A map approximation for the restricted three-body problem

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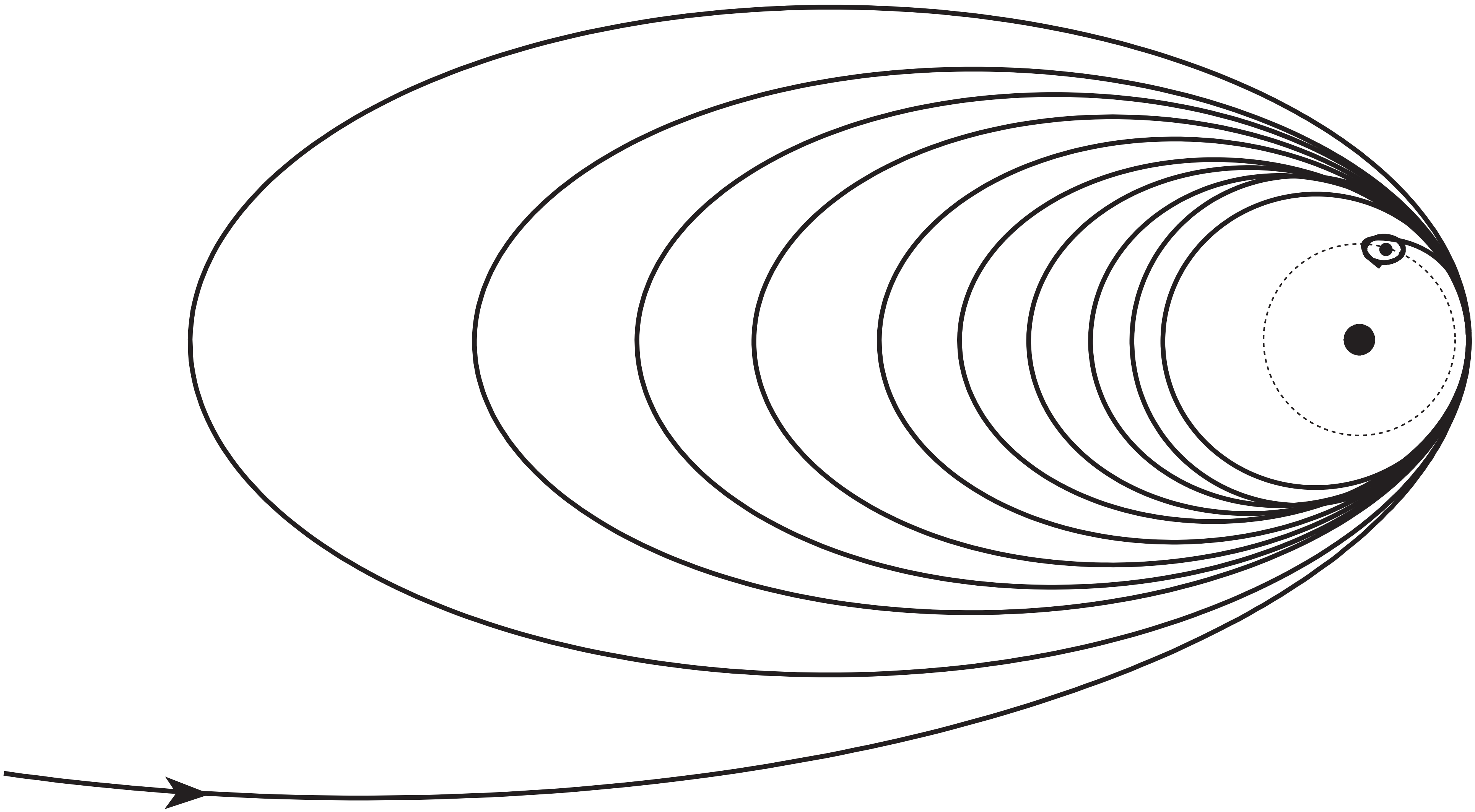
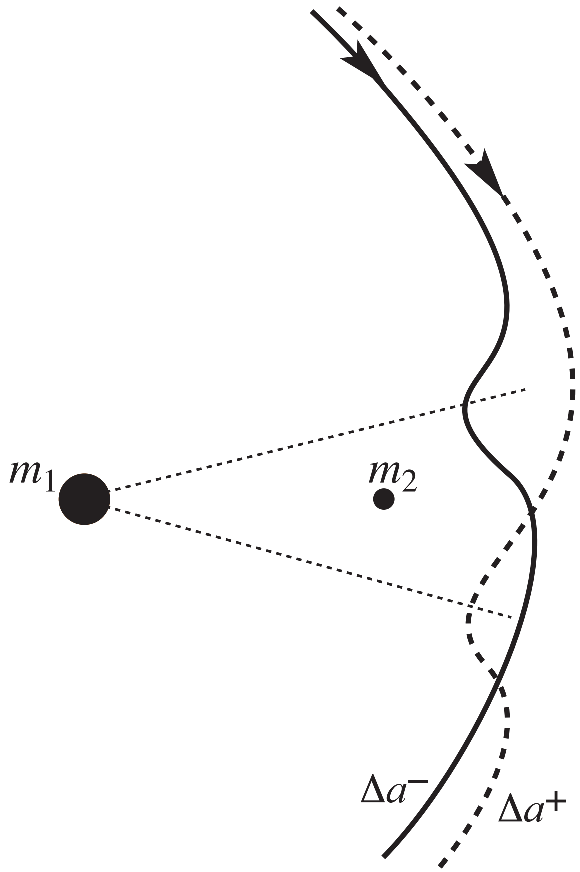
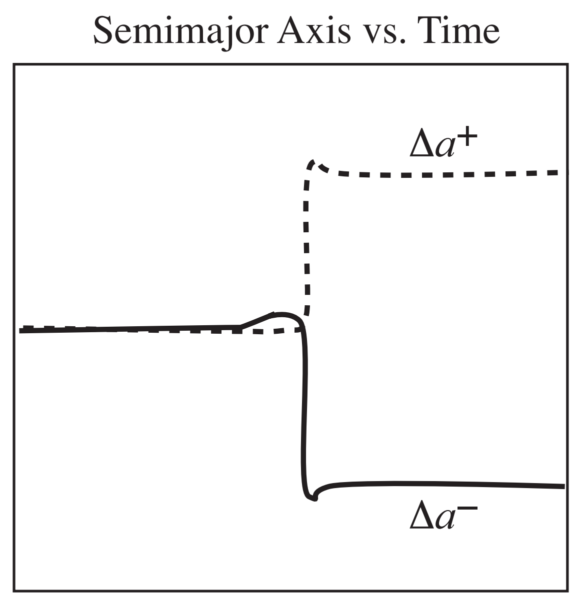
Infinity to capture about small companion in binary pair?



