + \sqrt{y^2+\gamma}) - \xi xy + H_0.$$





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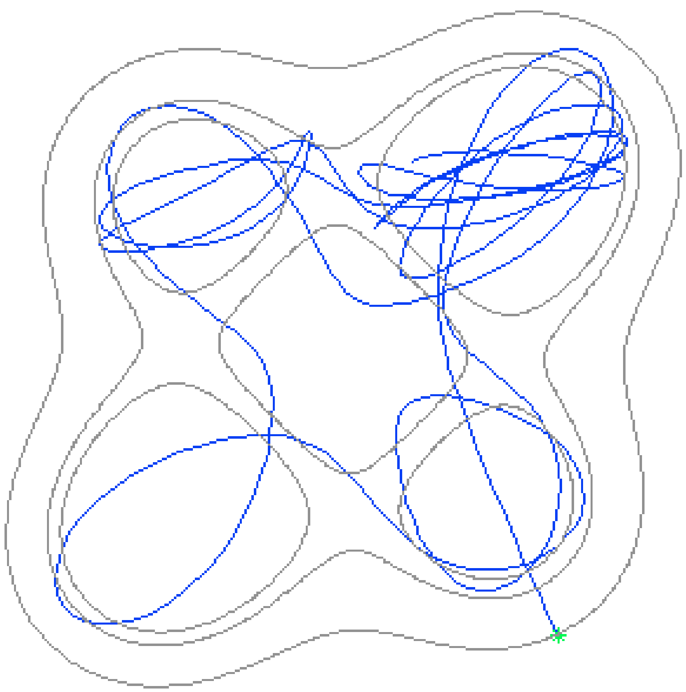
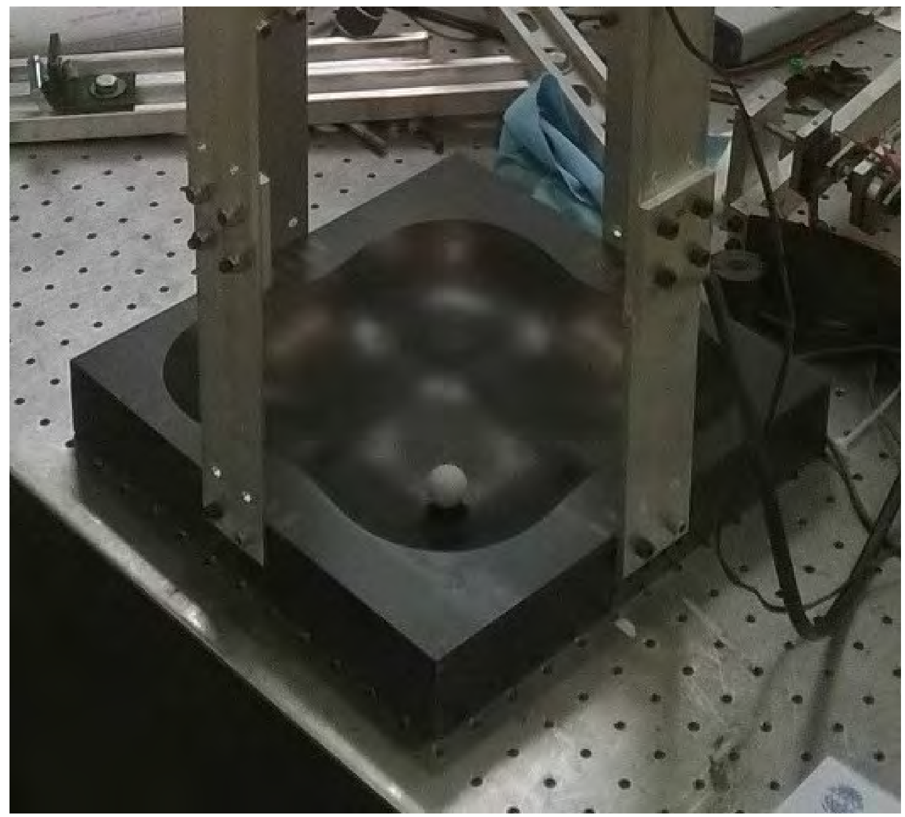
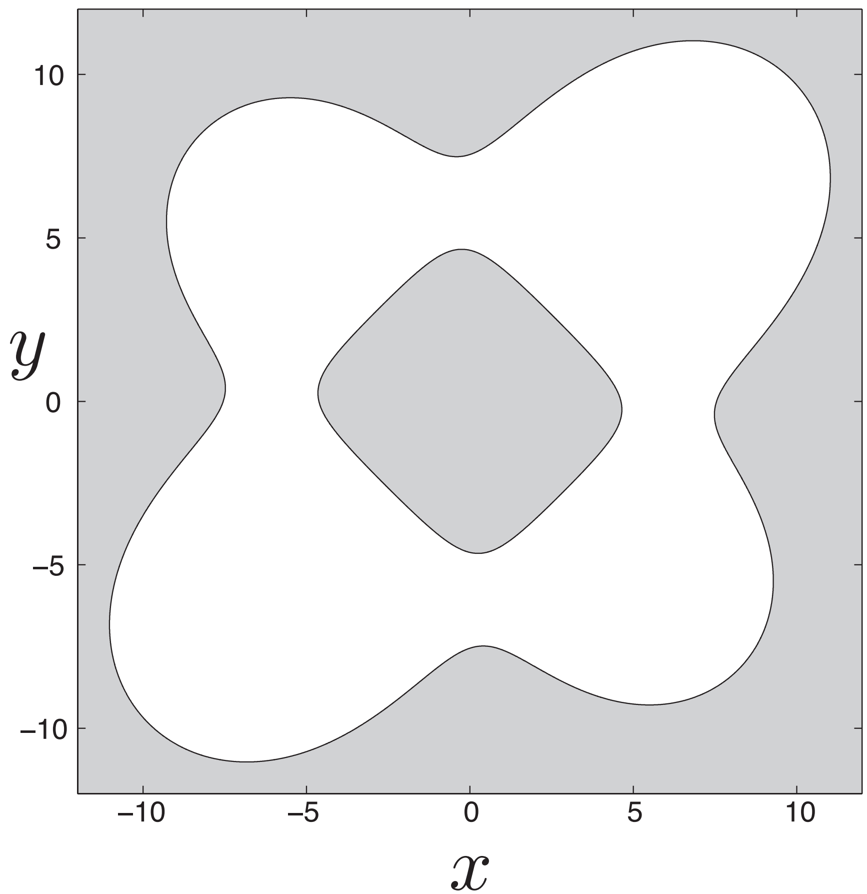




typical experimental trial

Transition tubes in the rolling ball system



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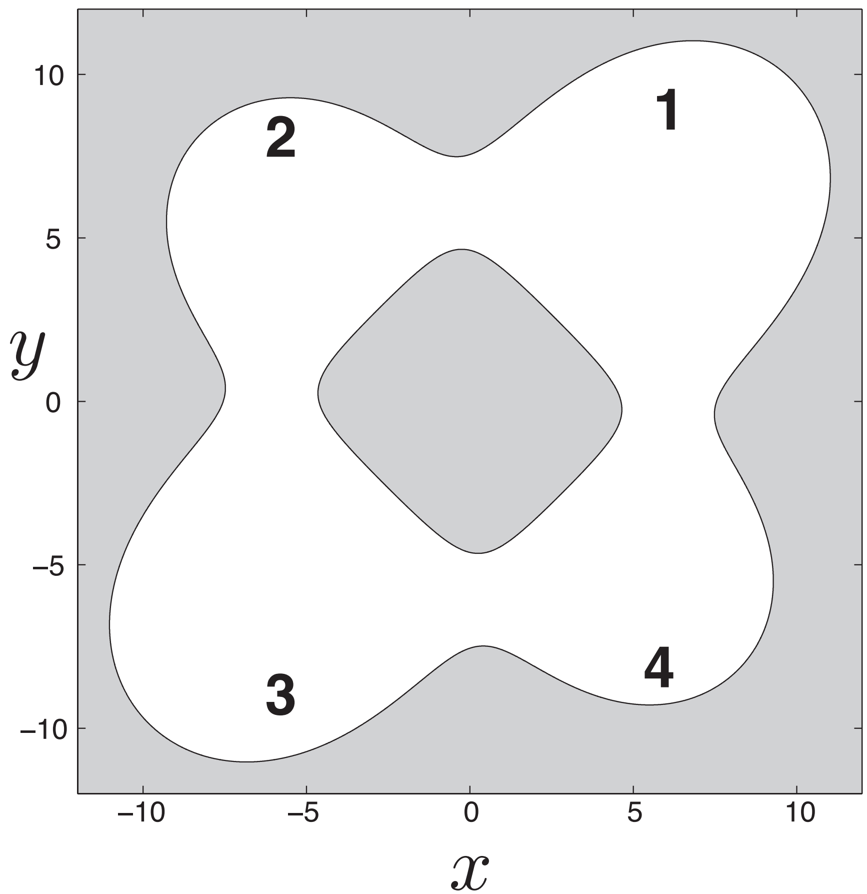

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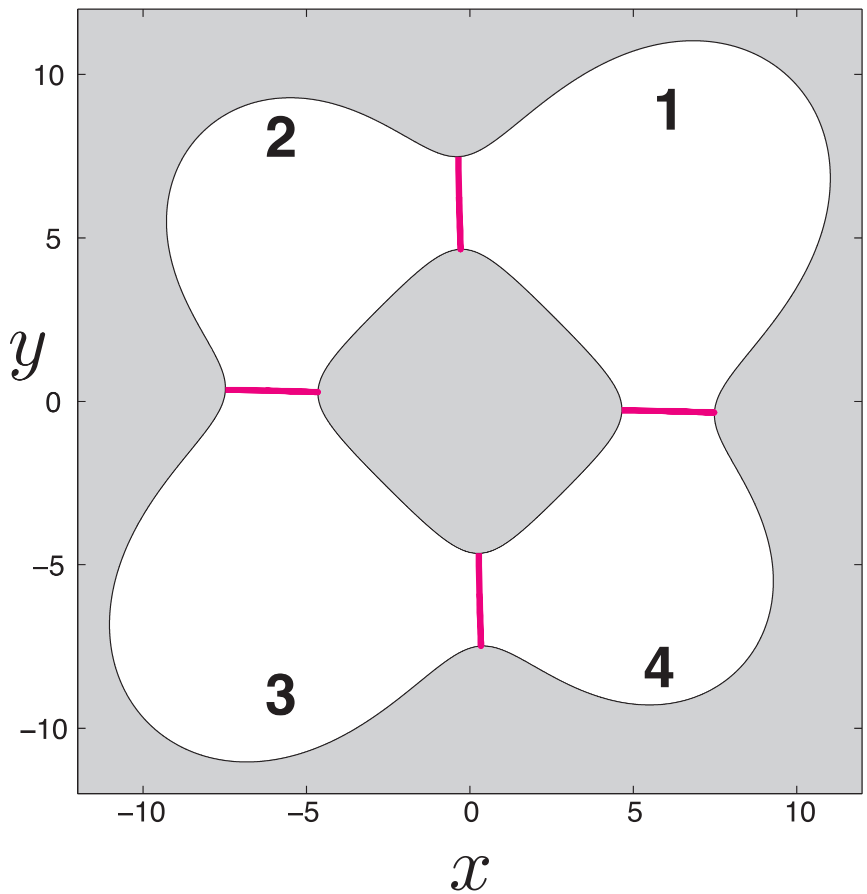

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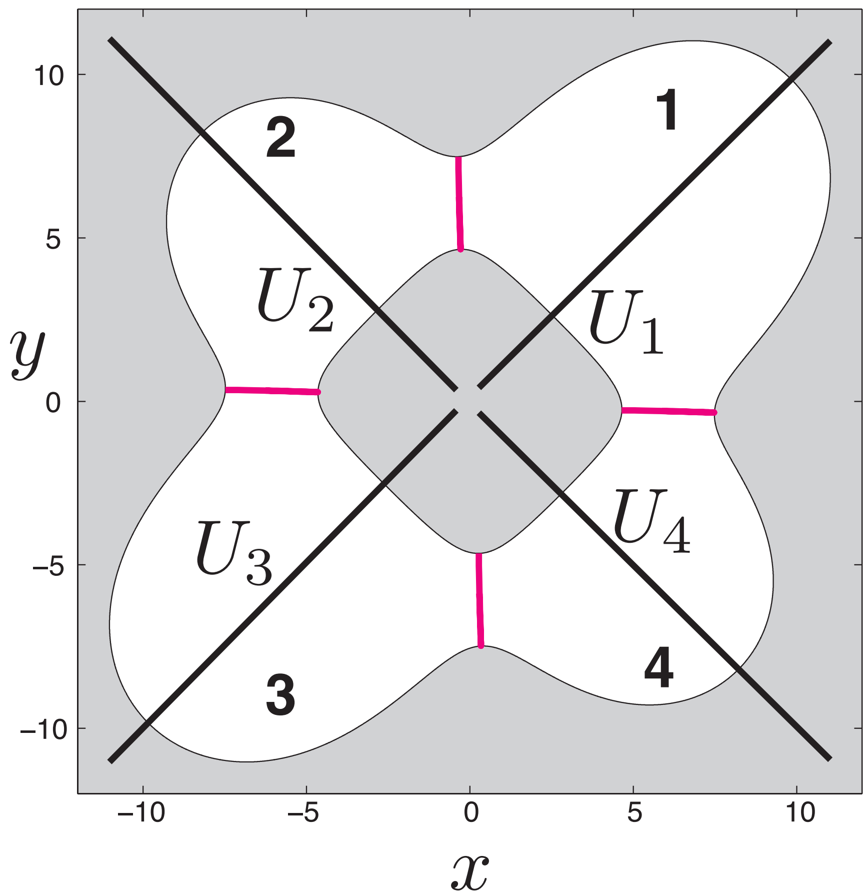

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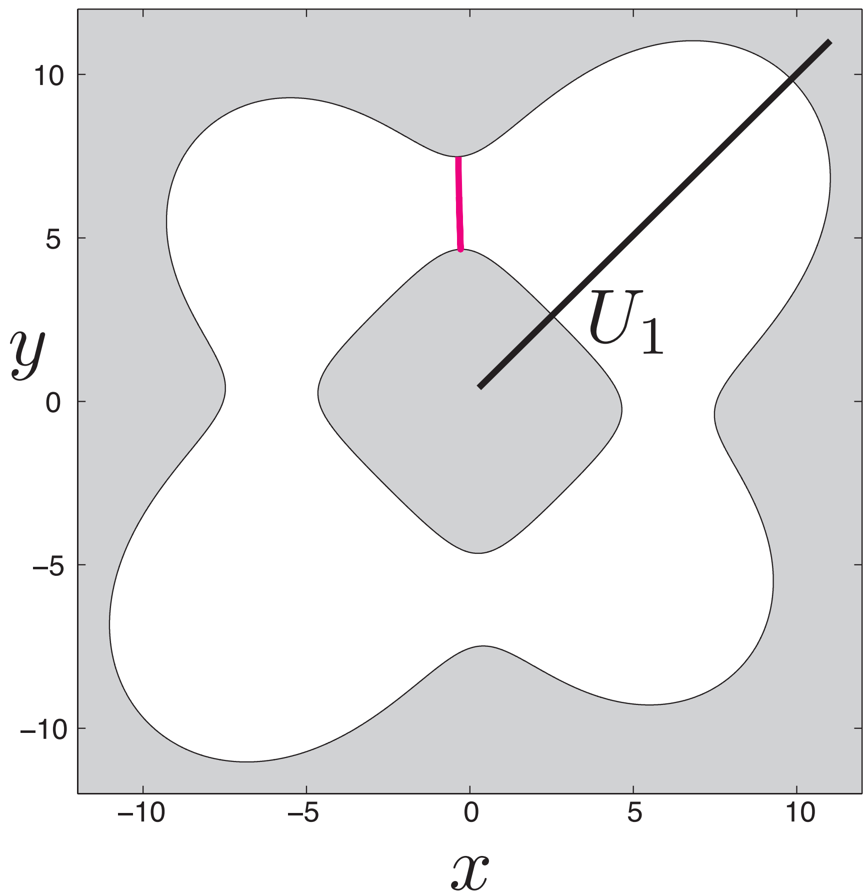

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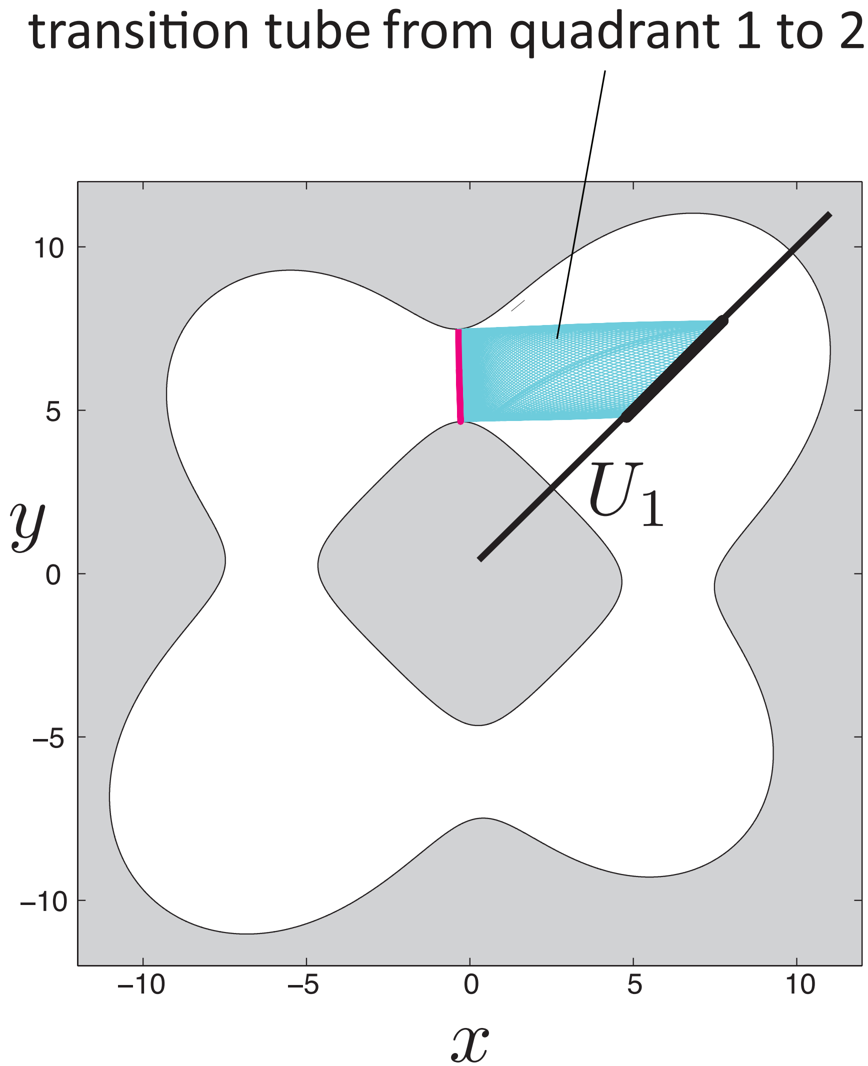

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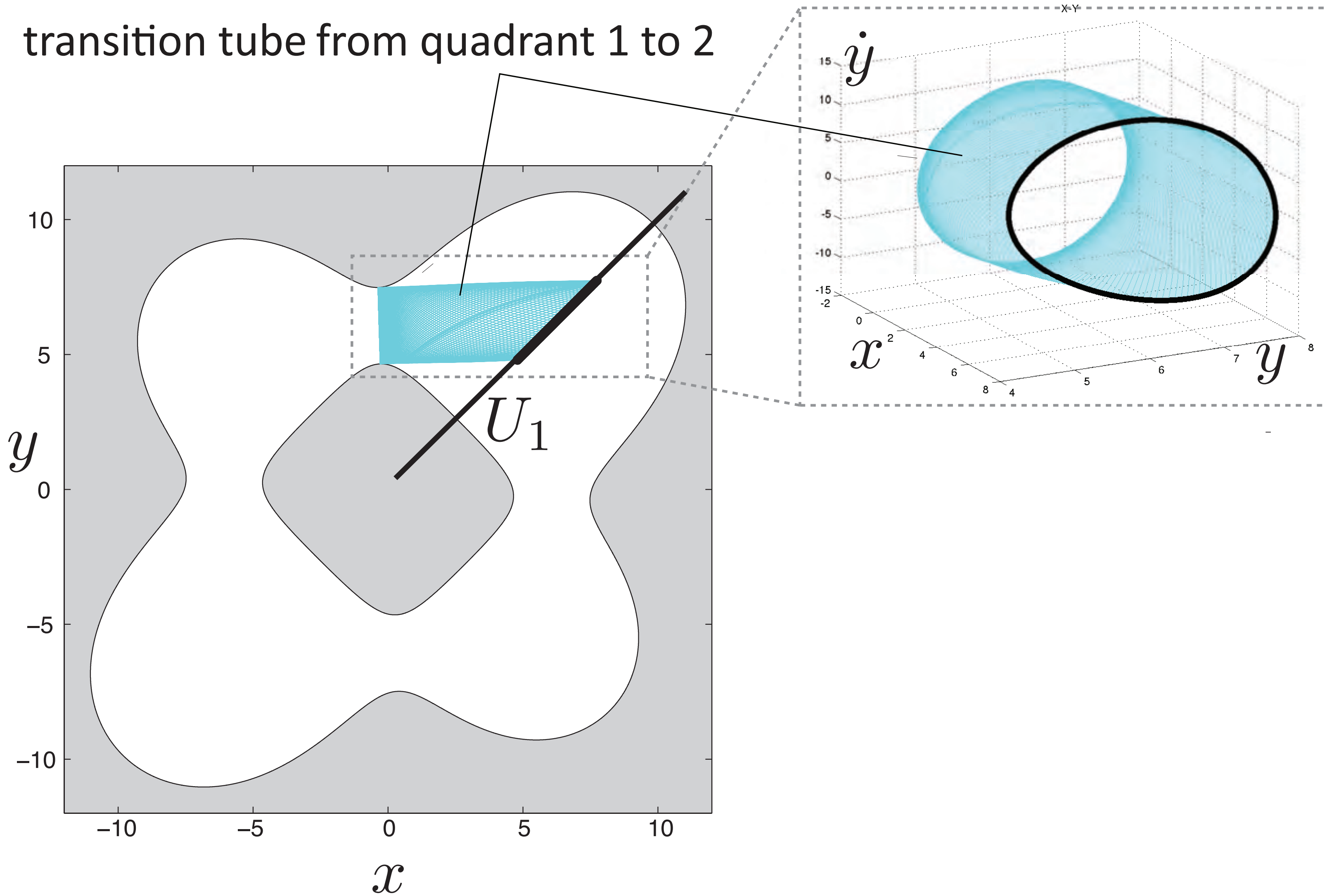

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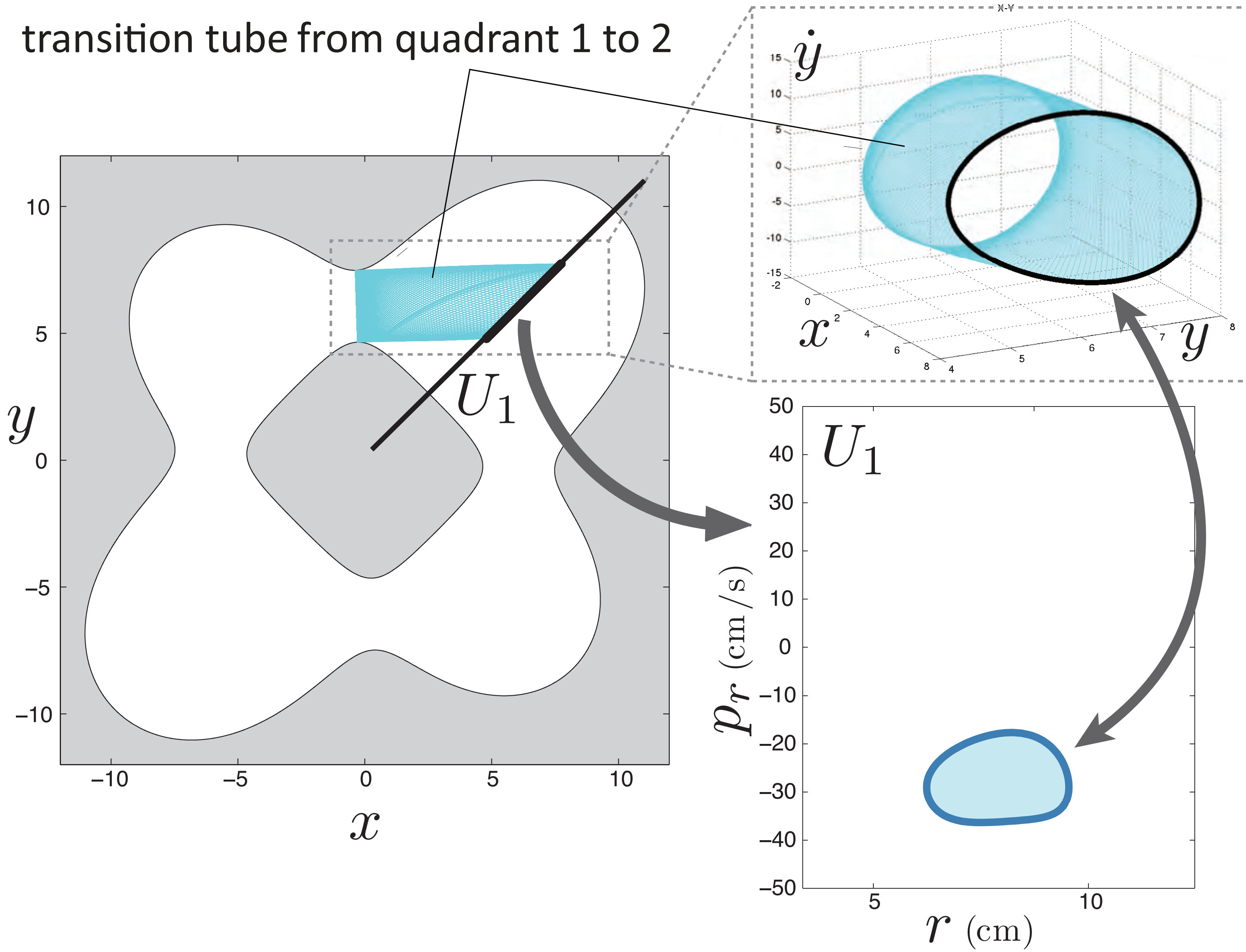