- After **consecutive gravity assists**, large orbit changes

- Key idea: model particle motion as “**kicks**” at periapsis

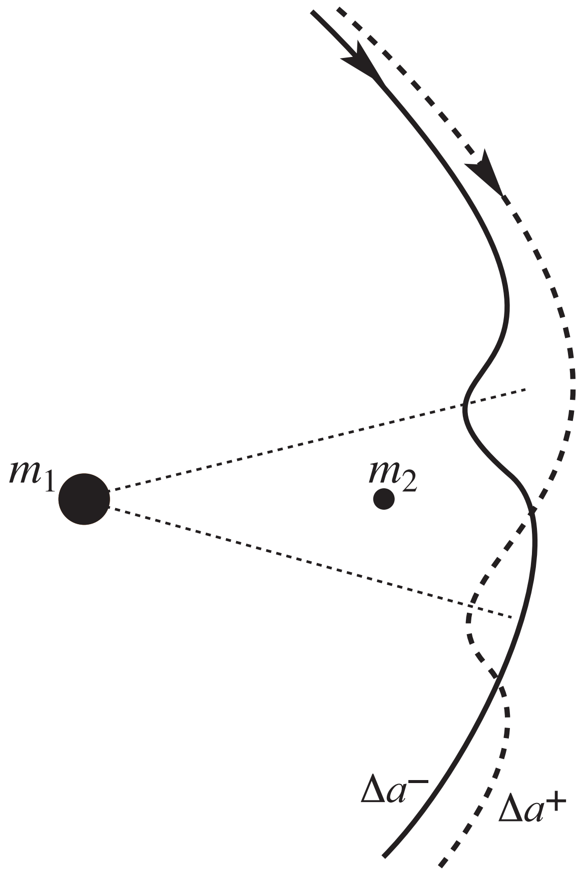


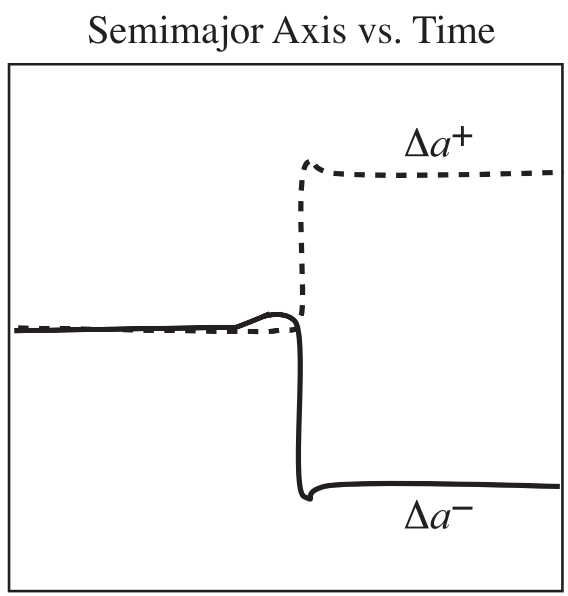


In rotating frame where m_1, m_2 are fixed

Kicks at periapsis

- Sensitive dependence on **argument of periapse** ω

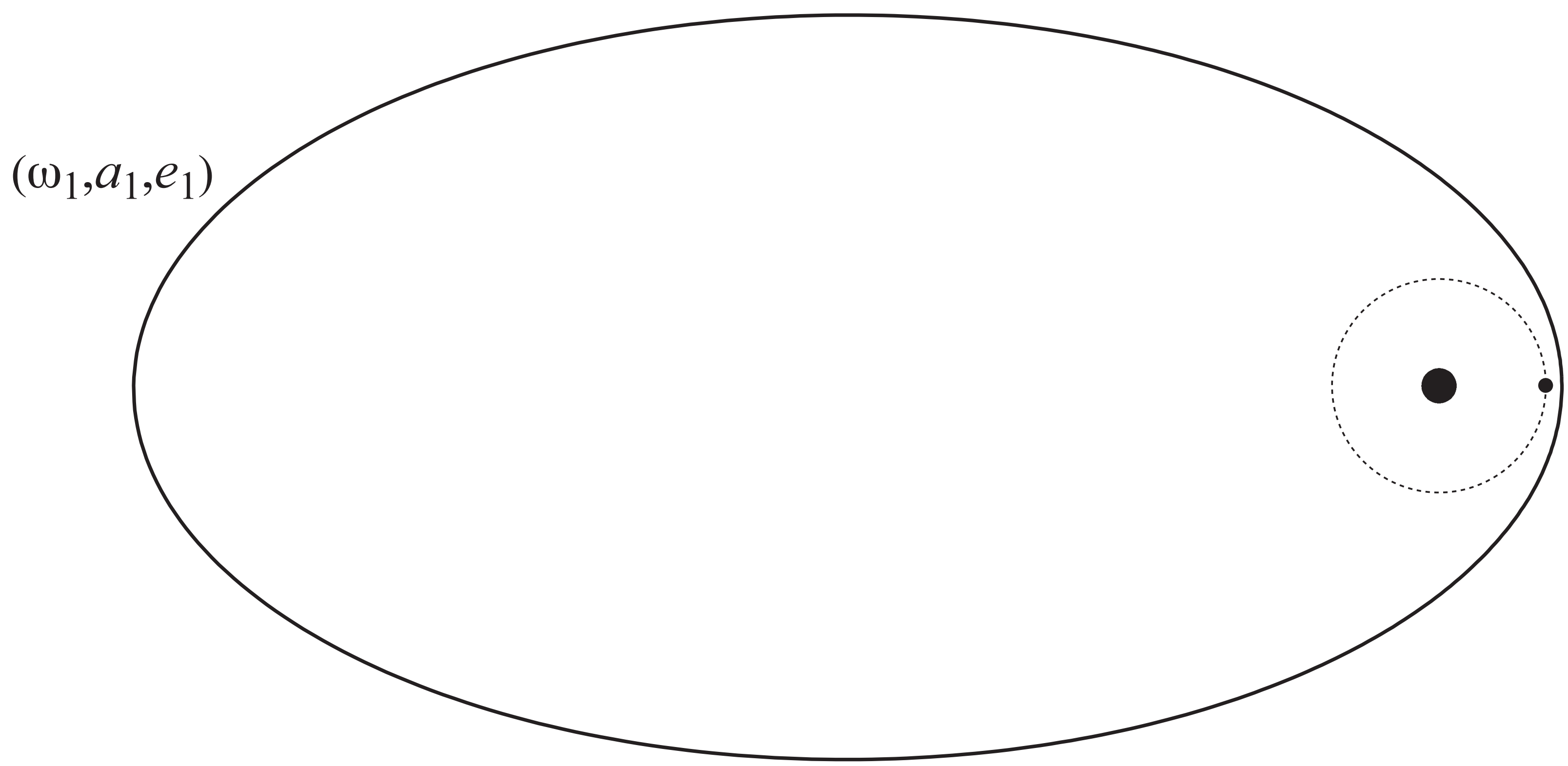




In rotating frame where m_1, m_2 are fixed

Kicks at periapsis

- Construct **update map** $(\omega_1, a_1, e_1) \mapsto (\omega_2, a_2, e_2)$ using average perturbation per orbit by smaller mass