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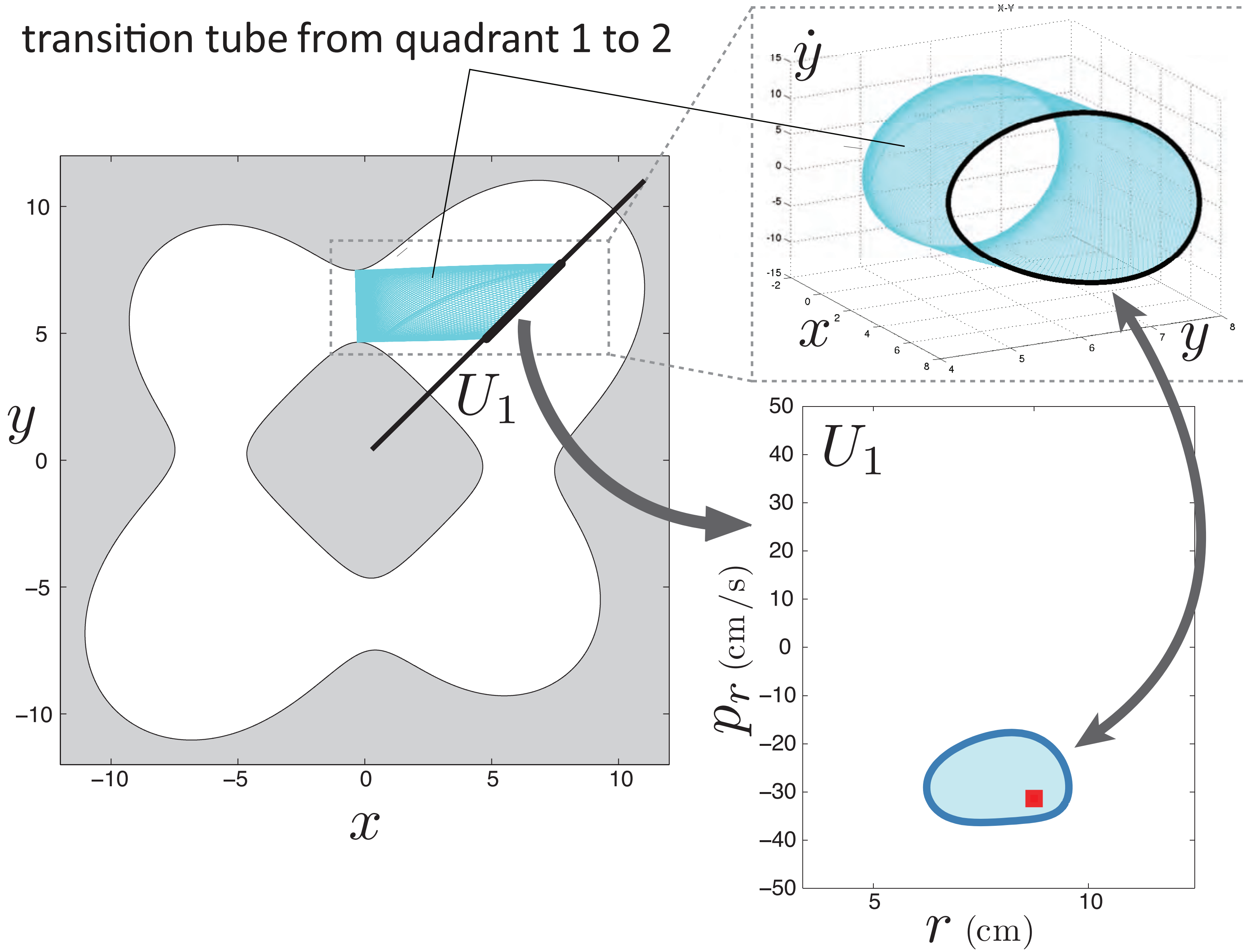

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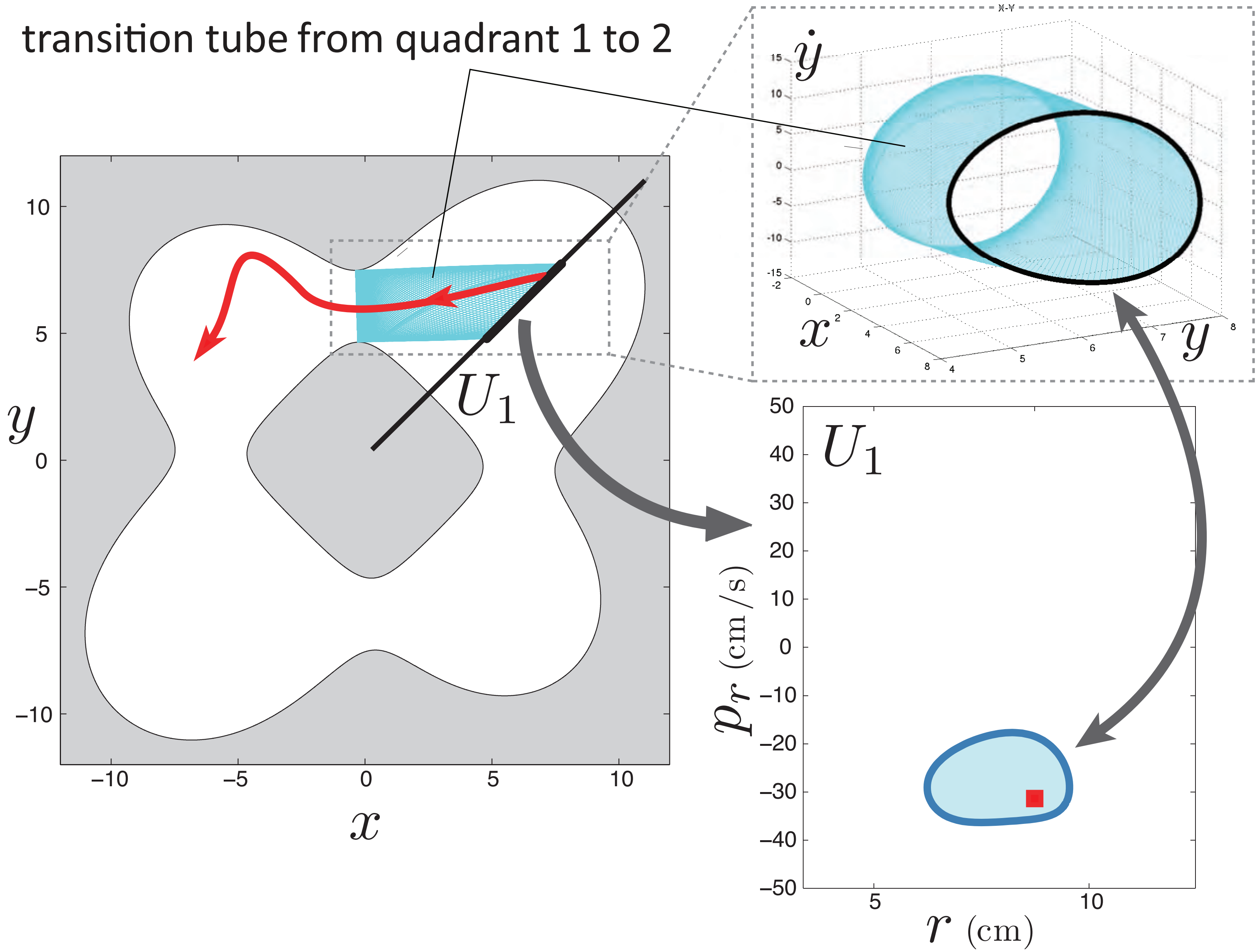

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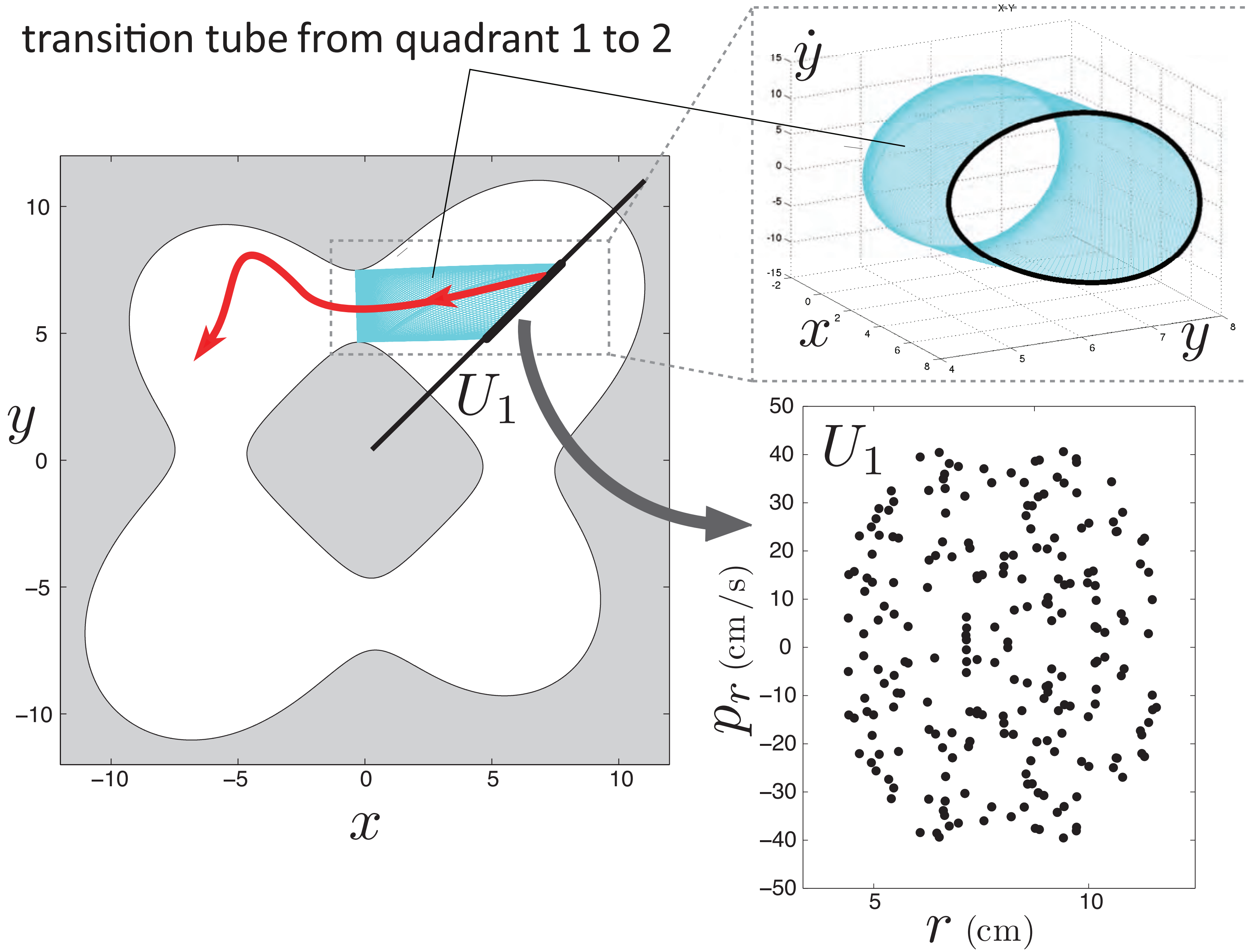

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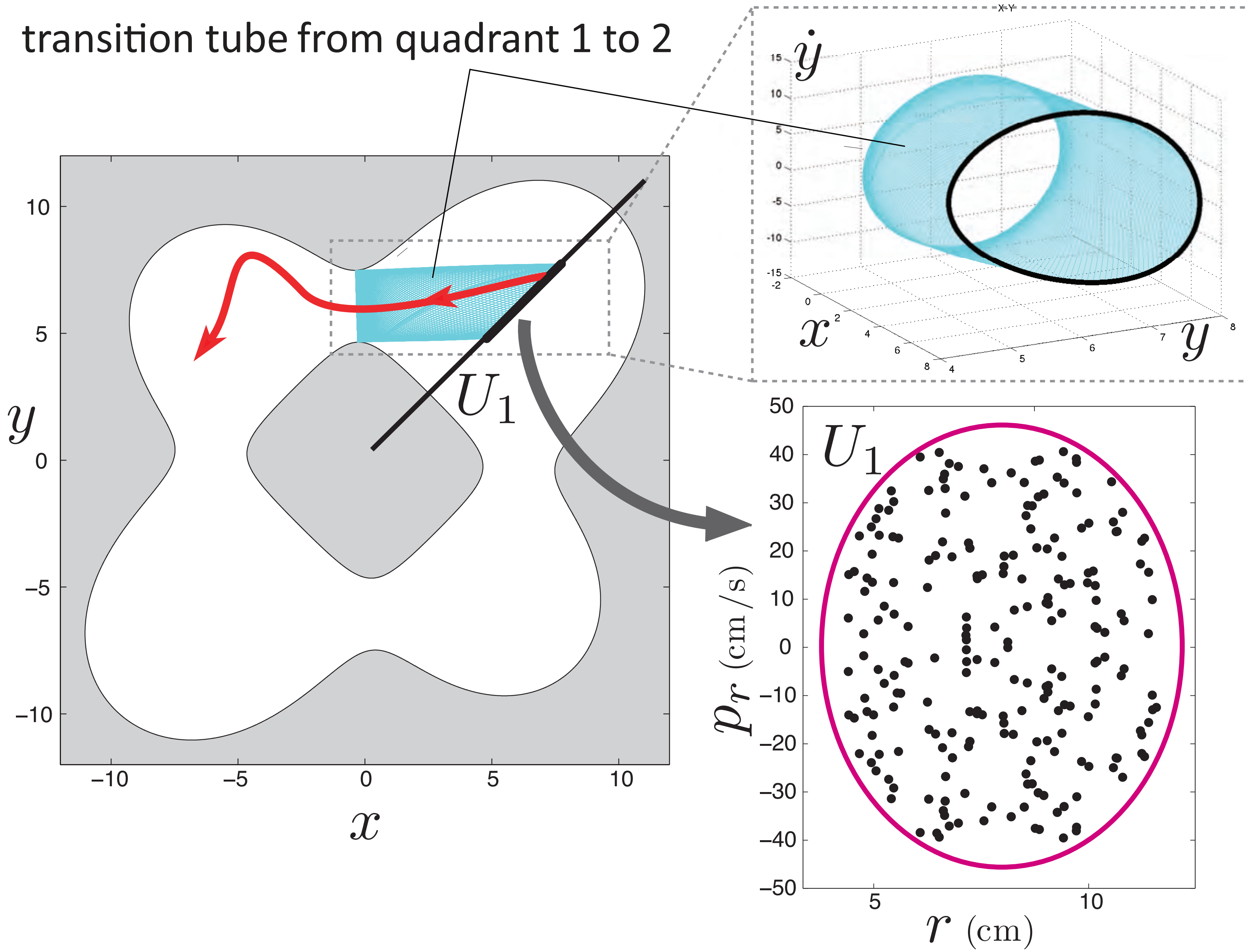

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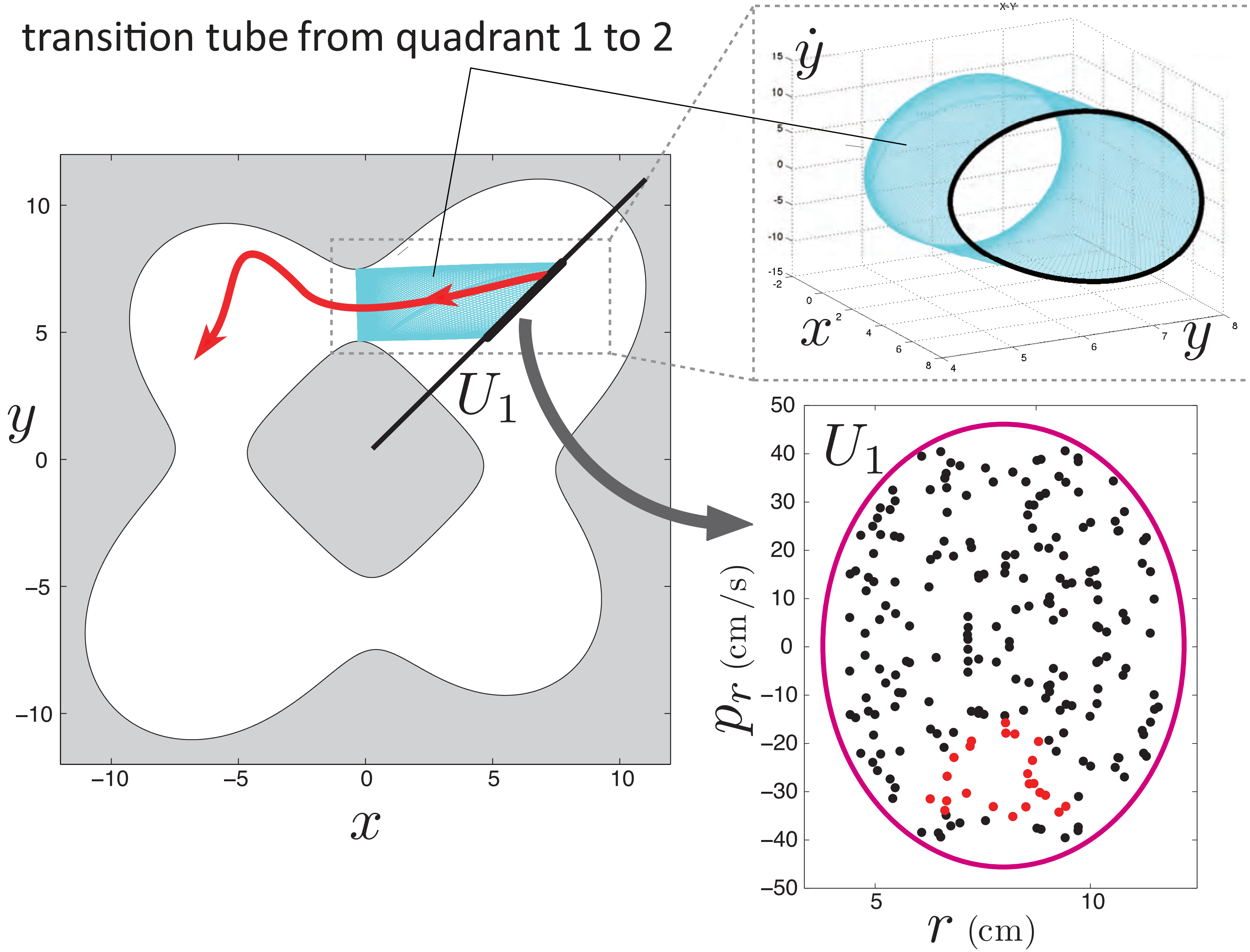

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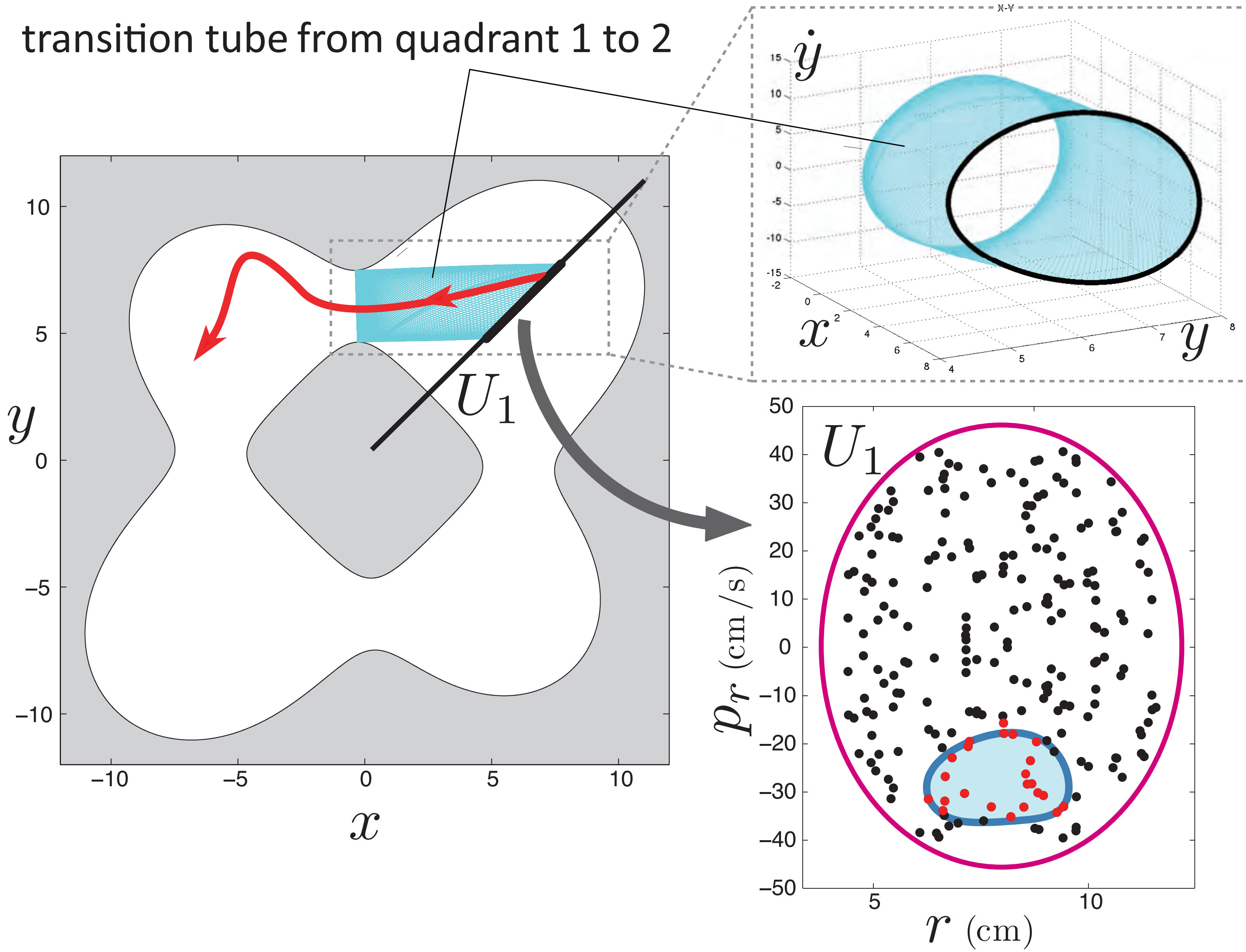
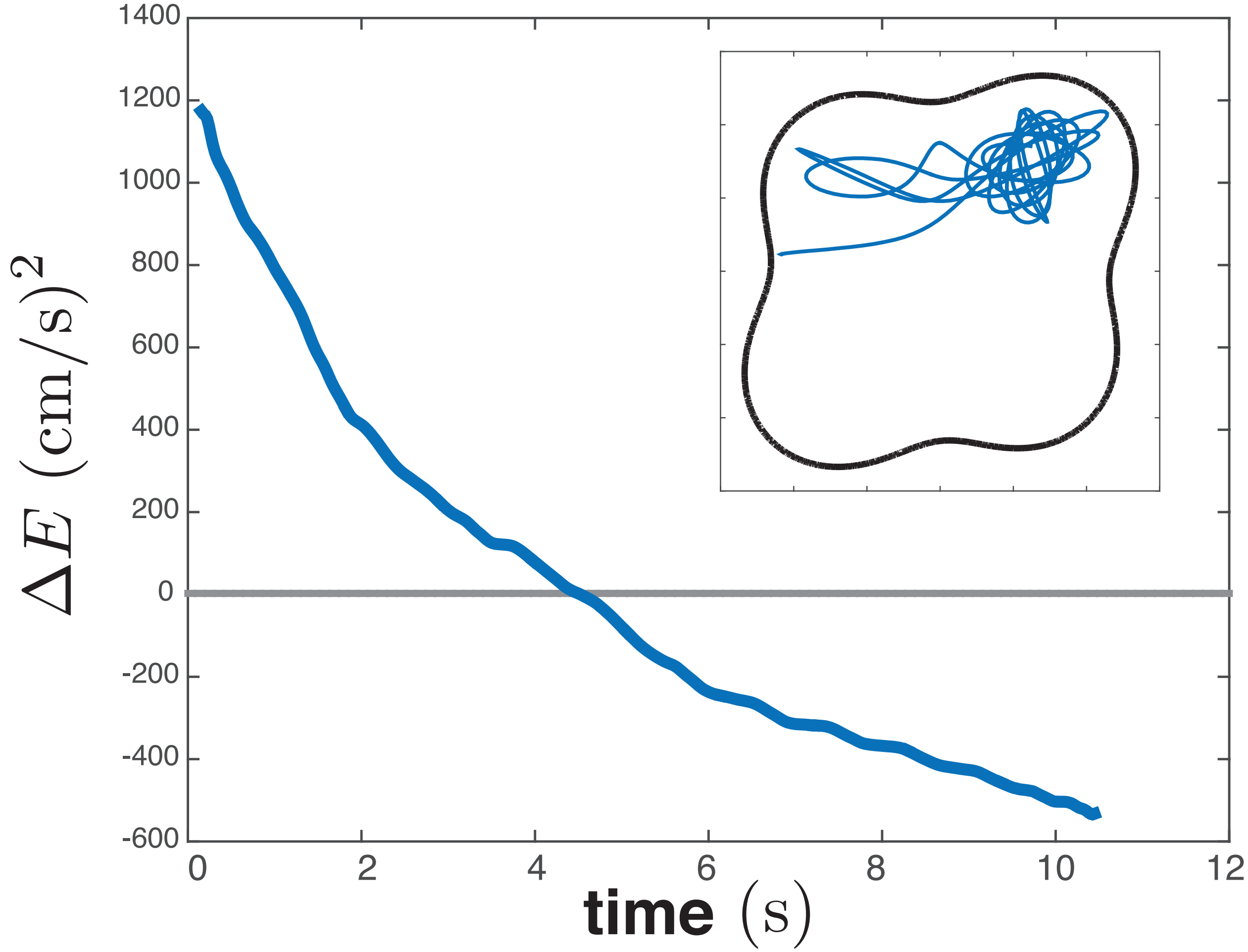


Analysis of experimental data



- 120 experimental trials of about 10 seconds each, recorded at 50 Hz

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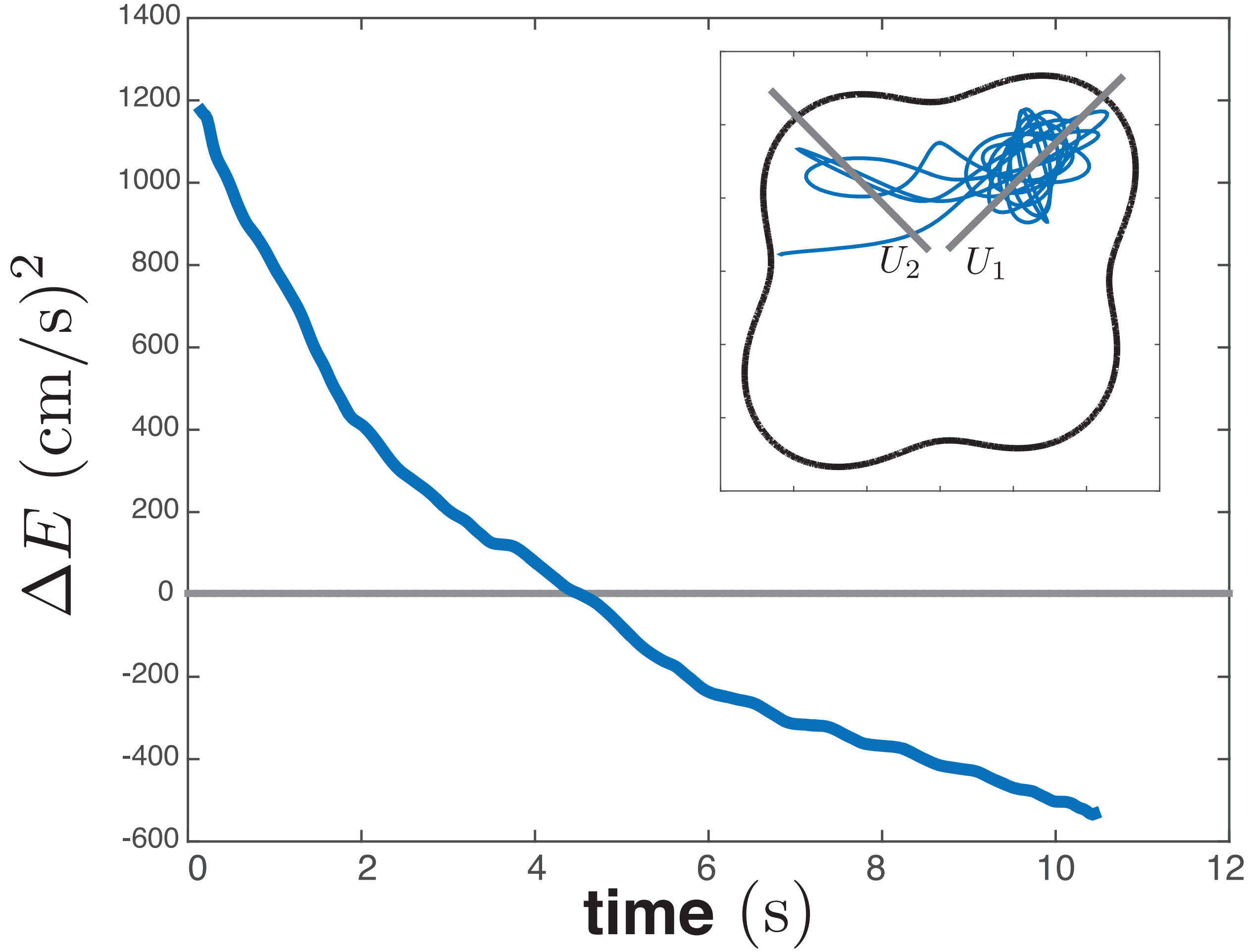

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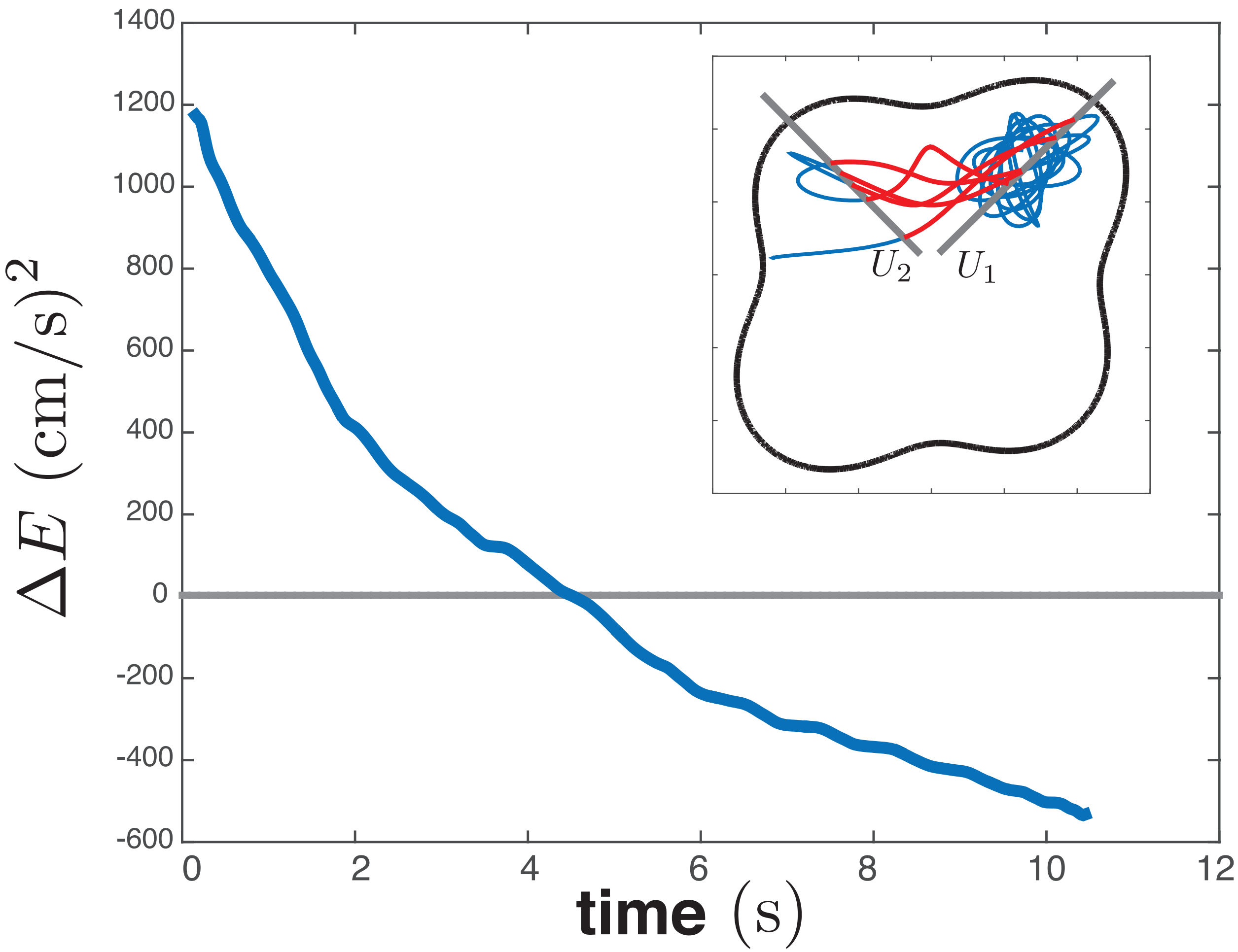

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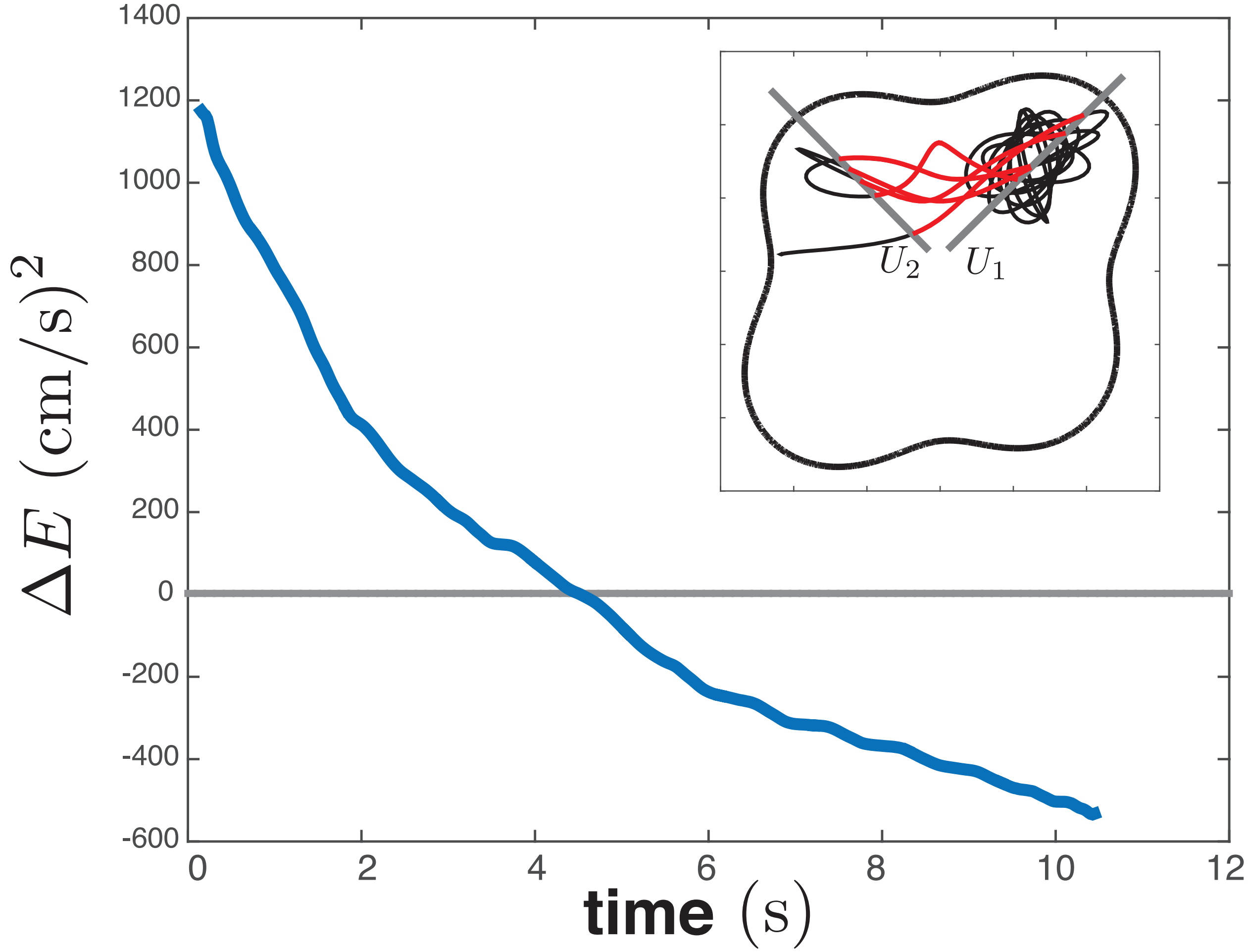

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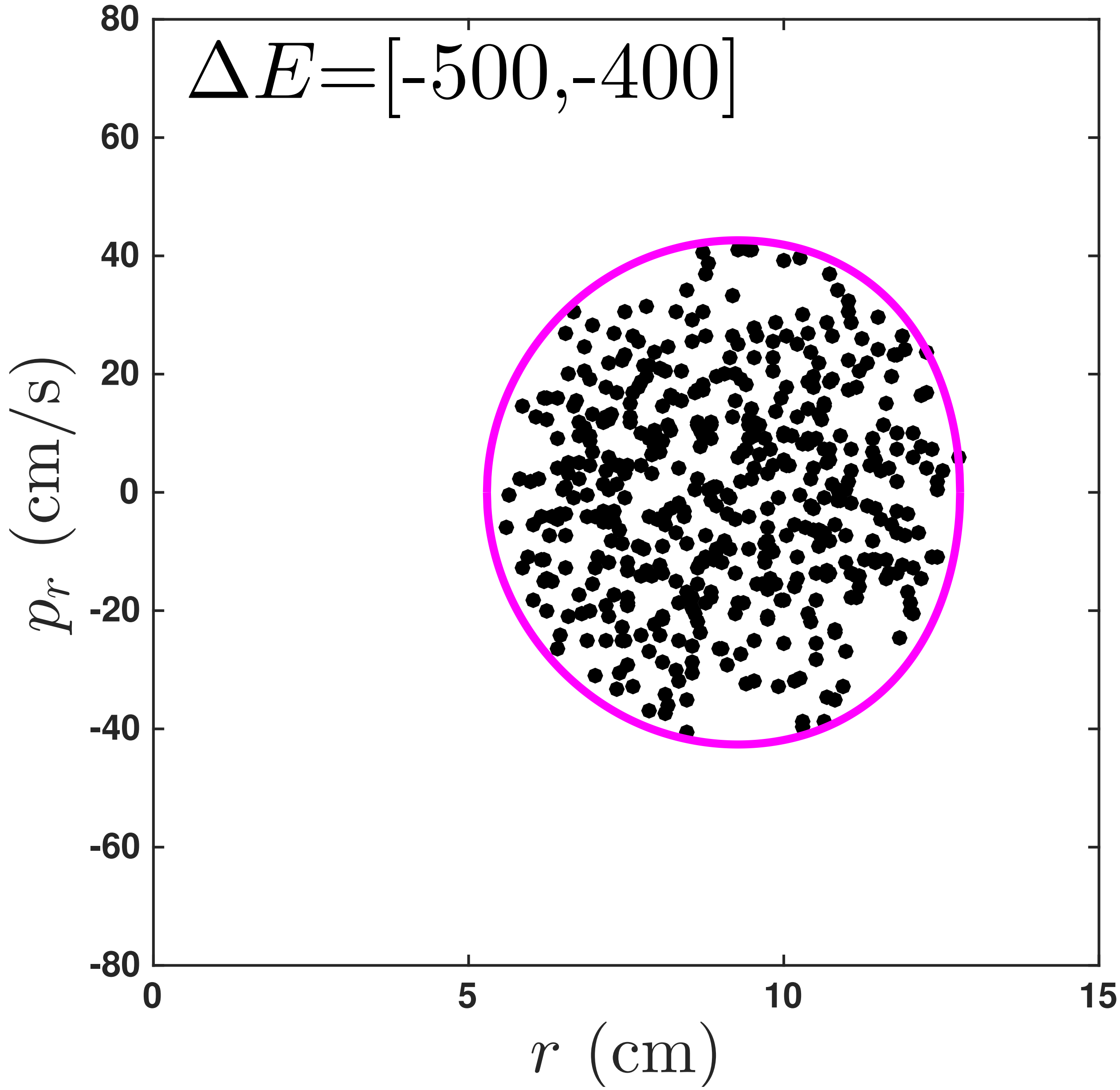
Poincaré sections at various energy ranges



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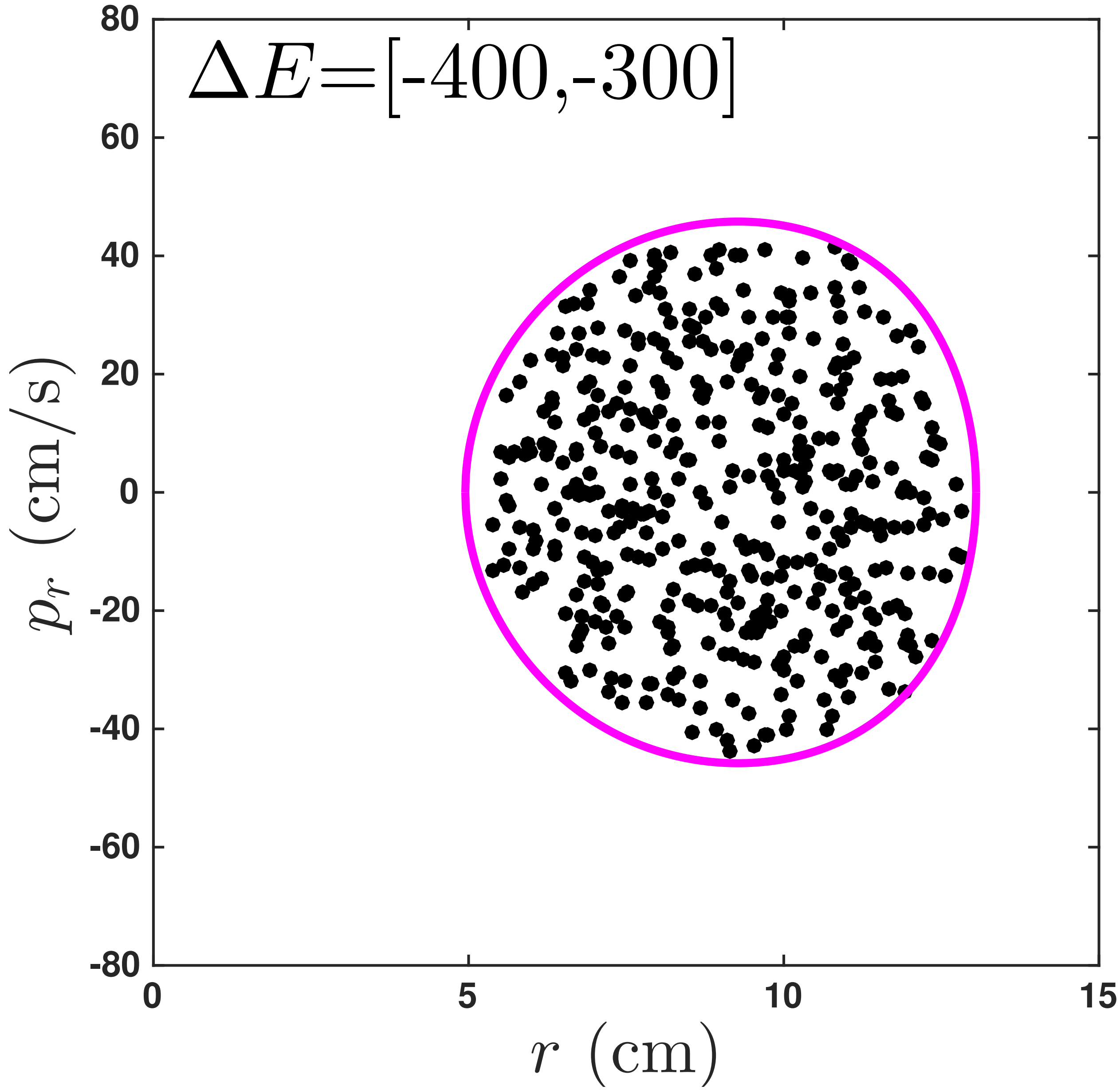

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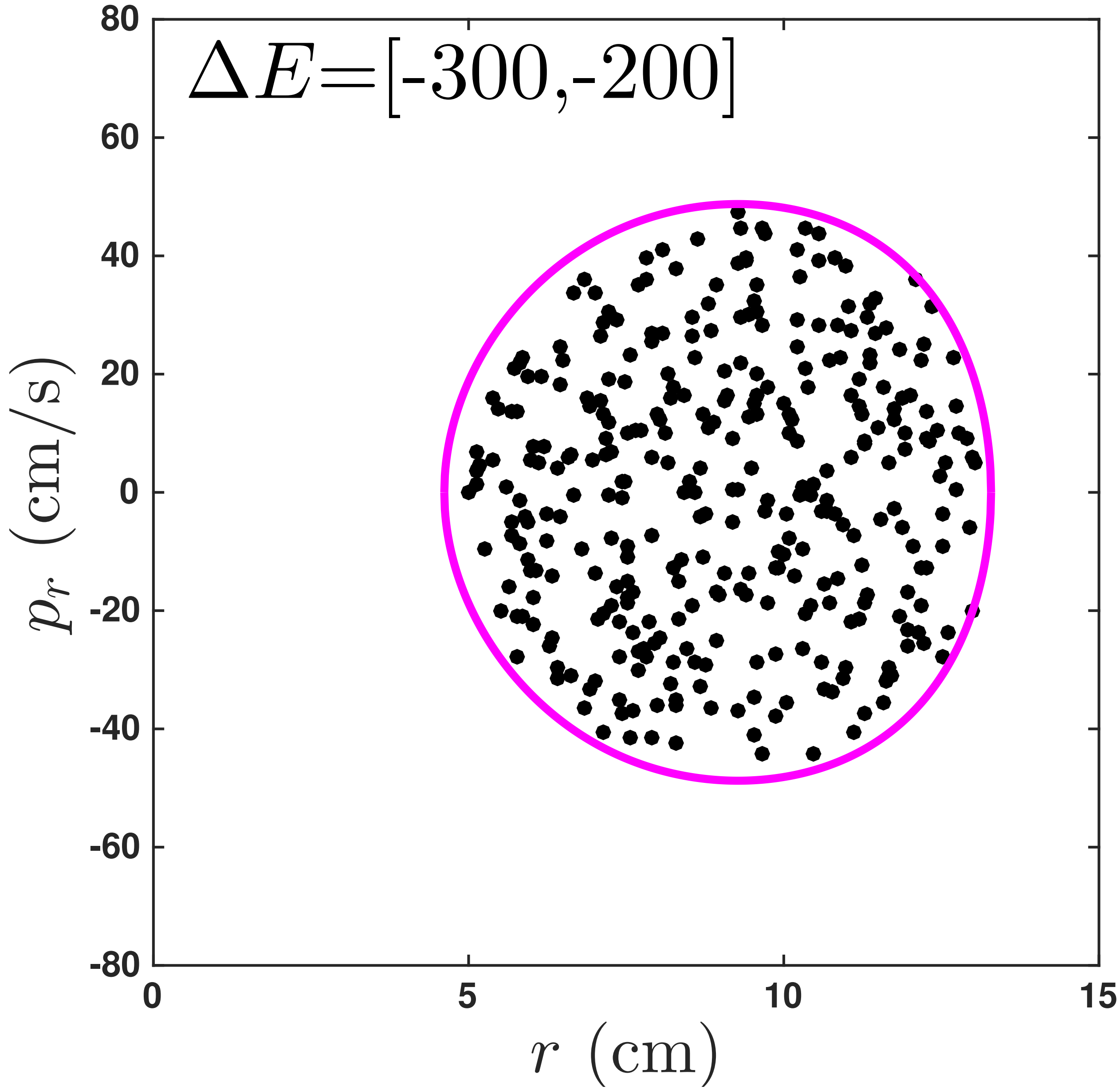