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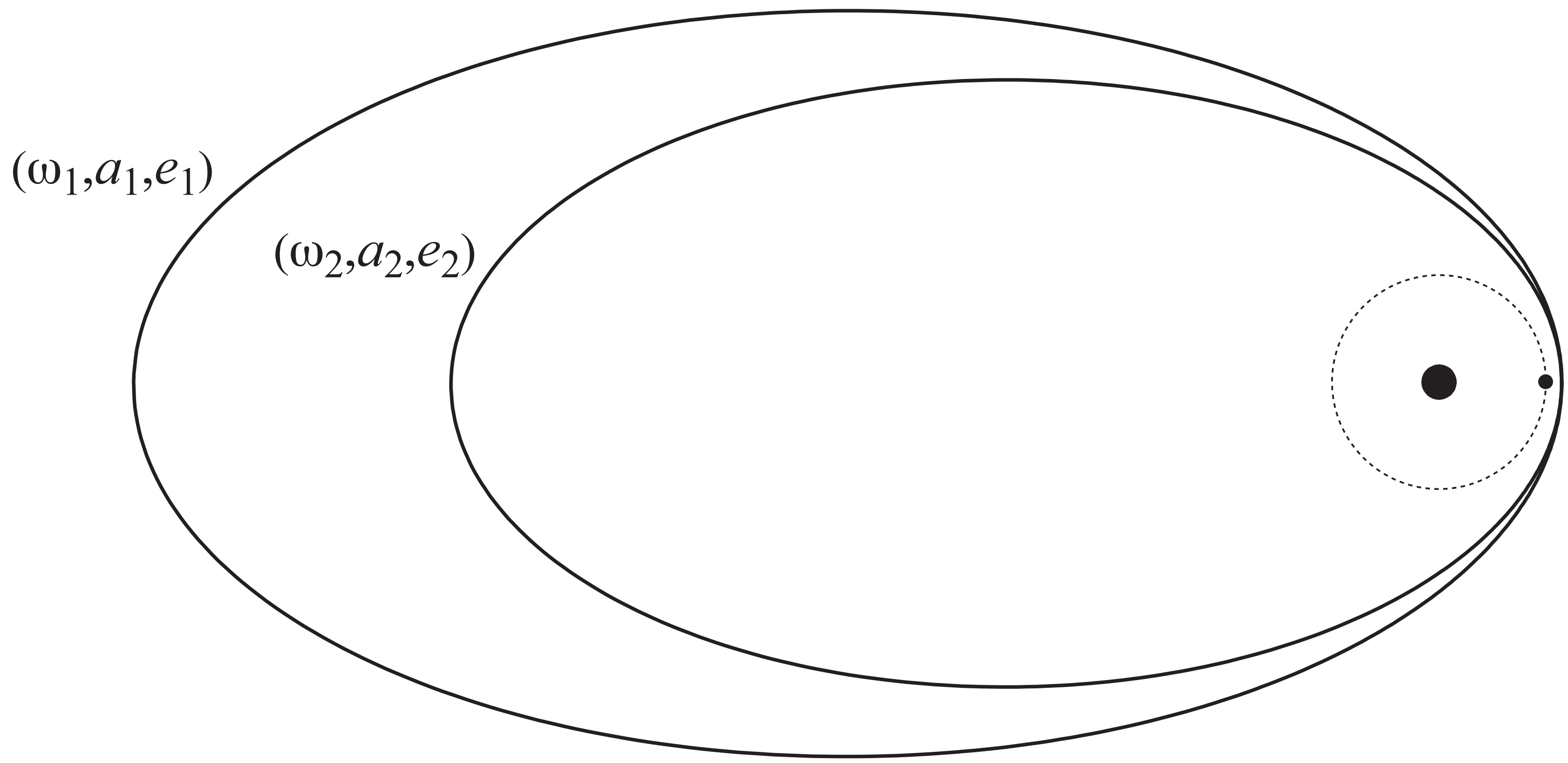
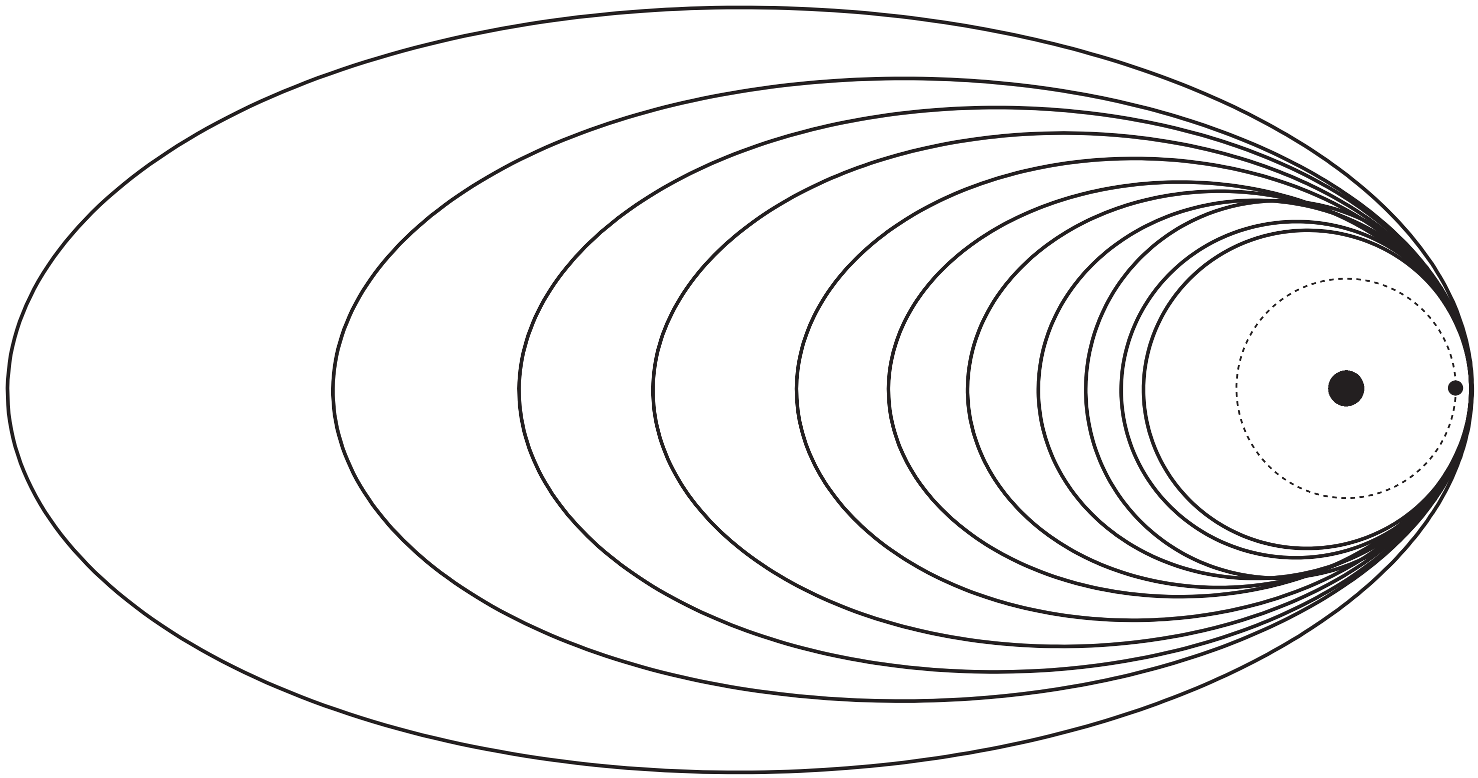