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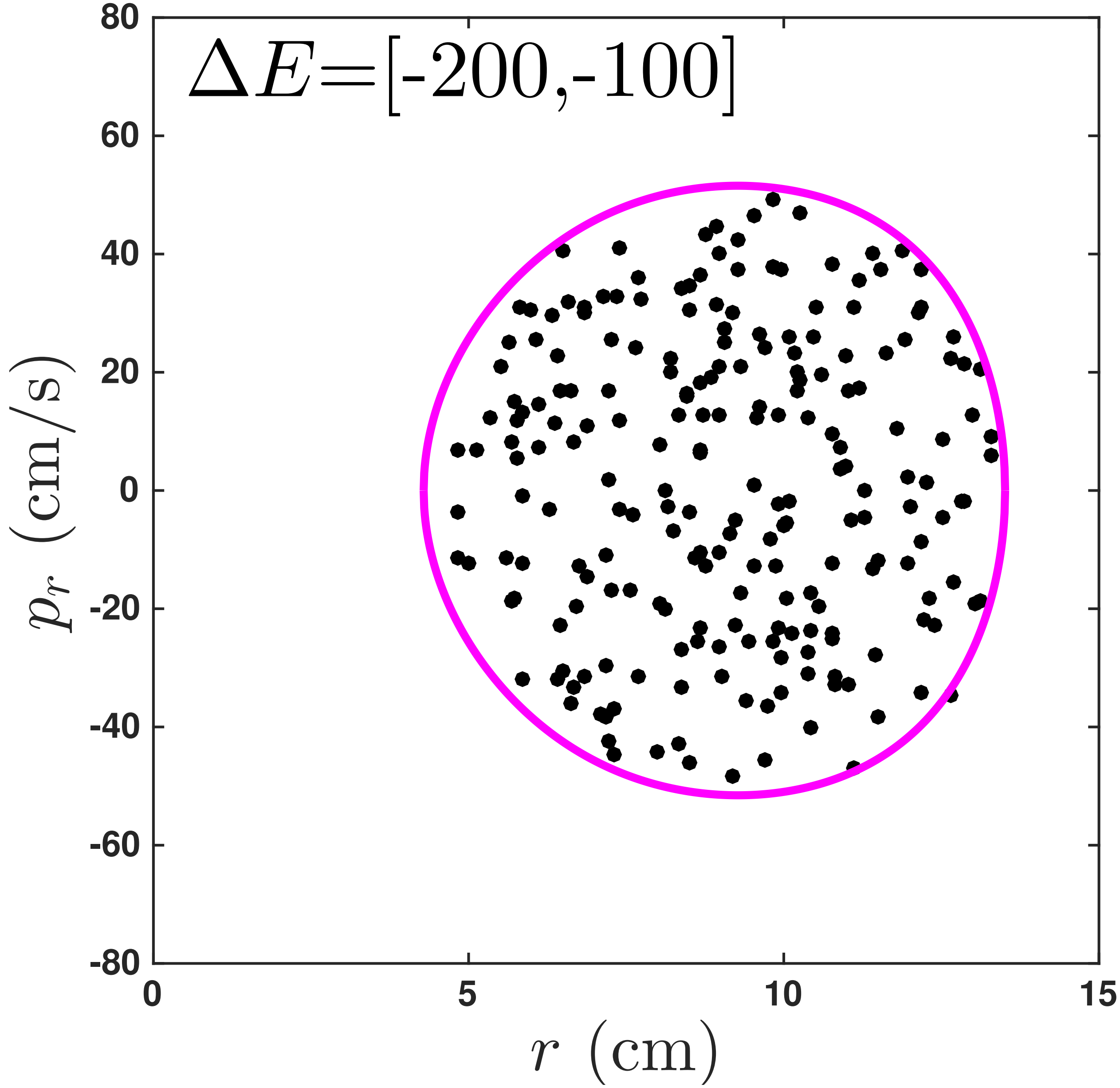

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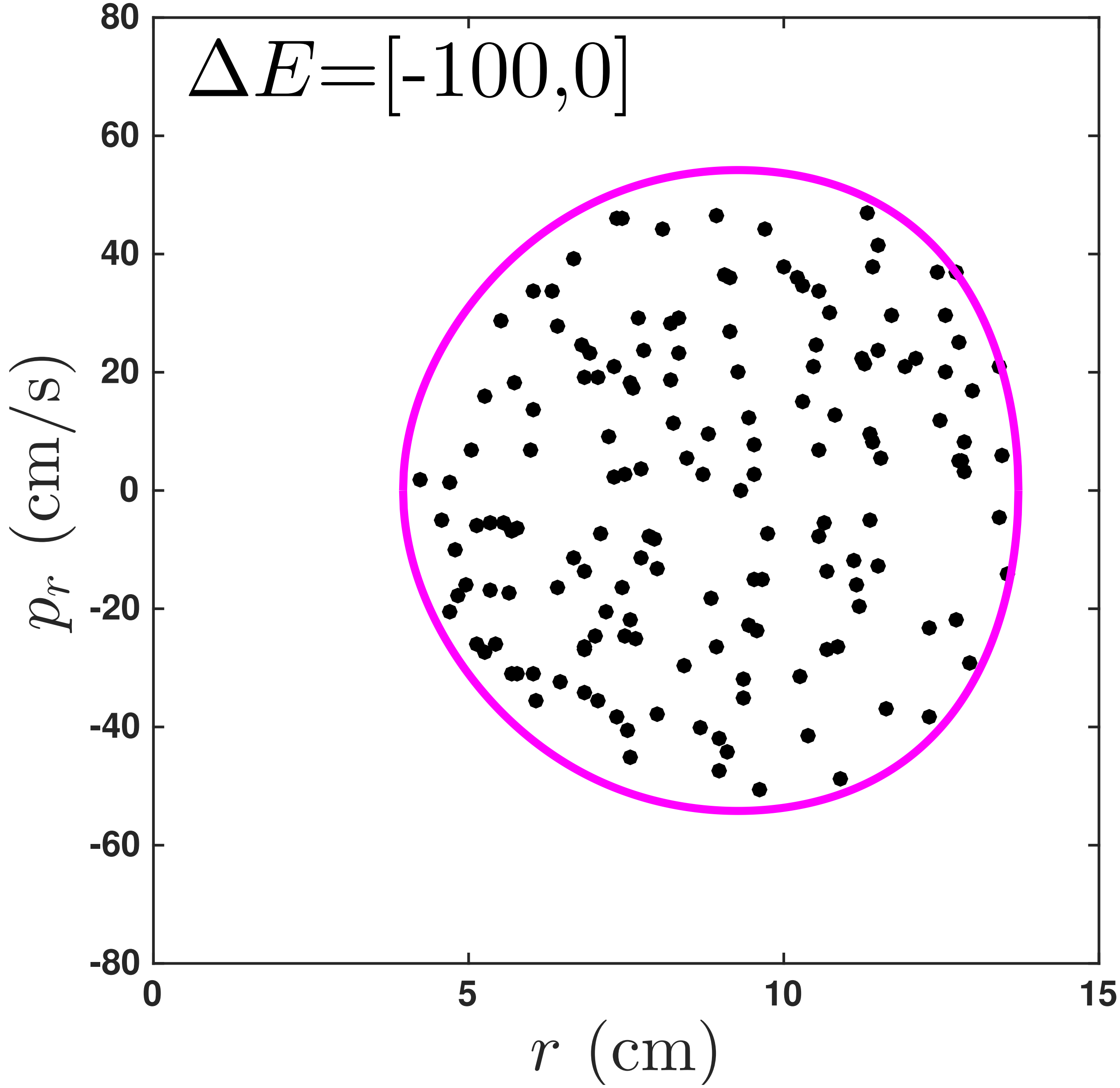

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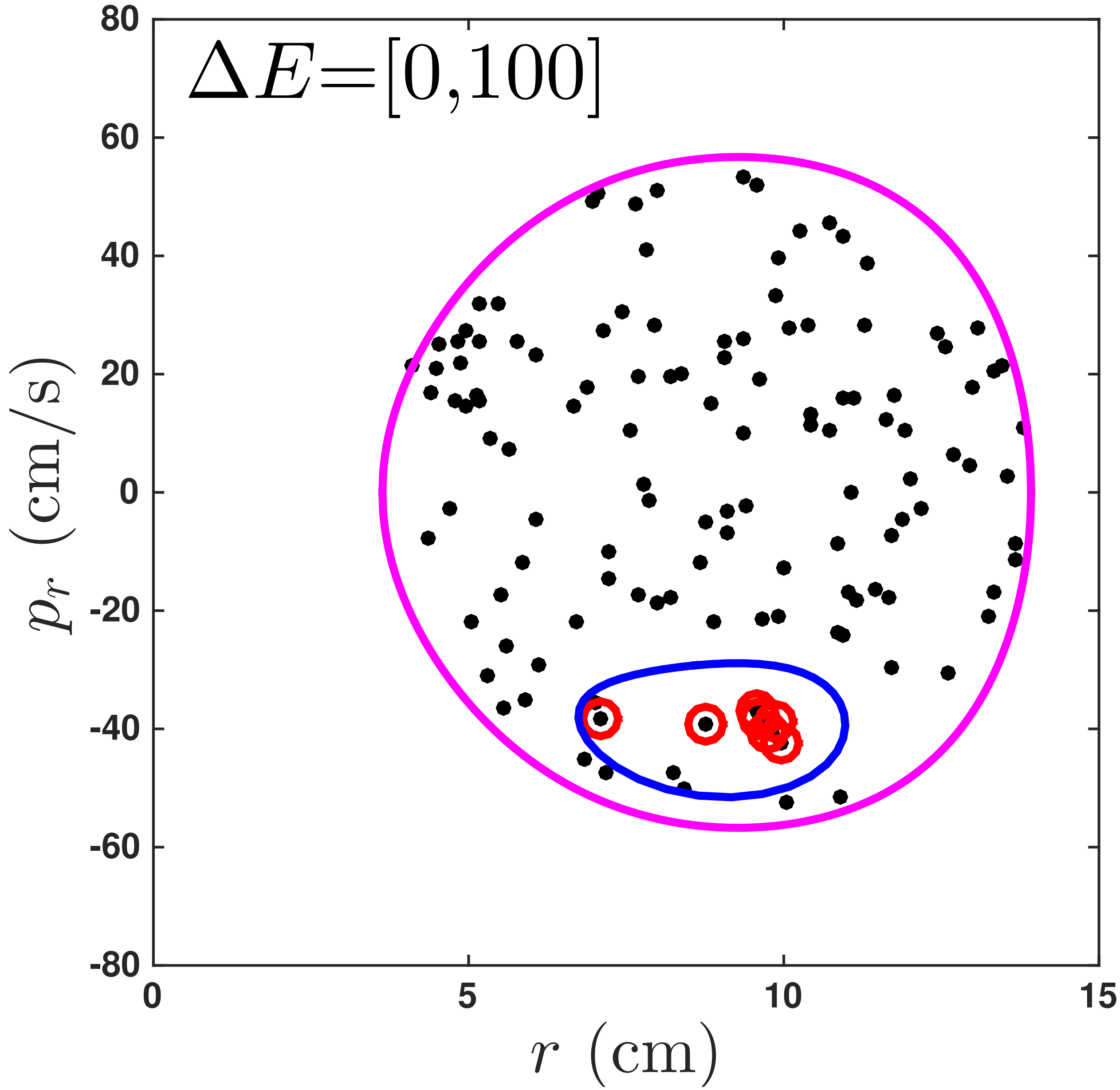

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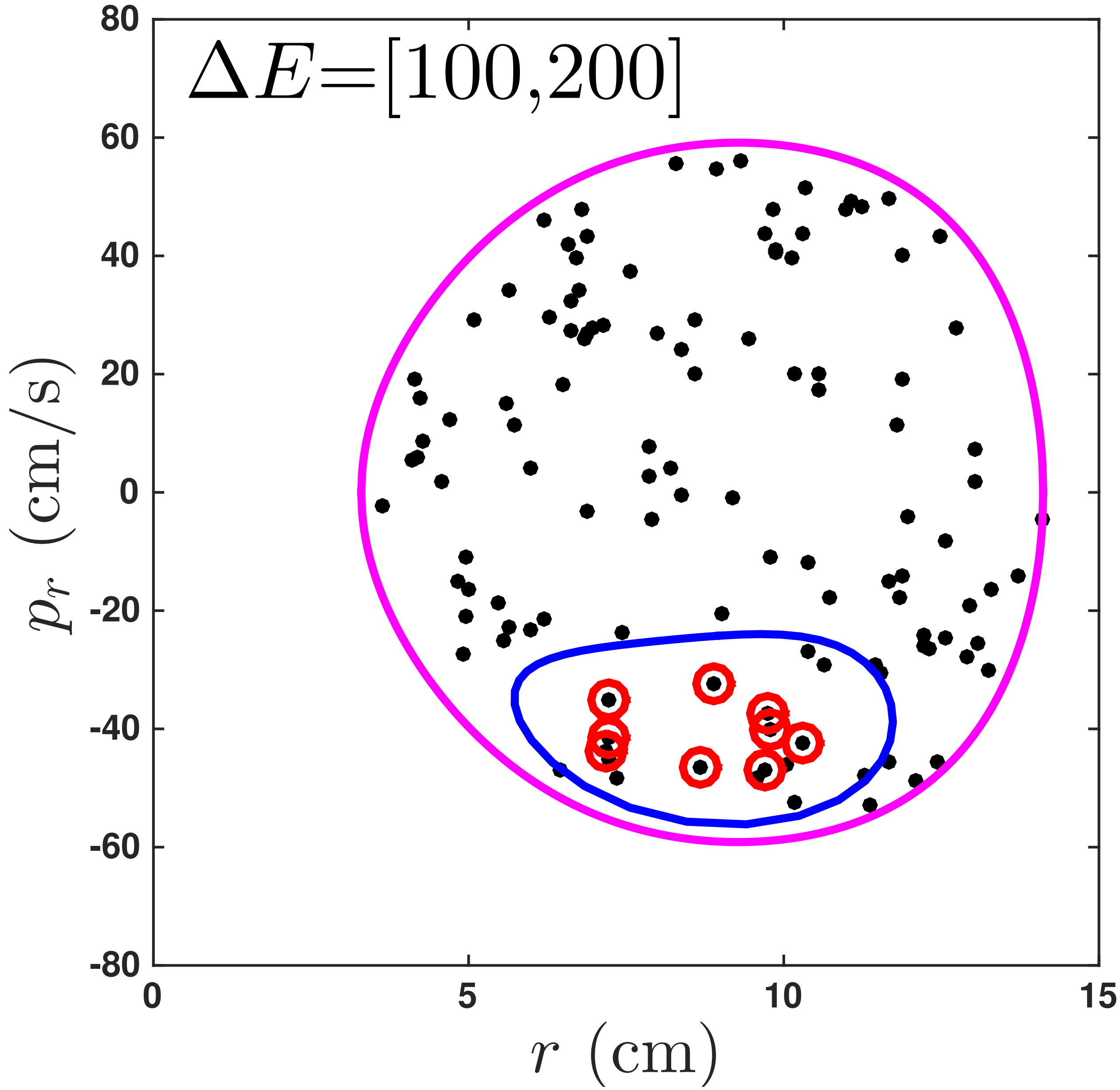

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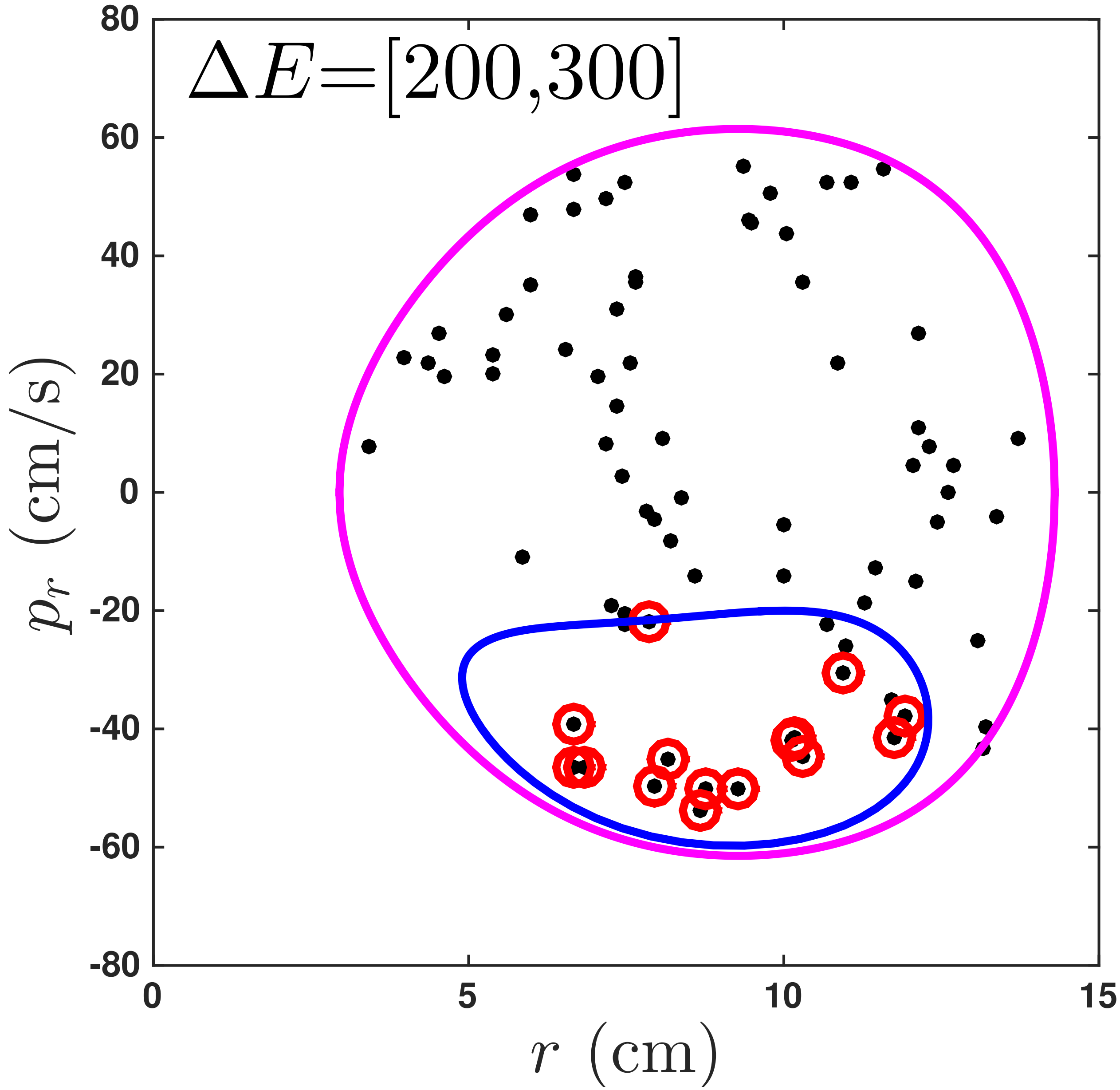

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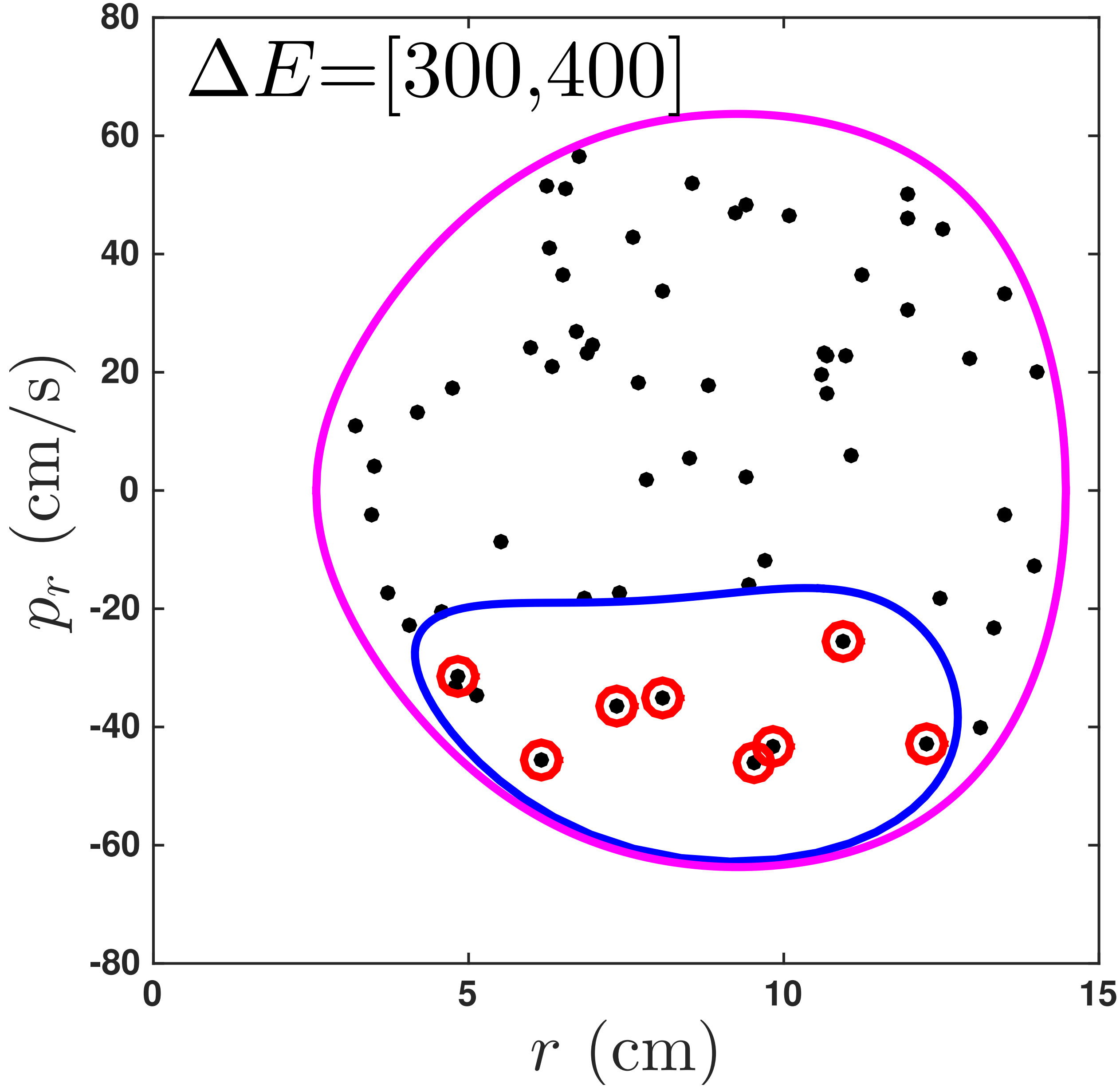

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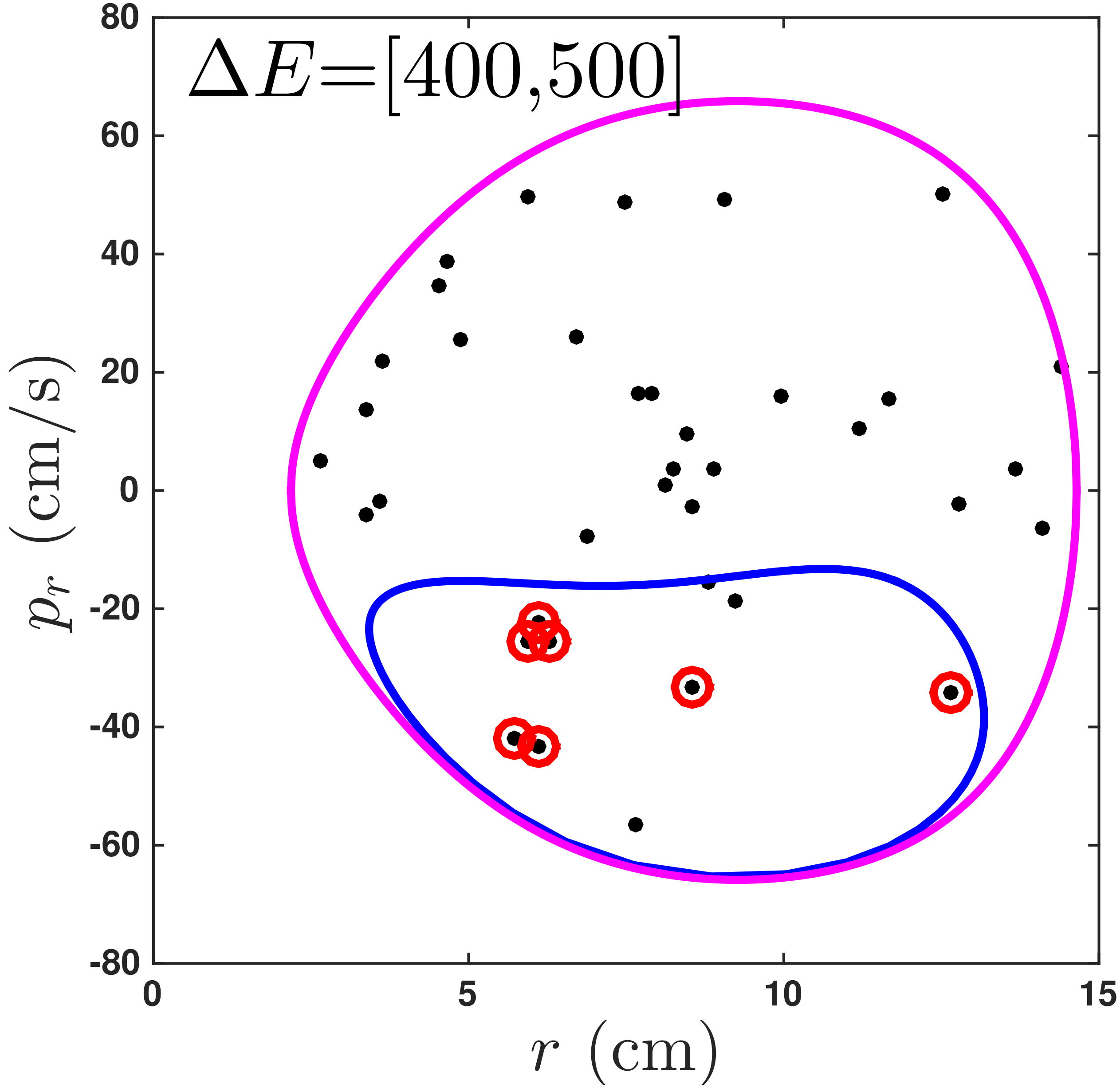