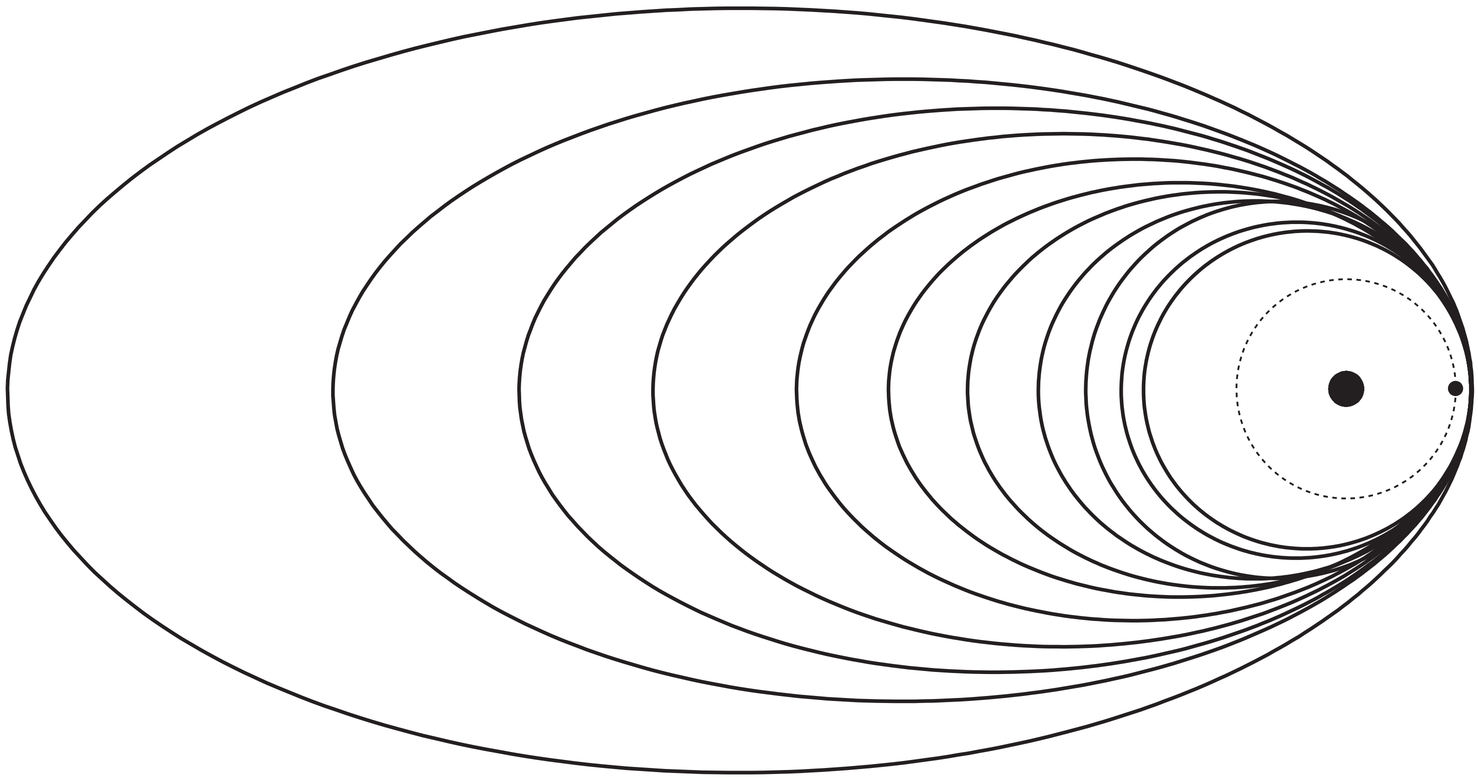
Kicks at periapsis

- Cumulative effect of **consecutive gravity assists** can be dramatic.



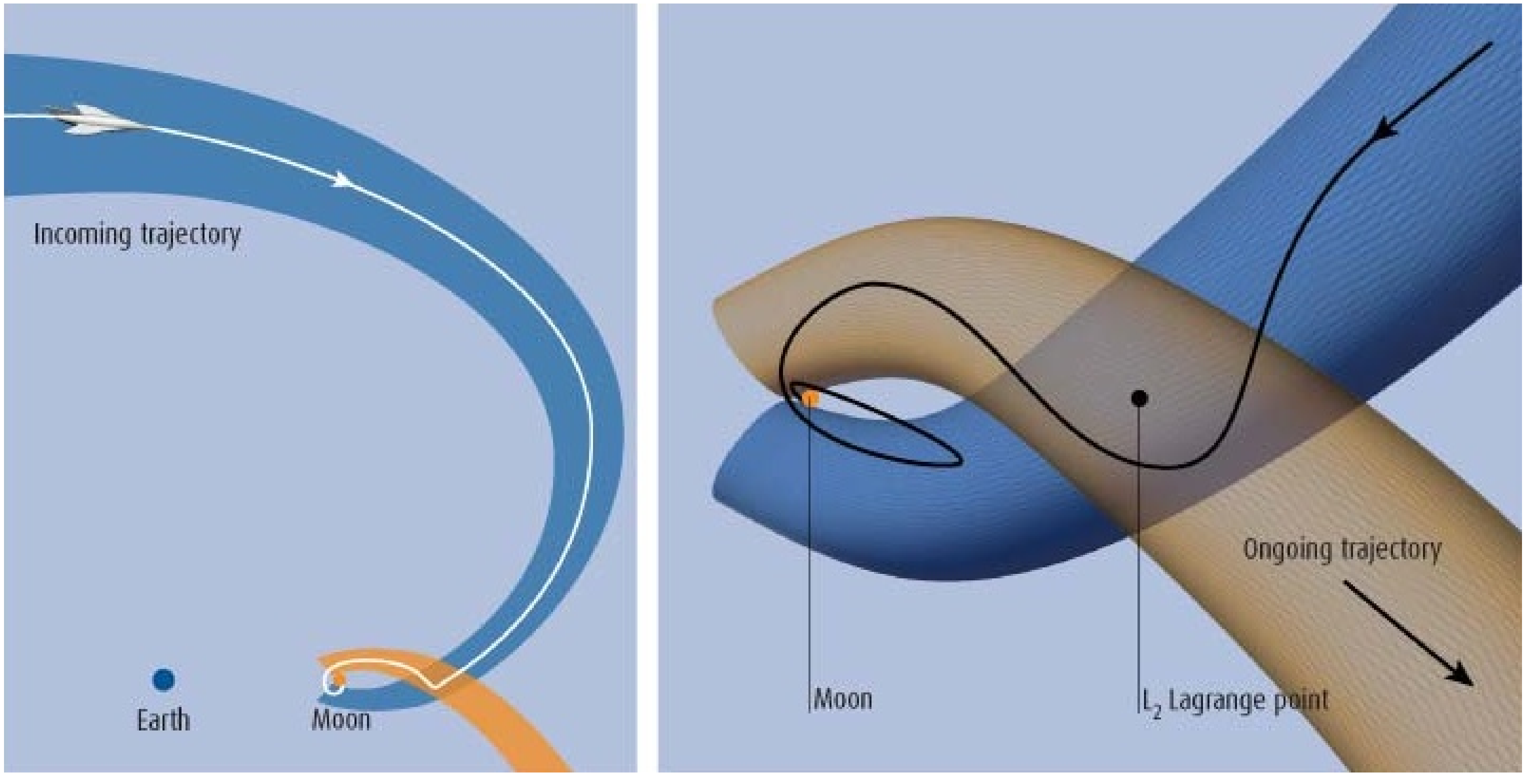
Not hyperbolic swing-by

- Occur **outside sphere of influence** (Hill radius) – not the close, hyperbolic swing-bys of Voyager