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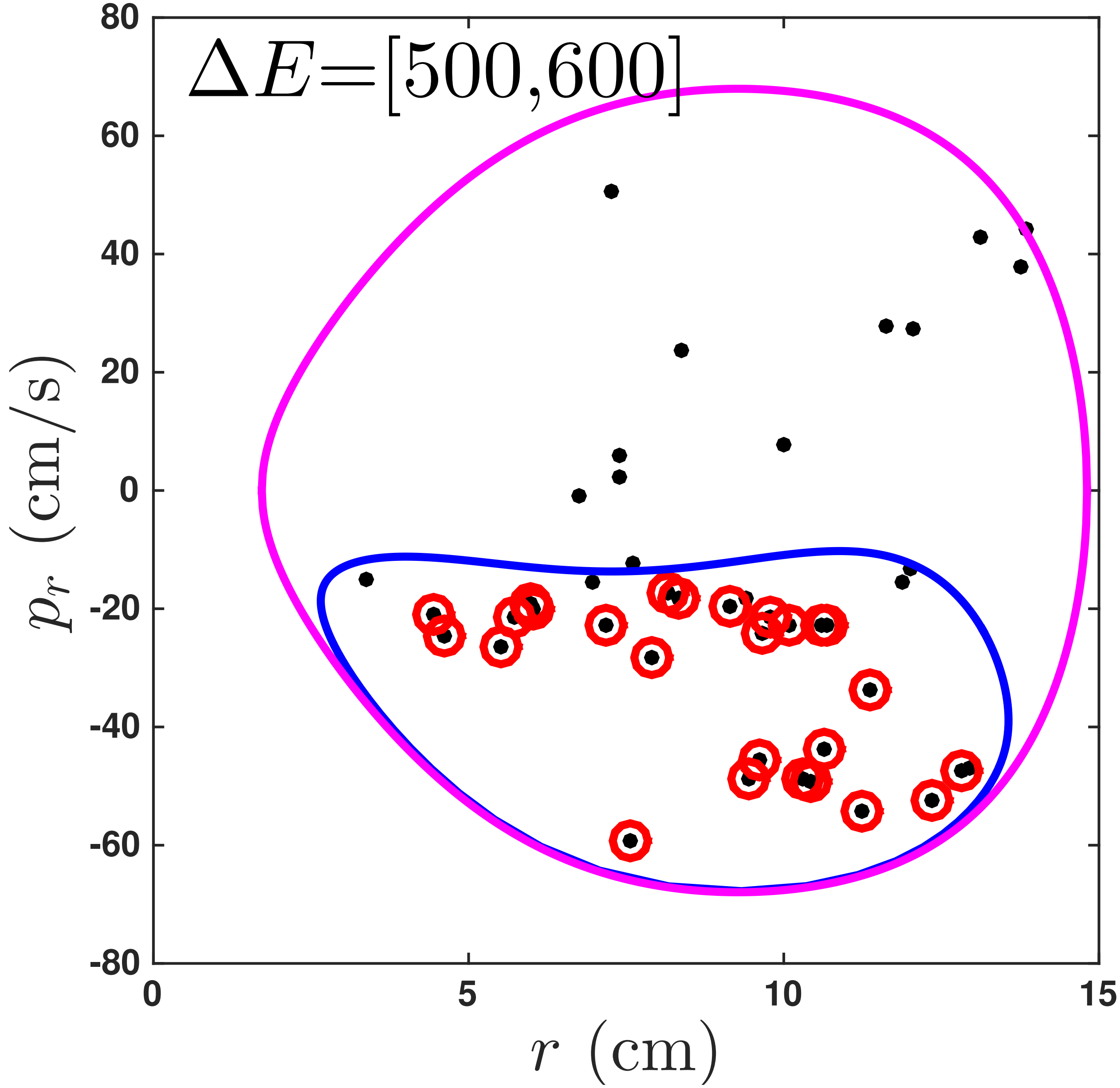

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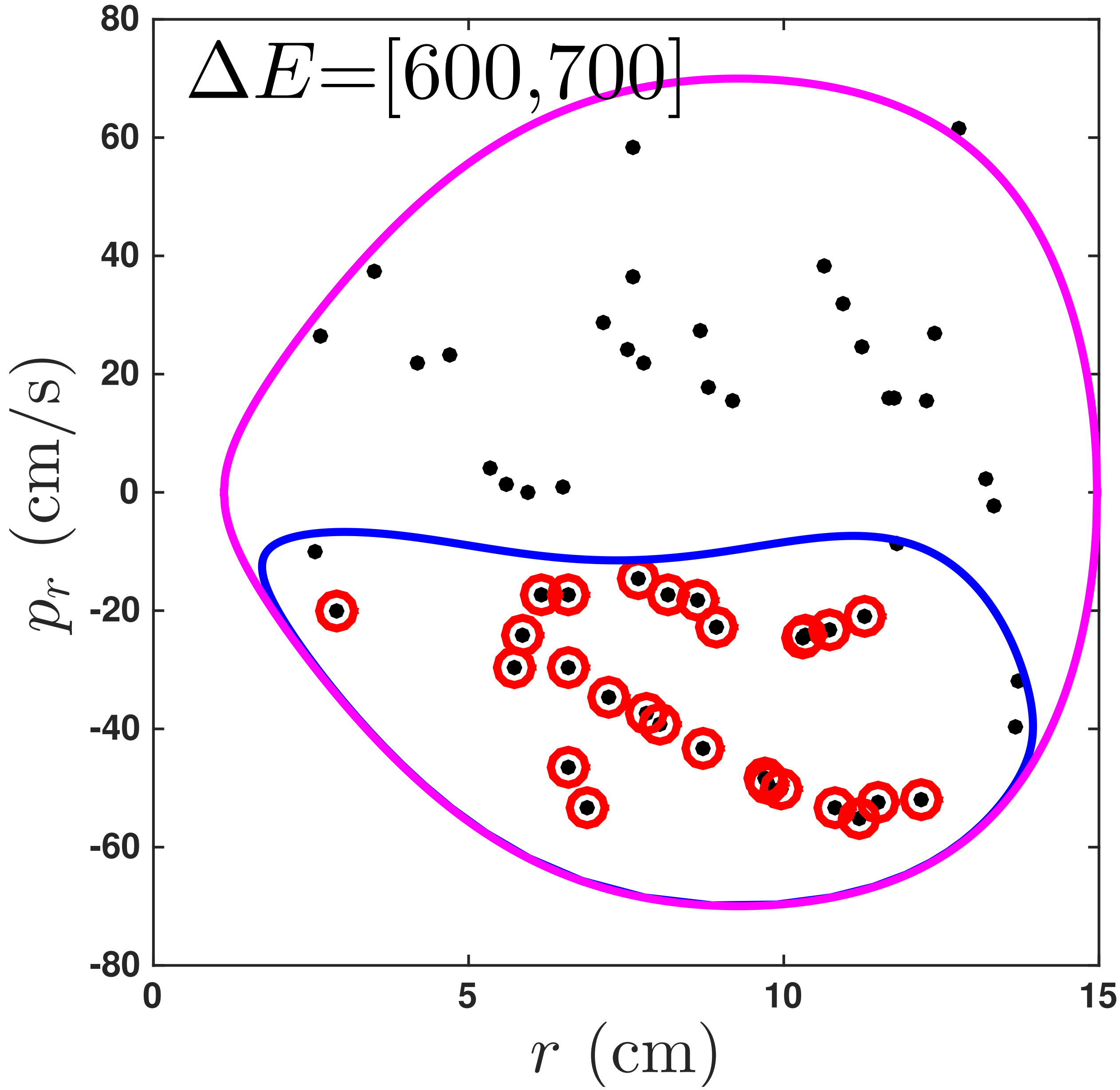

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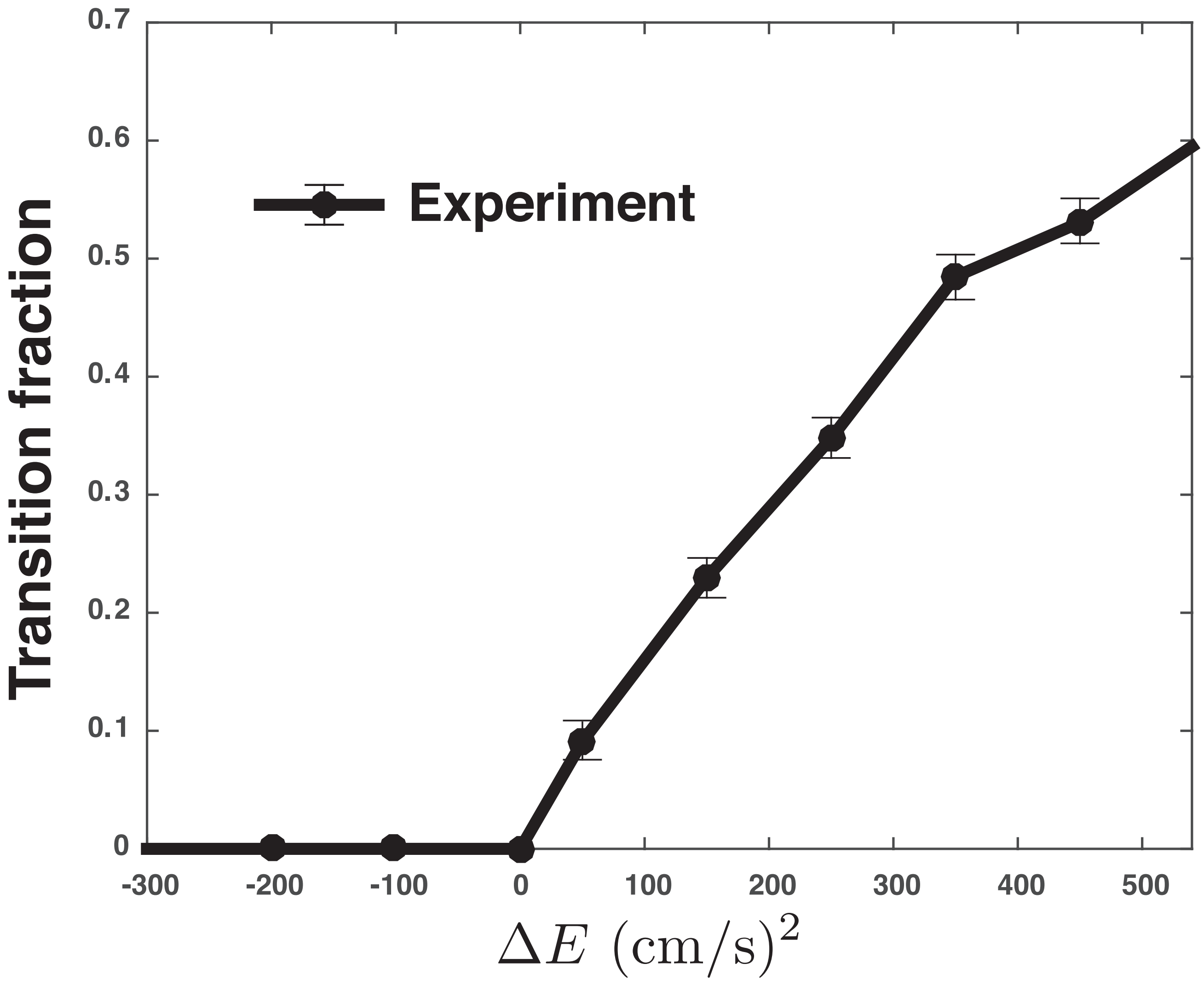

Theory for small excess energy, ΔE

- Area of the transitioning region, the tube cross-section (MacKay [1990])

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Theory for small excess energy, ΔE

- The transitioning fraction, under well-mixed assumption,

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- For slightly larger values of ΔE, there will be a correction term leading to a decreasing slope,

$$\frac{\partial p_{\text{trans}}}{\partial \Delta E} = \frac{T_{\text{po}}}{A_0}\left(1 - 2\frac{\tau}{A_0}\Delta E\right)$$

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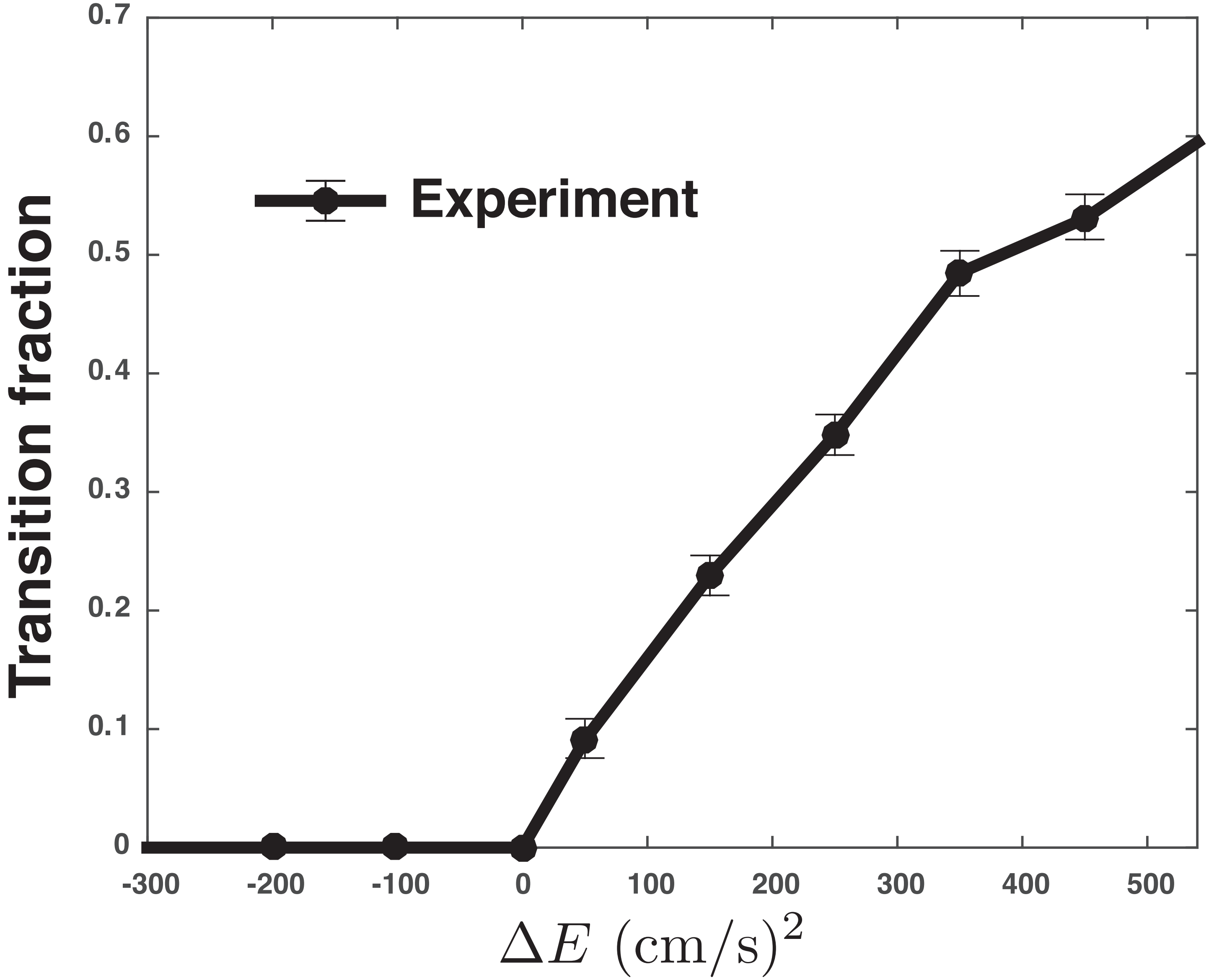

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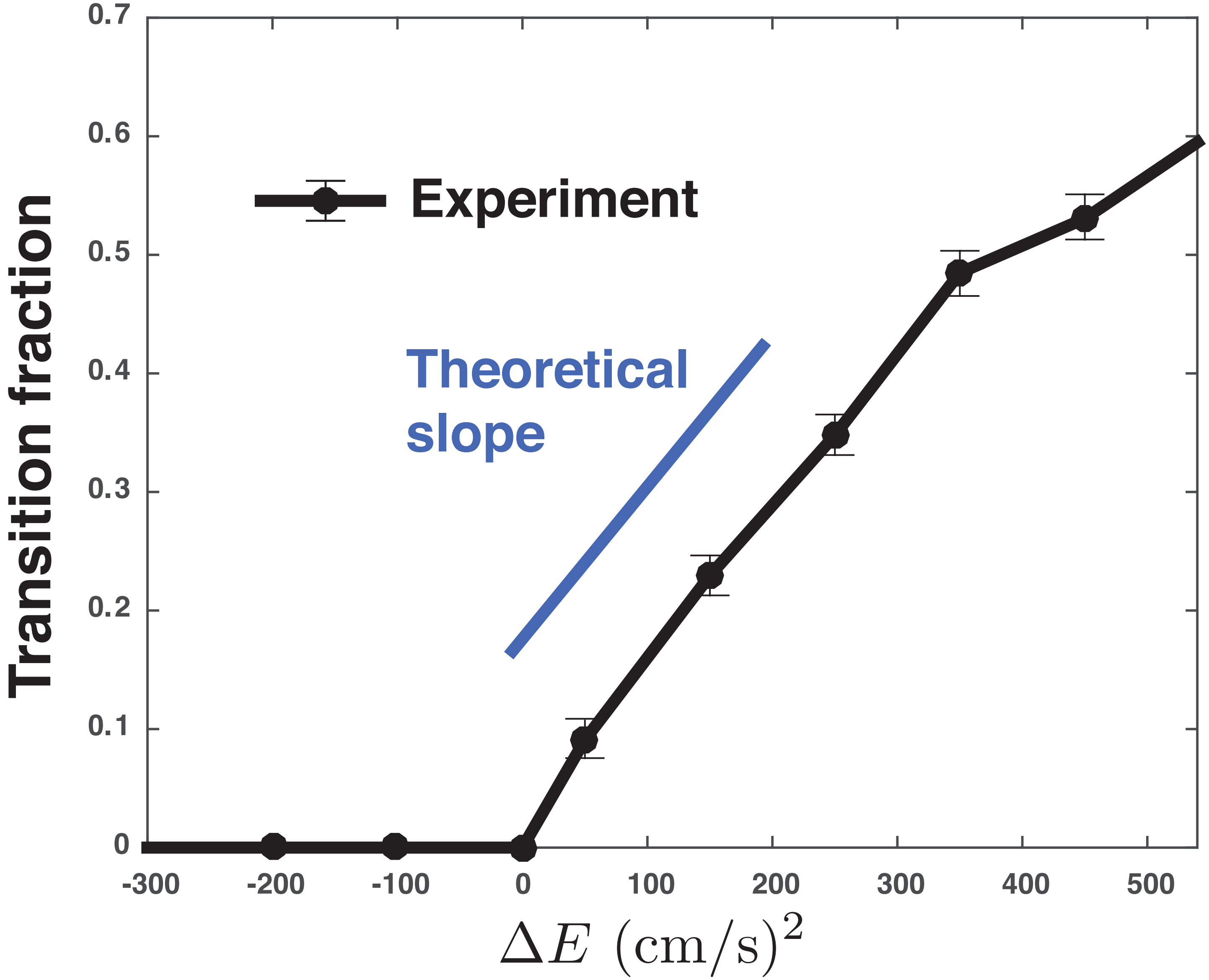

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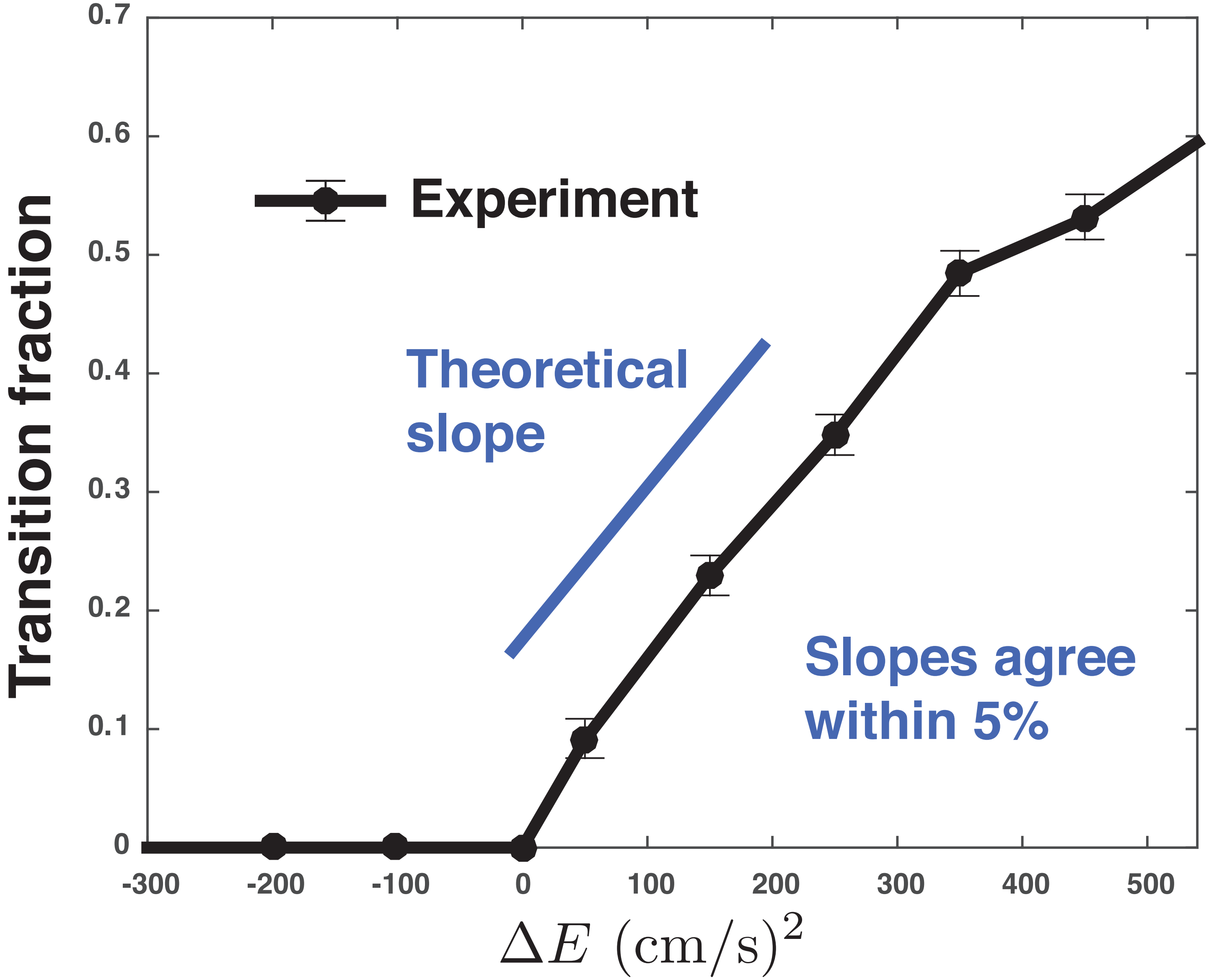

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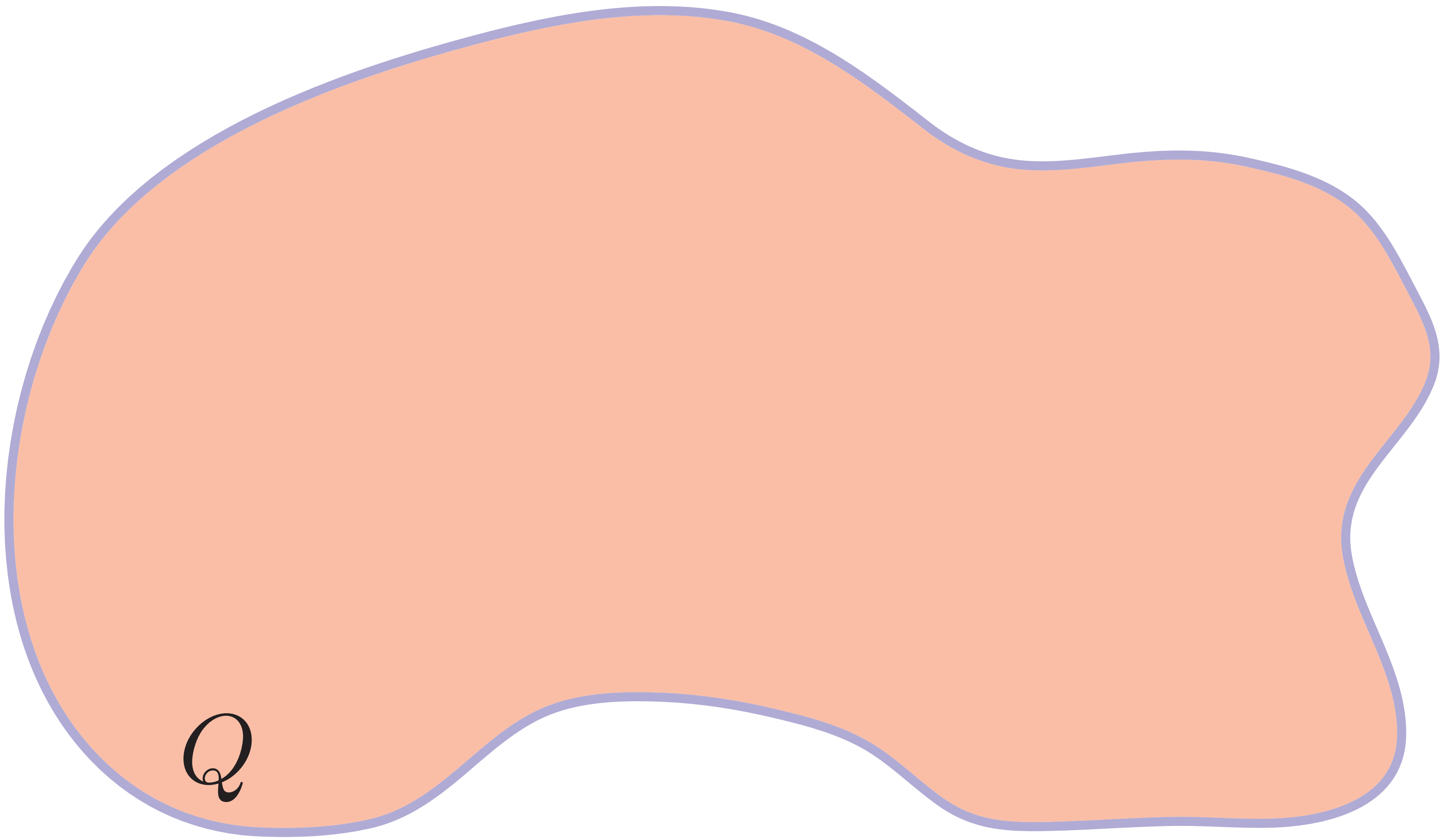
  — avoid a transition in the presence of a disturbance which is larger than the control

Partial control - safe set S



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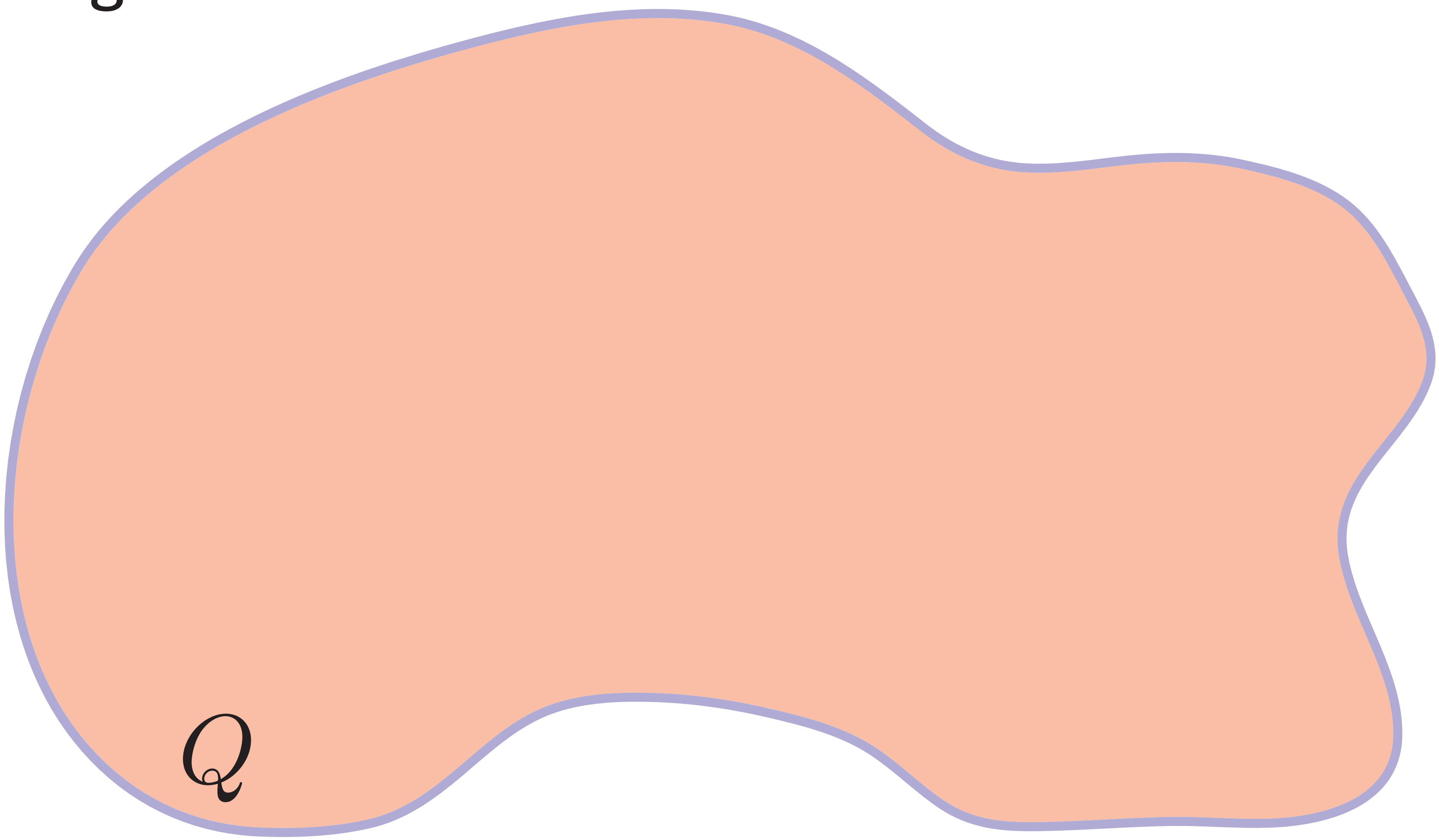
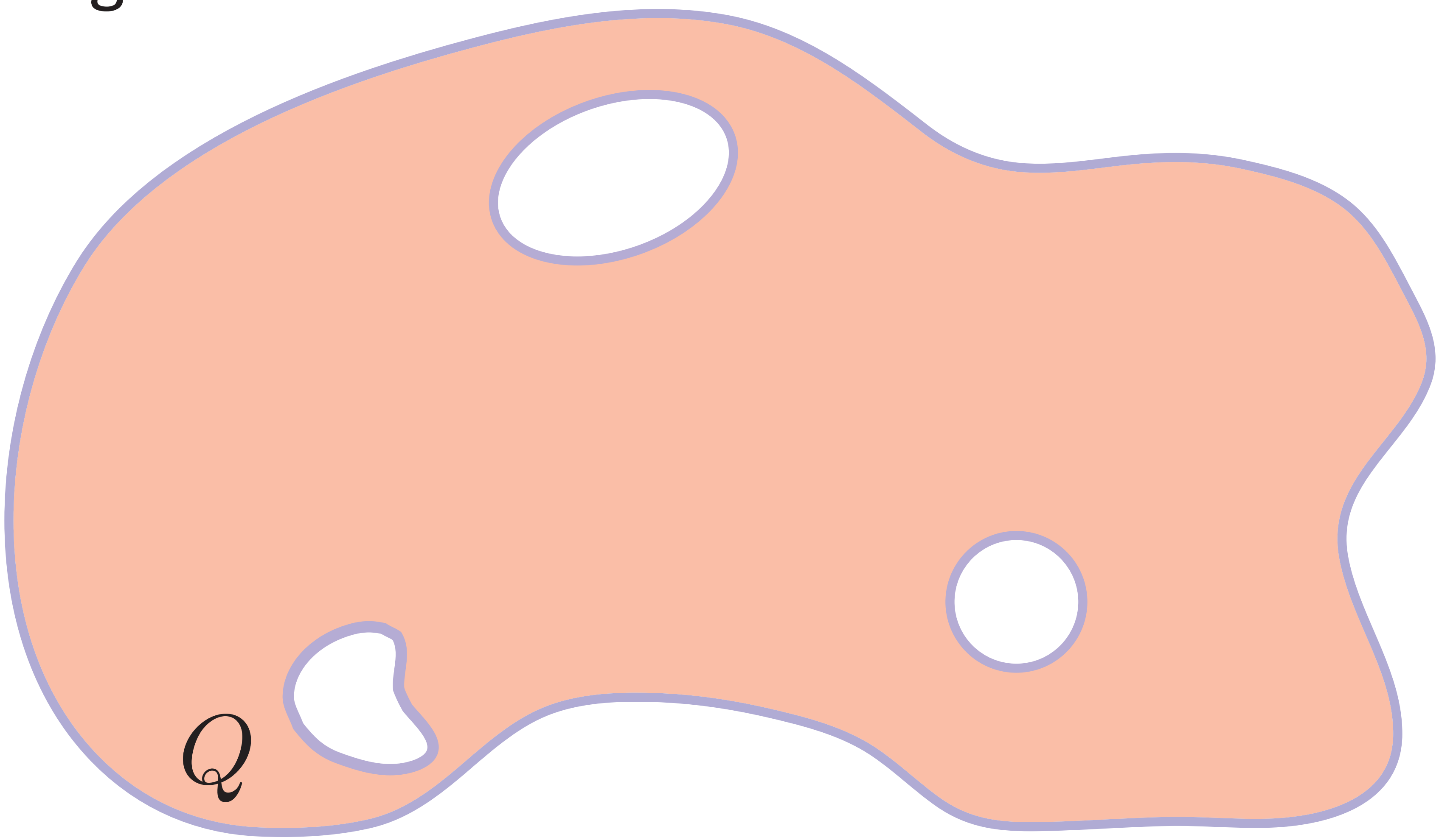
Region to be avoided in white



Partial control - safe set S

Region to be avoided in white - could include holes



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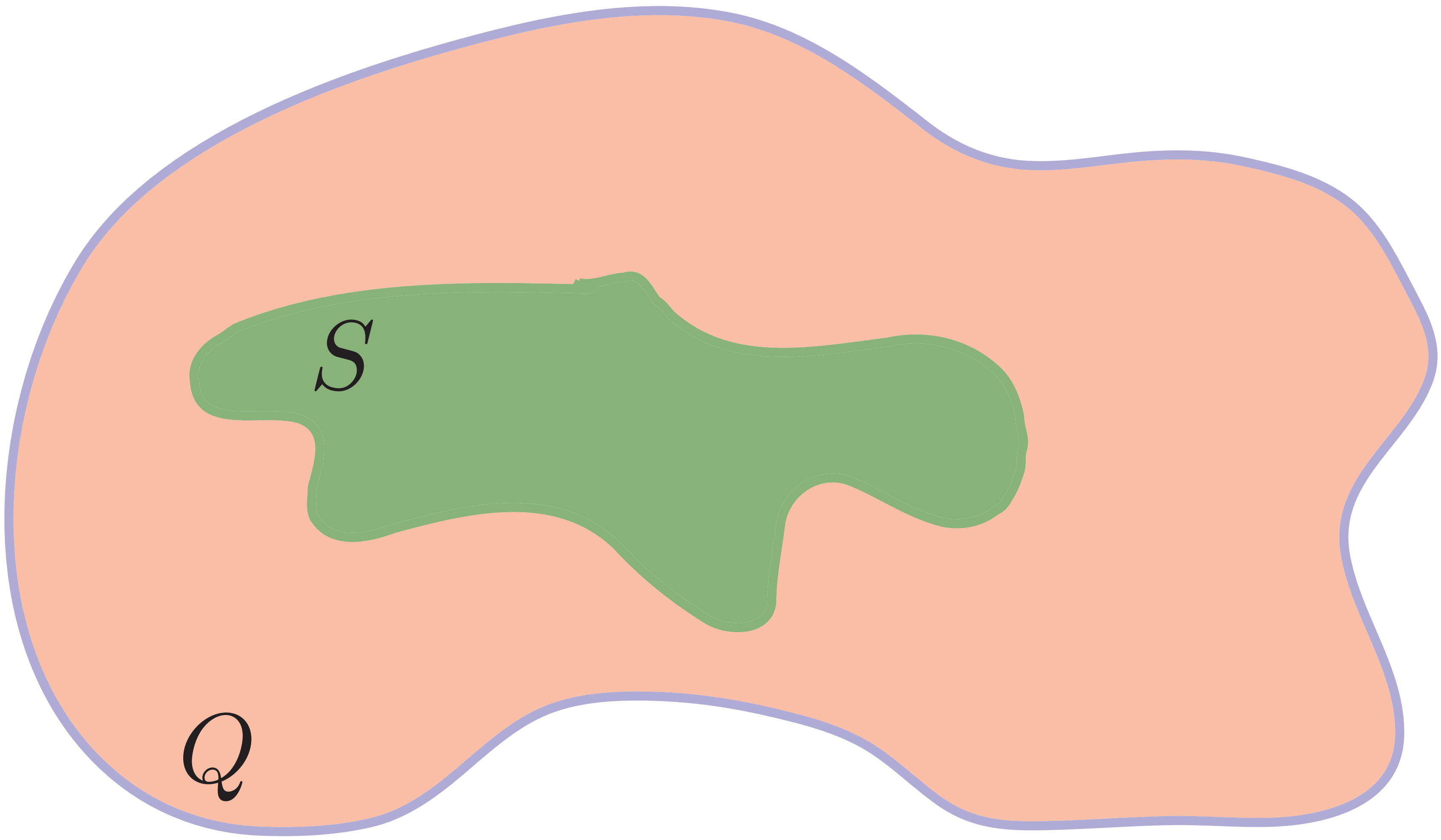

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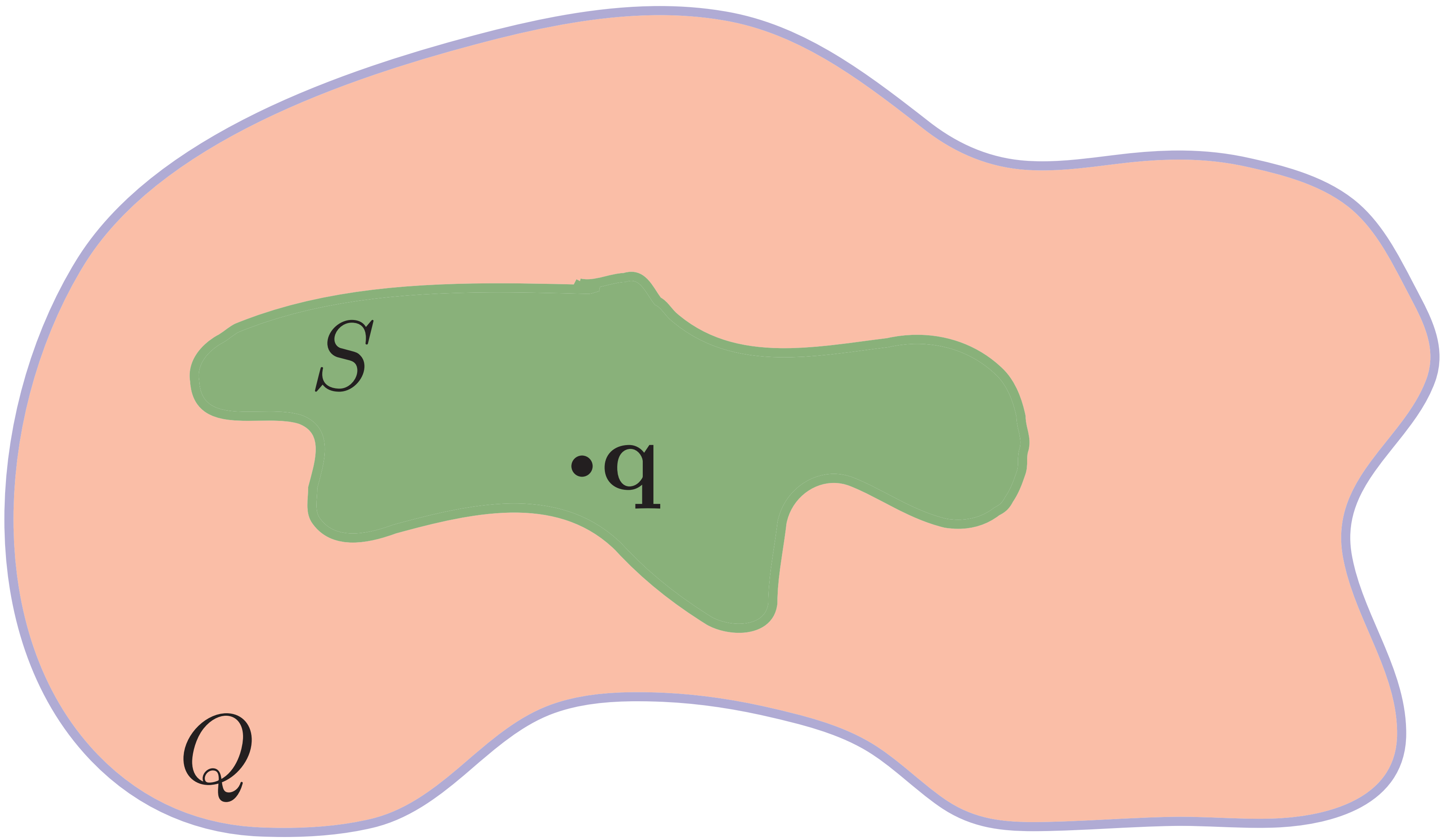

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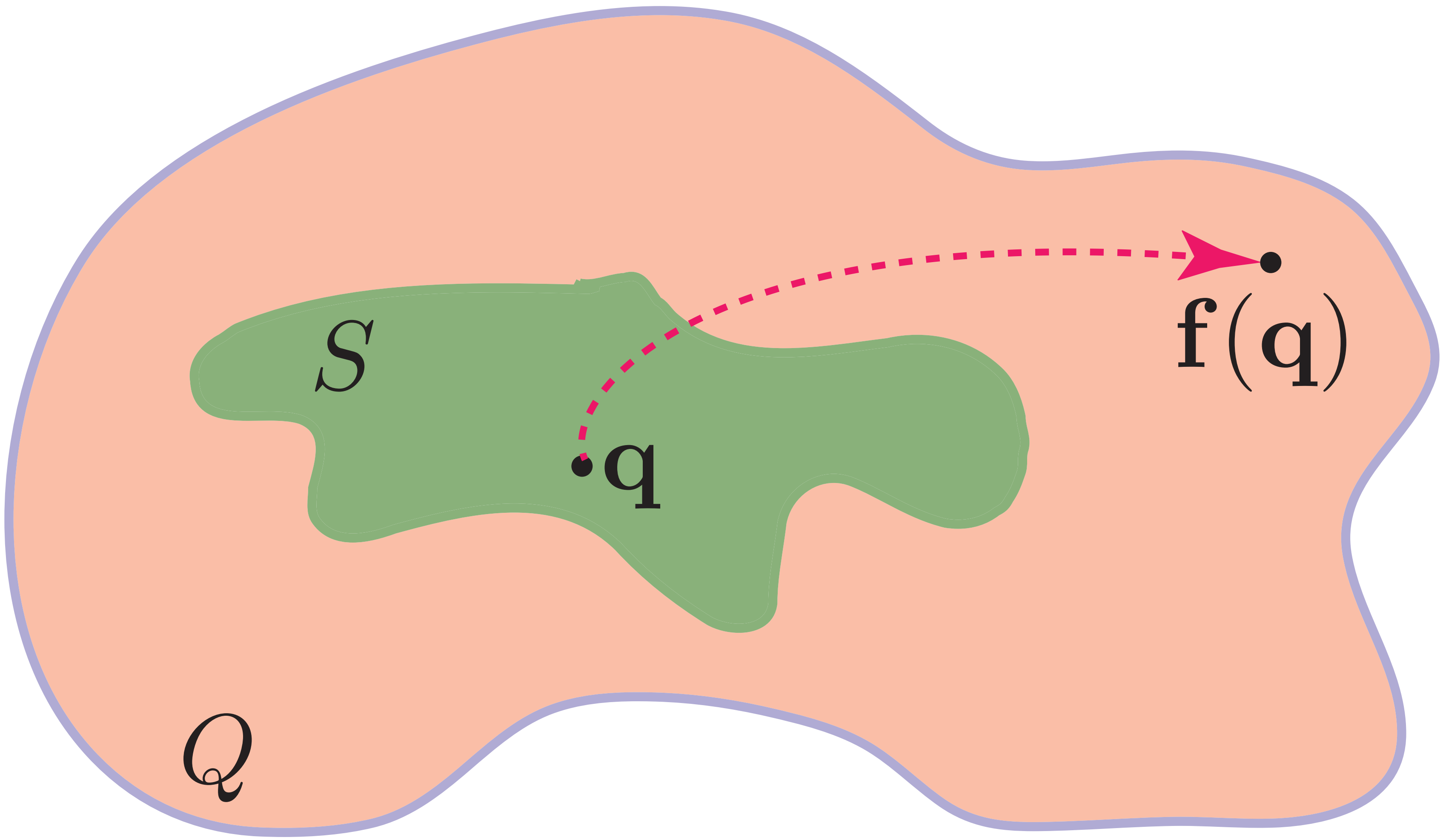

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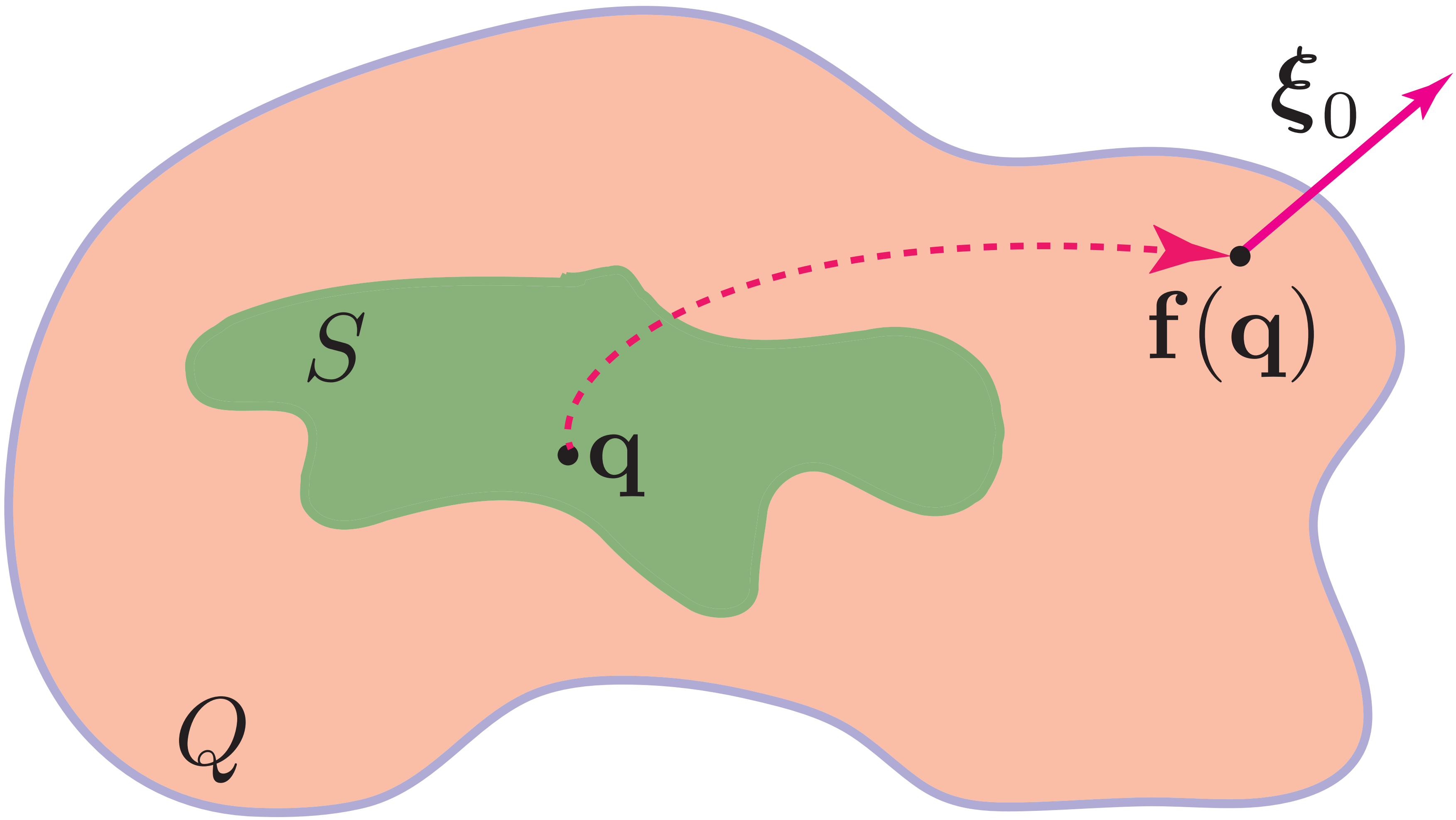

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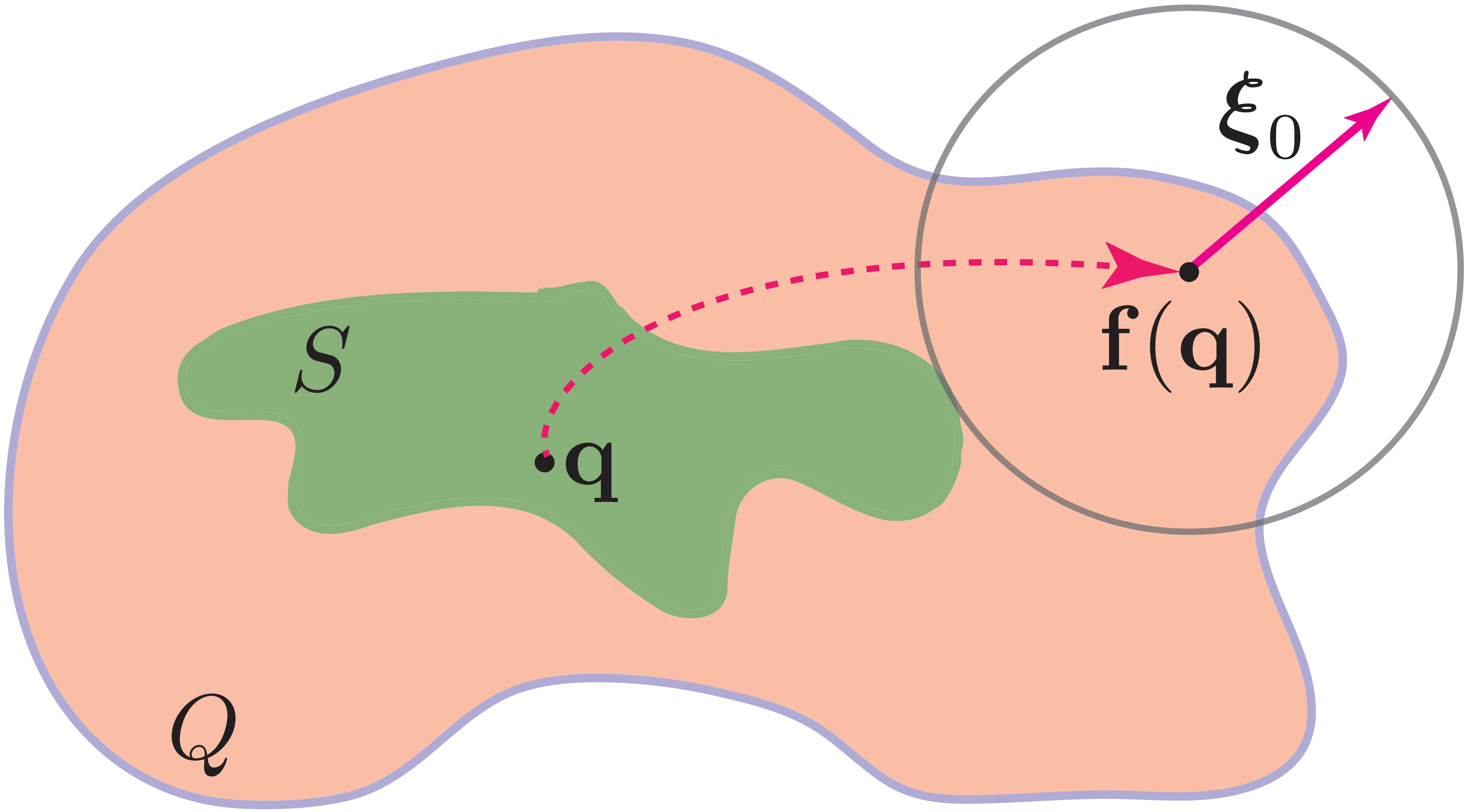

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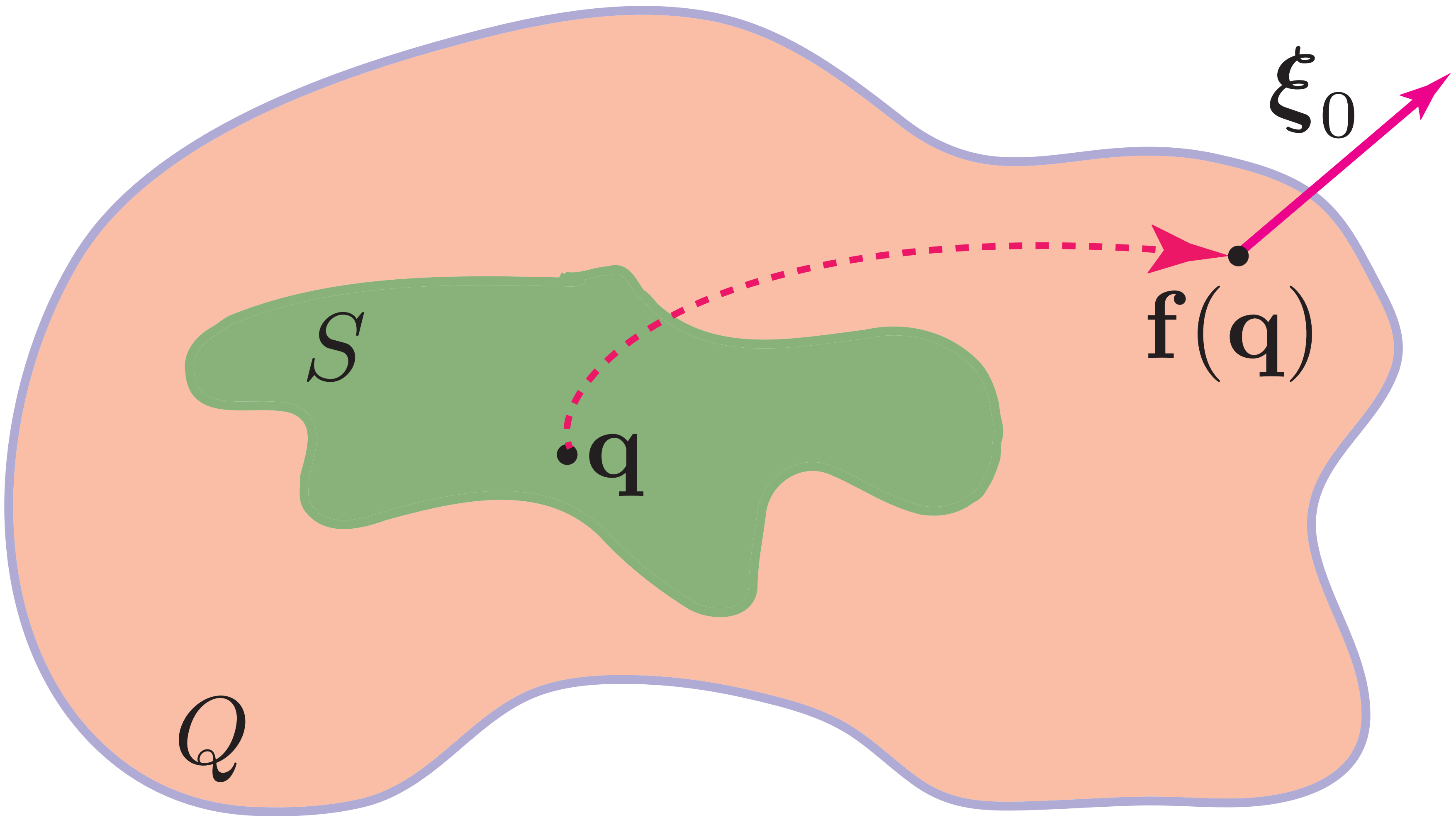

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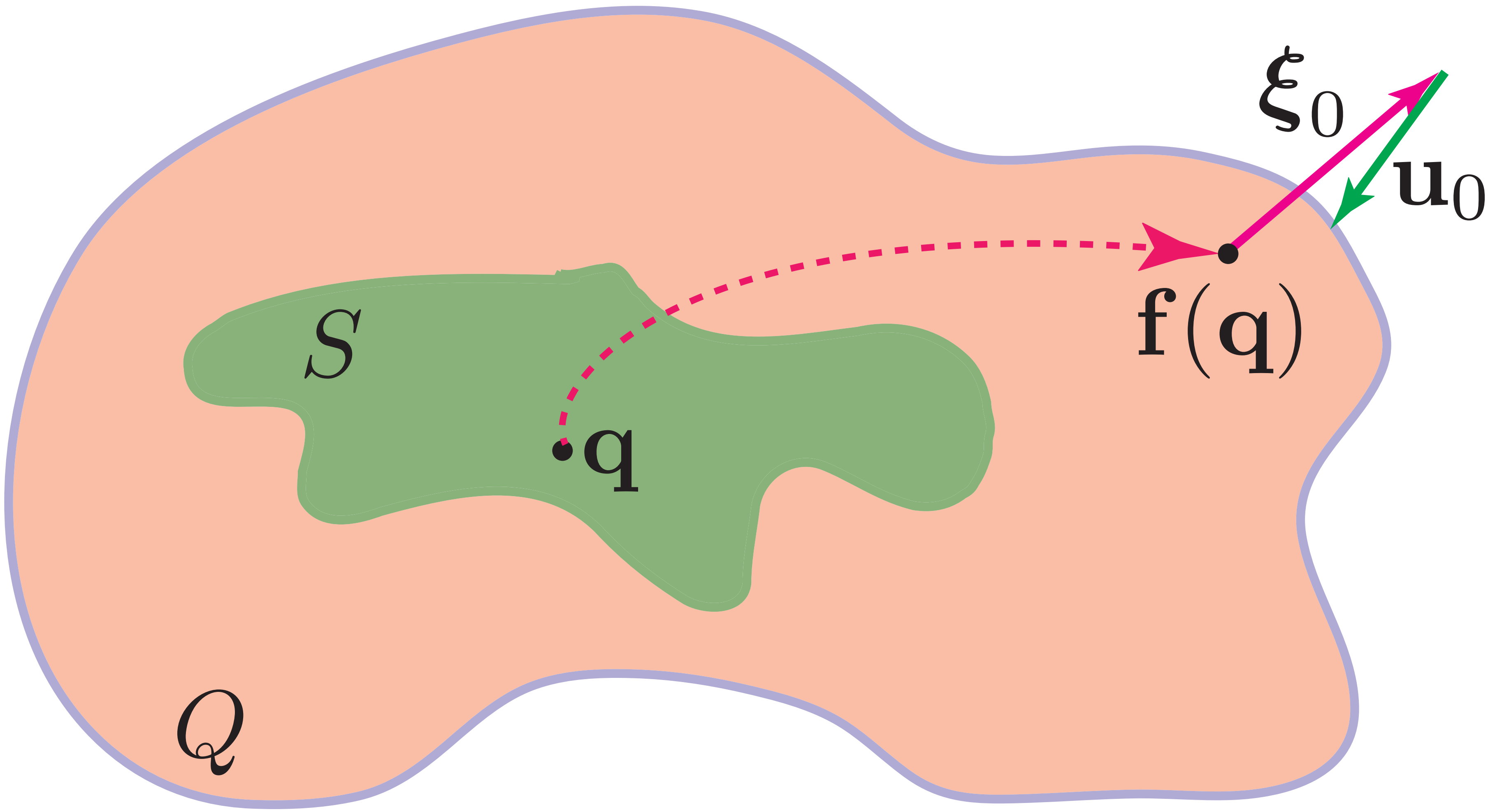

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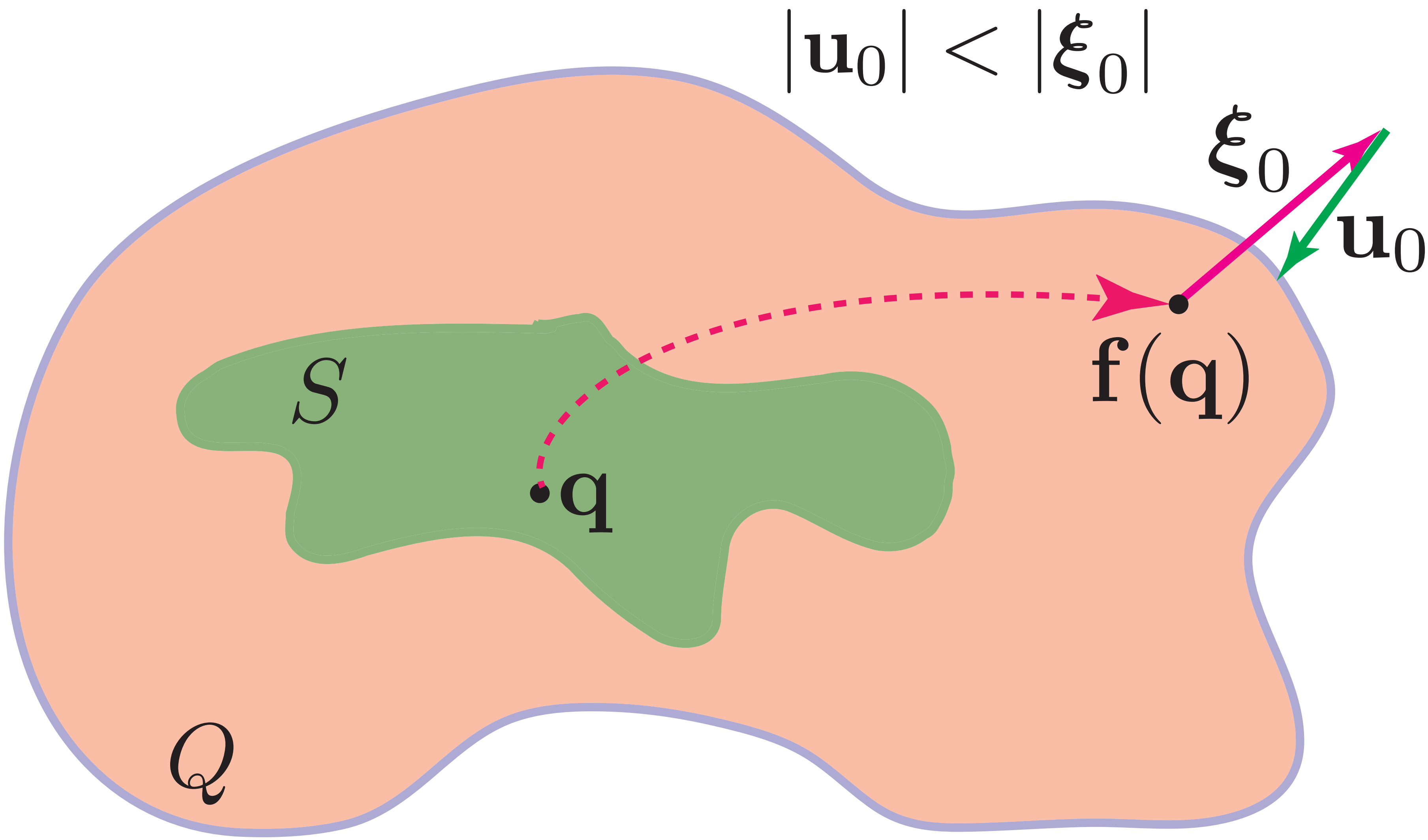


Control smaller than disturbance

Ship motion and capsize

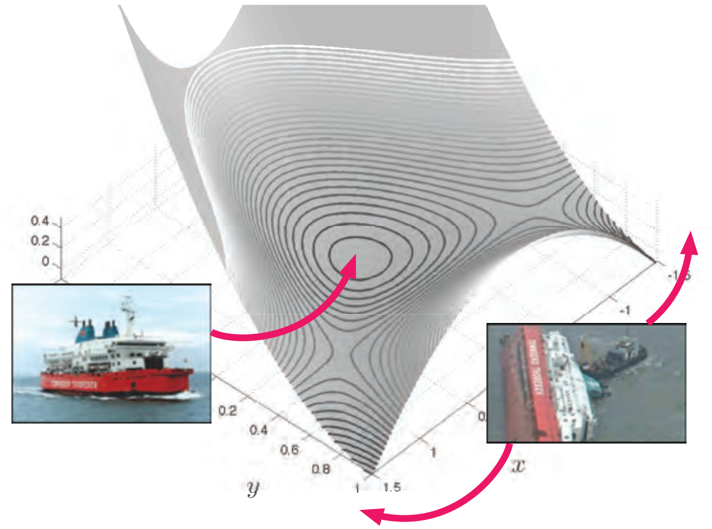
Ship motion and capsize

Ship motion and capsize

- Model built around Hamiltonian,

  $$H = p_x^2/2 + R^2 p_y^2/4 + V(x,y),$$

  where $x = \mathrm{roll}$ and $y = \mathrm{pitch}$ are coupled



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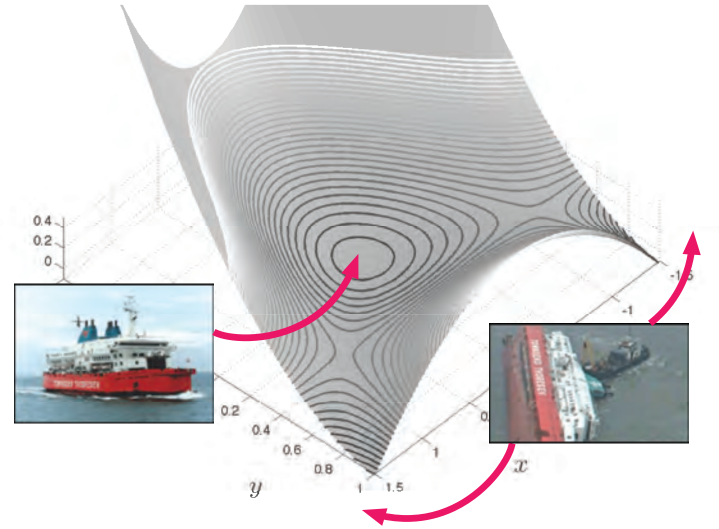

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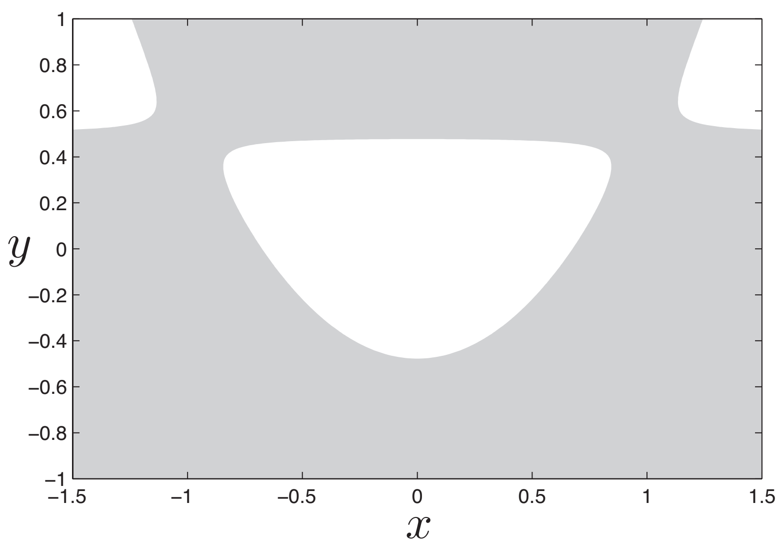

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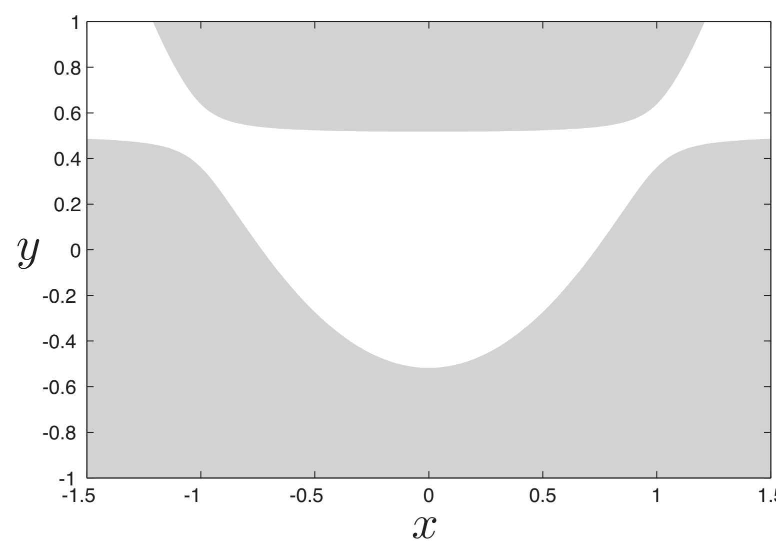

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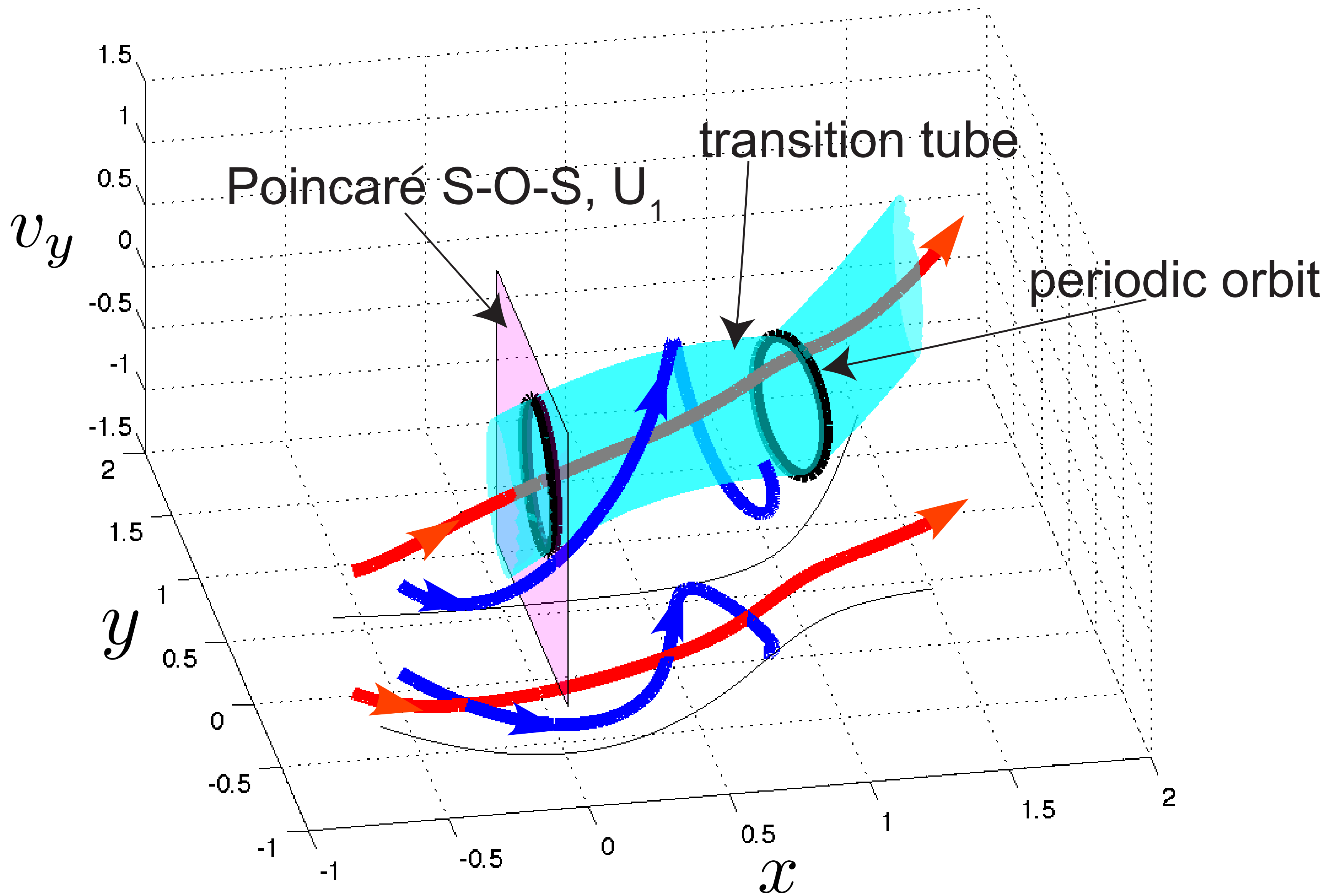
Tubes leading to capsize



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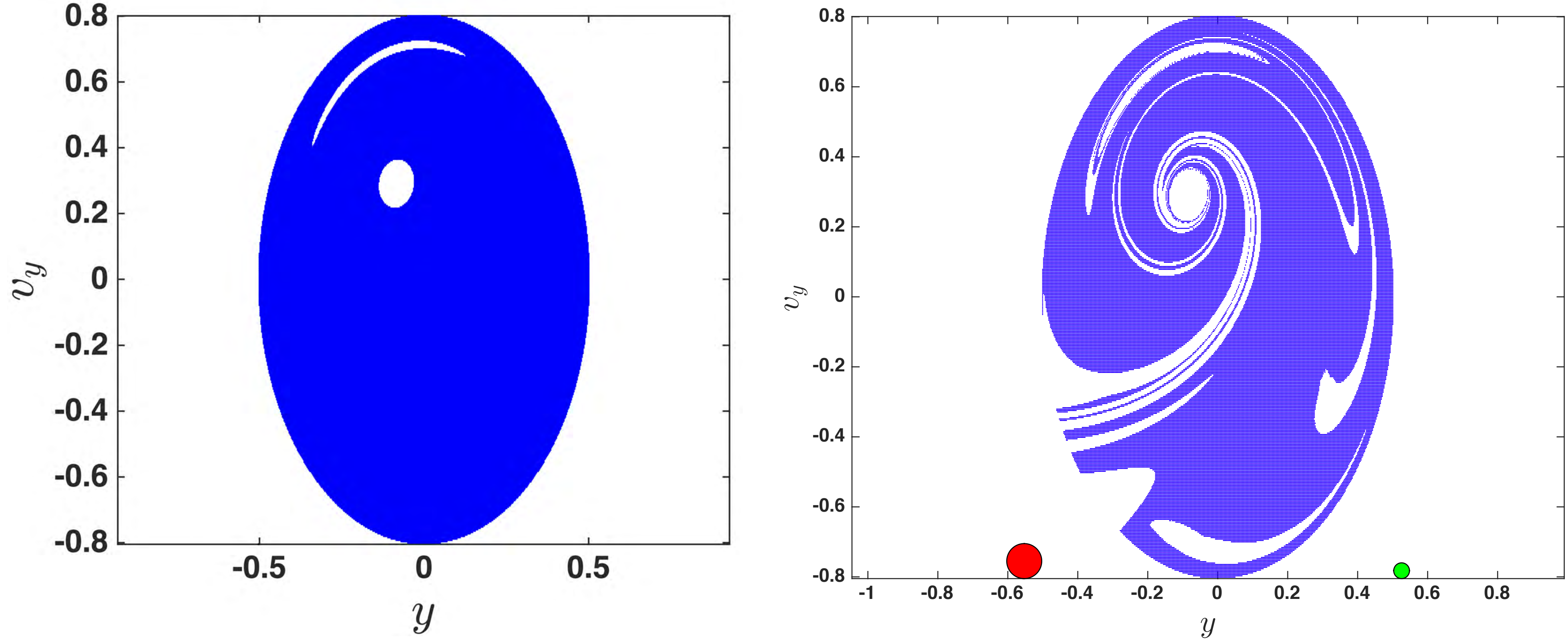

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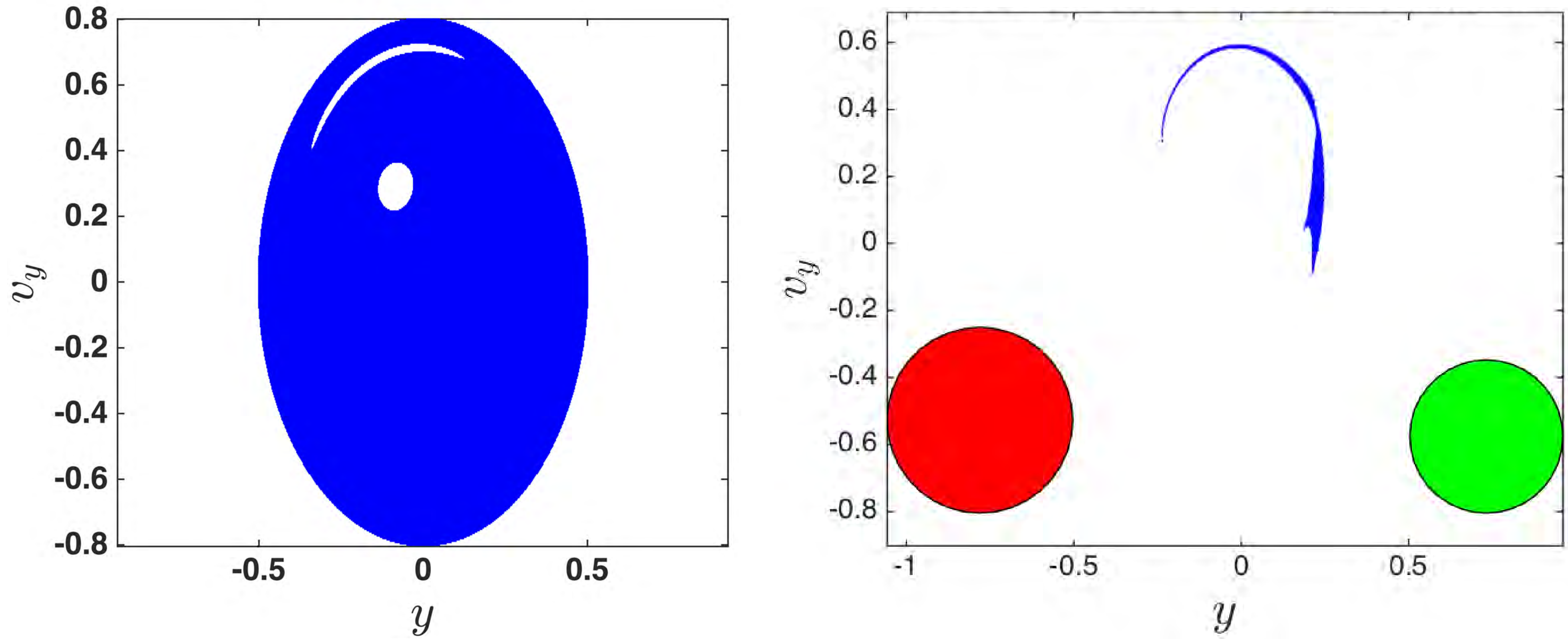
- Safe set shown when disturbance (red) is random ocean waves and smaller control (green) is via steering or control moment gyroscope
- Could inform **control schemes to avoid capsize** in rough seas

Partial control to avoid capsize



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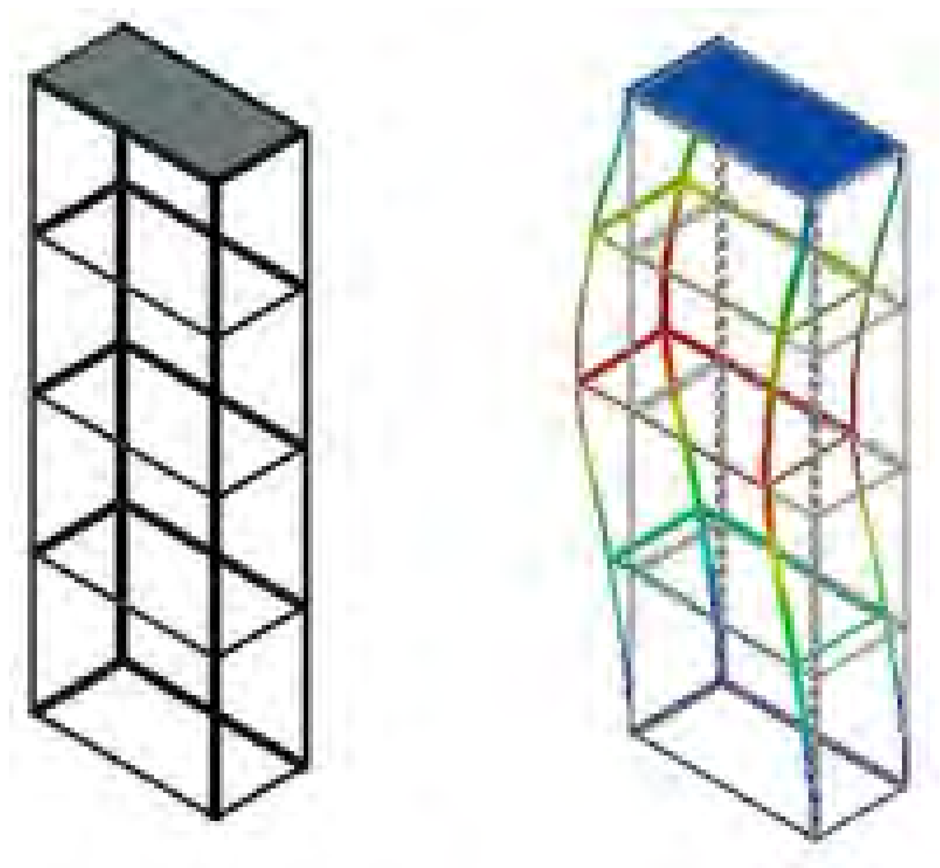
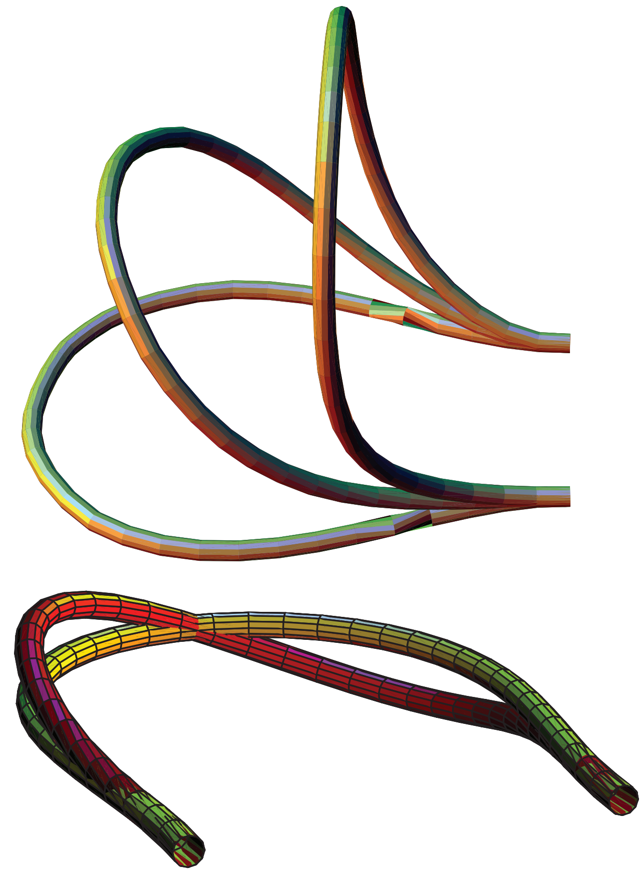
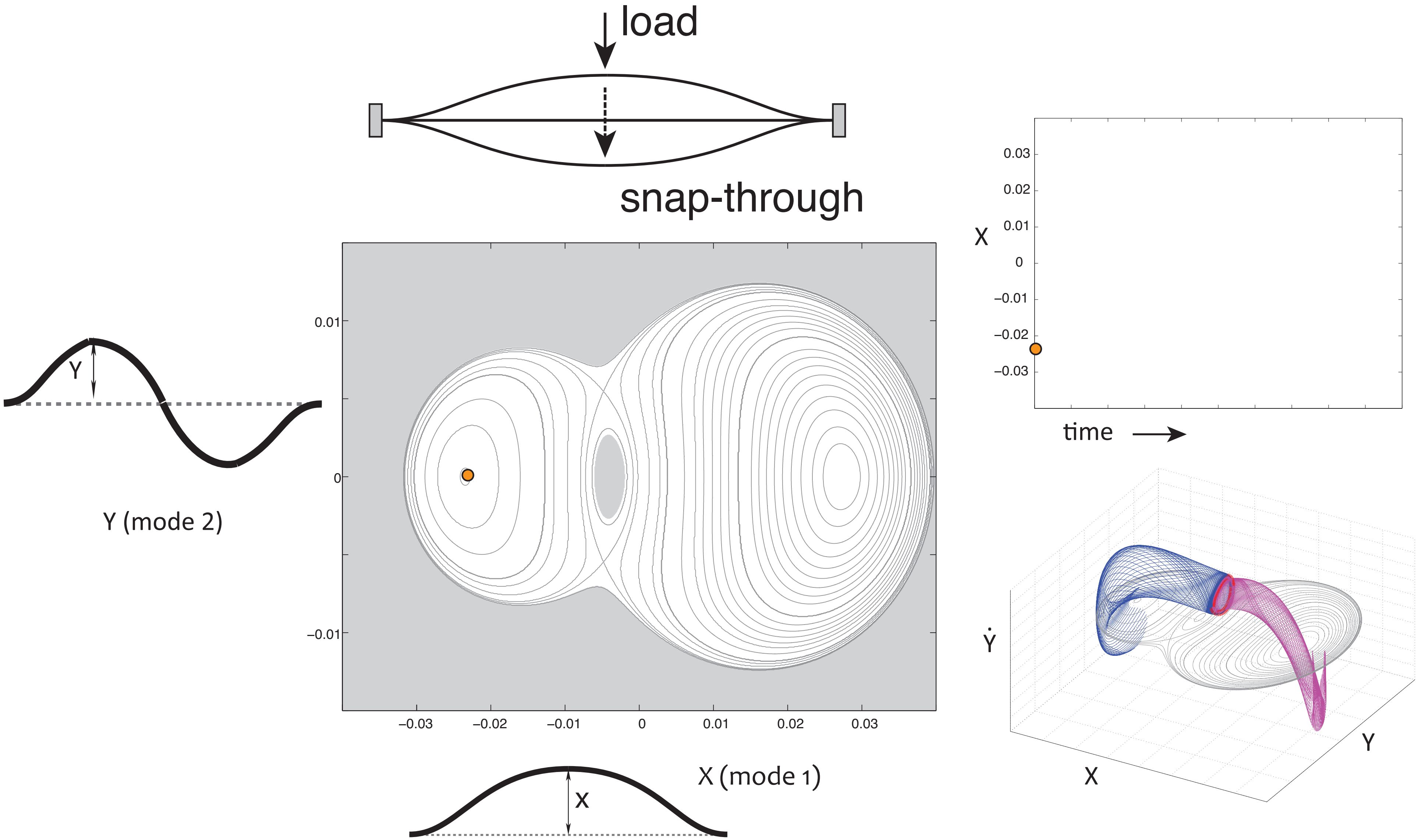




Buckling, bending, twisting, and crumpling of flexible bodies

- adaptive structures that can bend, fold, and twist to provide advanced engineering opportunities for deployable structures, mechanical sensors

Next steps — structural mechanics



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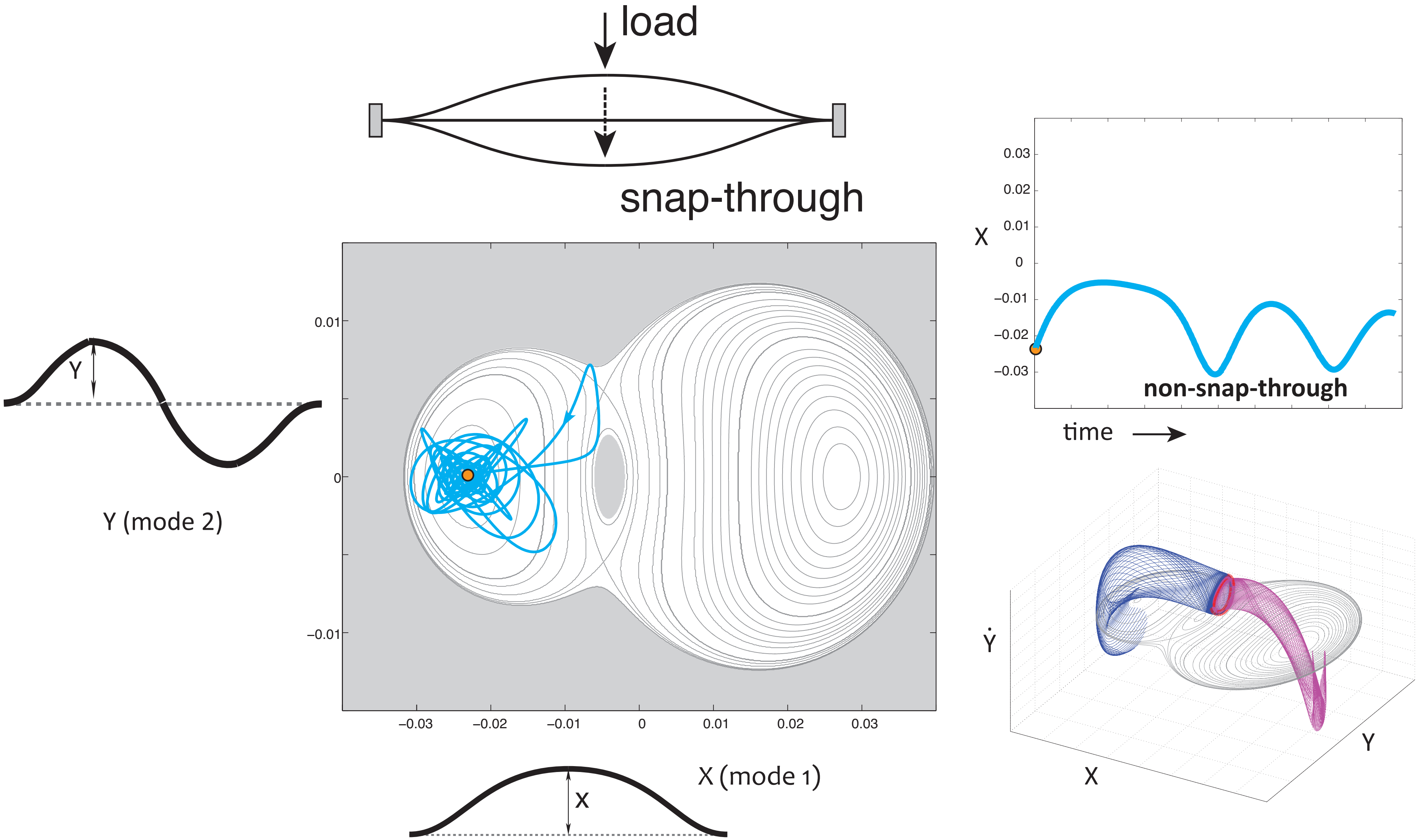

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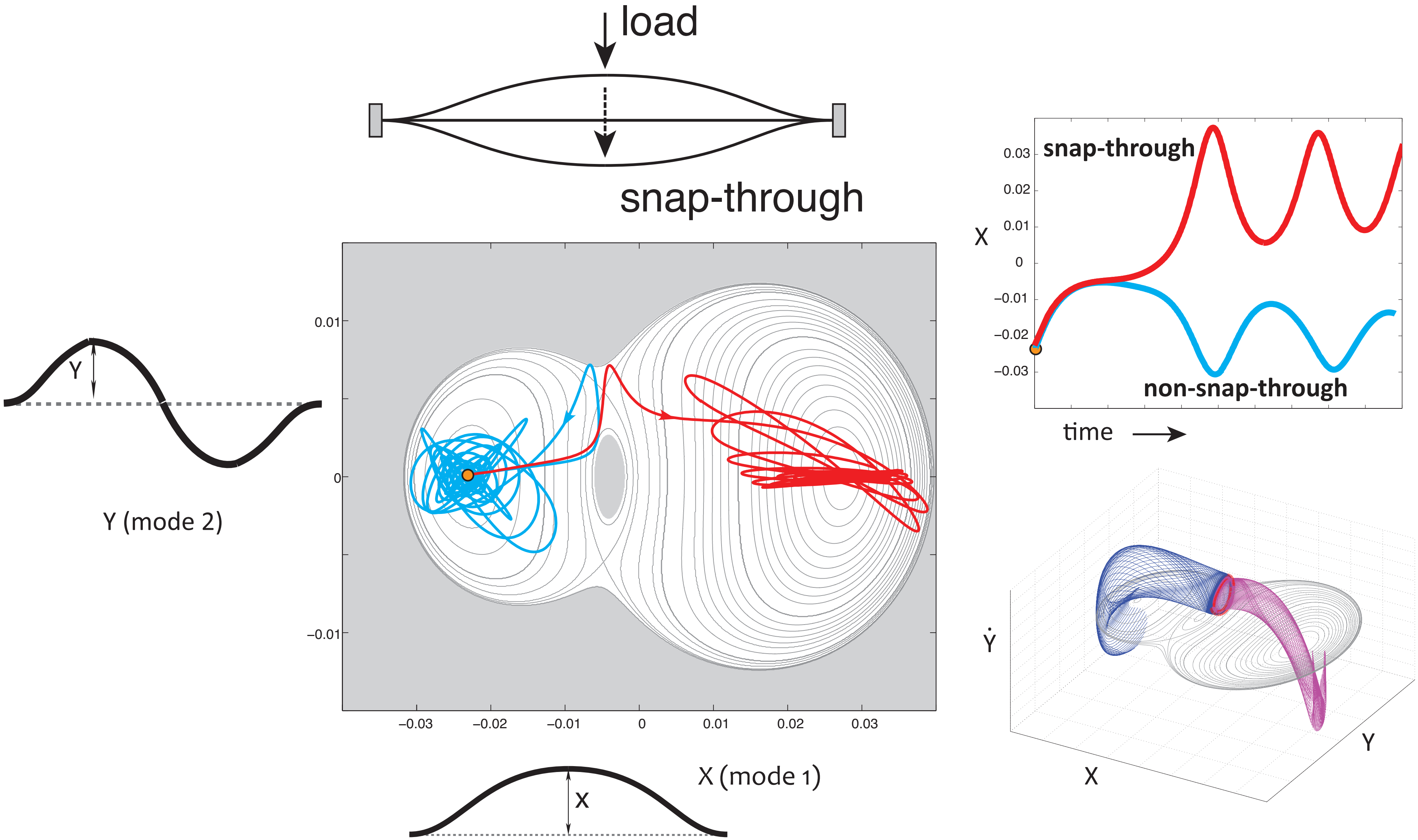

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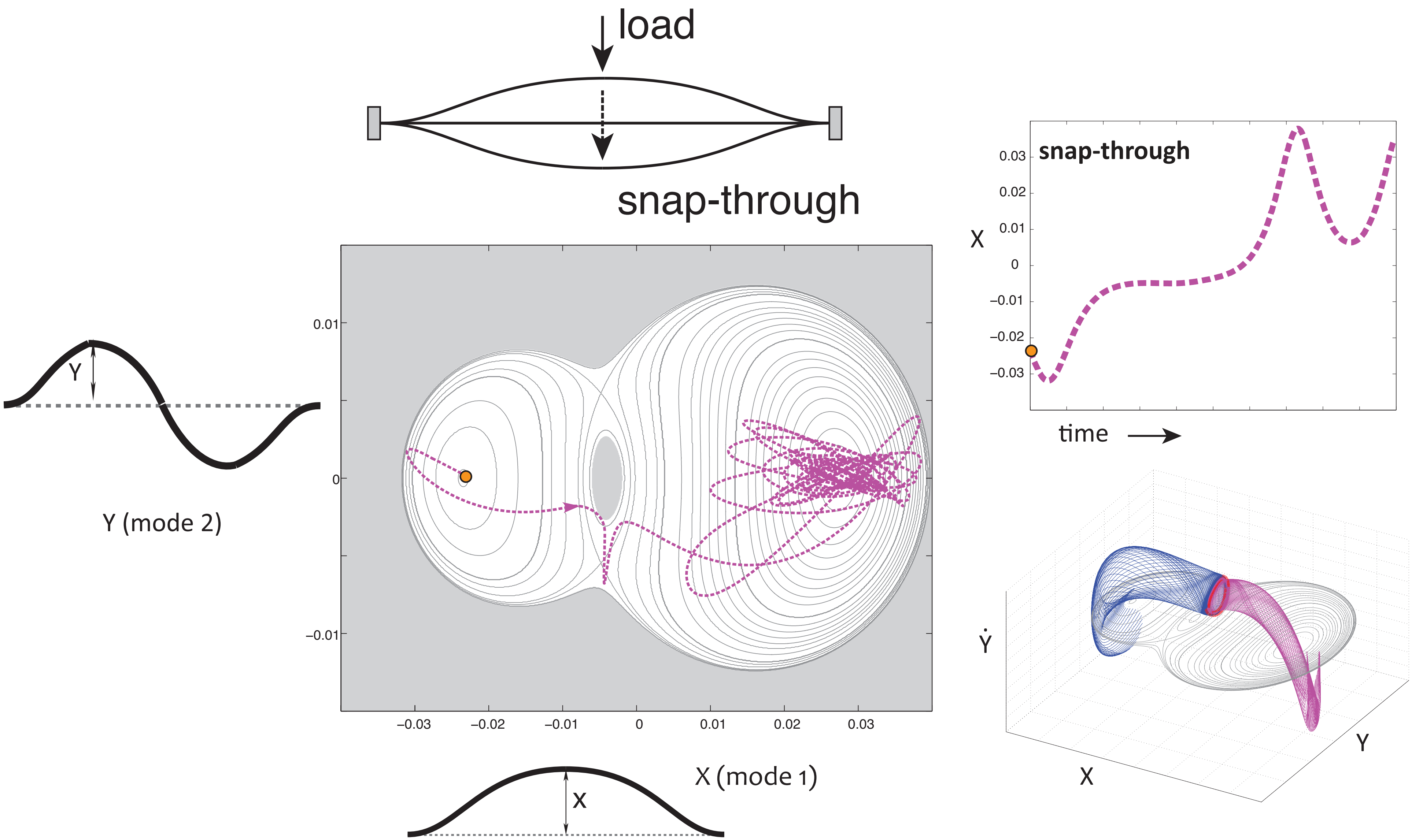

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Papers in preparation; check status at:
shaneross.com