Capture by secondary

- Dynamically connected to capture thru tubes



Starting model: restricted 3-body problem

- Particle assumed on **near-Keplerian orbit** around m_1
- In the frame co-rotating with m_2 and m_1,

$$H_{\rm rot}(l, \omega, L, G) = K(L) + \mu R(l, \omega, L, G) - G,$$

  in Delaunay variables
- Evolution is Hamitlon's equations:

$$\frac{d}{dt}(l, \omega, L, G) = f(l, \omega, L, G)$$

- Jacobi constant, $C_J = -2H_{\rm rot}$
  conserved along trajectories

Change in orbital elements over one particle orbit

Picard's method of approximation

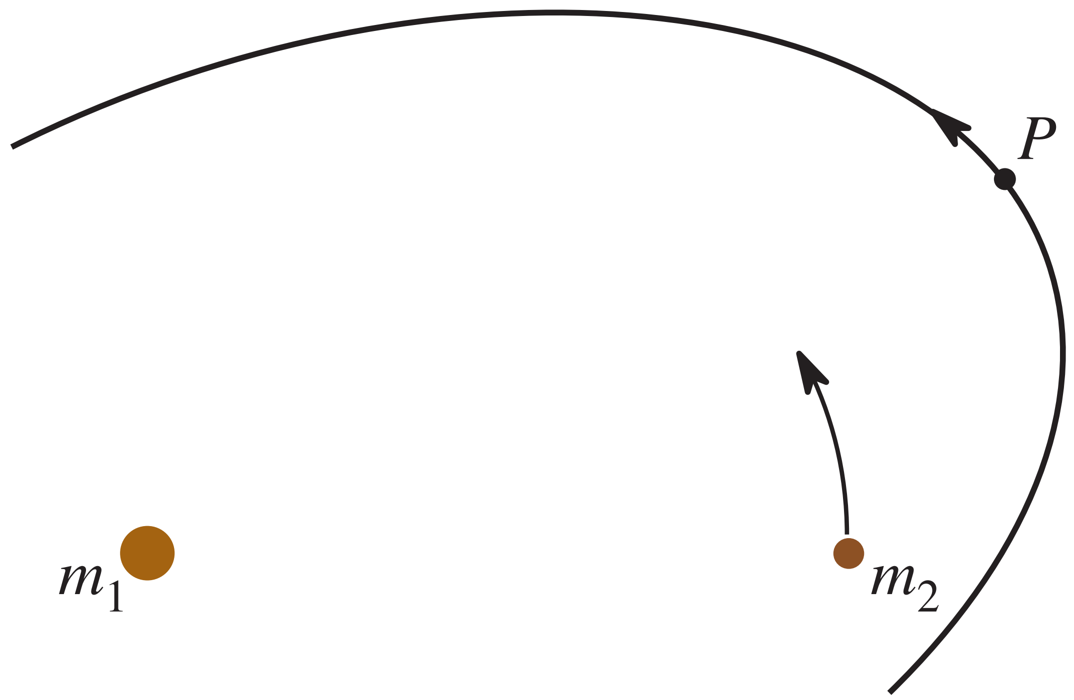
- Let $y(t) = x_0 =$ **unperturbed orbital elements**
- Approximate **change in orbital elements** over one particle orbit is

$$\Delta y = \int_{t_0}^{t_0+T} f(x_0, \tau)\, d\tau,$$

where $T =$ period of unperturbed orbit

Change in orbital elements over one particle orbit

- Assume greatest perturbation occurs at periapsis
  - Limits of integration, apoapsis to apoapsis



Change in orbital elements over one particle orbit

- Evolution of G (angular momentum)

$$\frac{dG}{dt} = -\mu \frac{\partial R}{\partial \omega},$$

- Picard's approximation:

$$\begin{aligned} \Delta G &= -\mu \int_{-T/2}^{T/2} \frac{\partial R}{\partial \omega}\, dt \\ &= -\frac{\mu}{G}\left[\left(\int_{-\pi}^{\pi} \left(\frac{r}{r_2}\right)^3 \sin(\omega + \nu - t(\nu))\, d\nu\right) - \sin\omega \left(2\int_0^{\pi} \cos(\nu - t(\nu))\, d\nu\right)\right] \end{aligned}$$

- $\Delta K =$ **Keplerian energy change** over an orbit

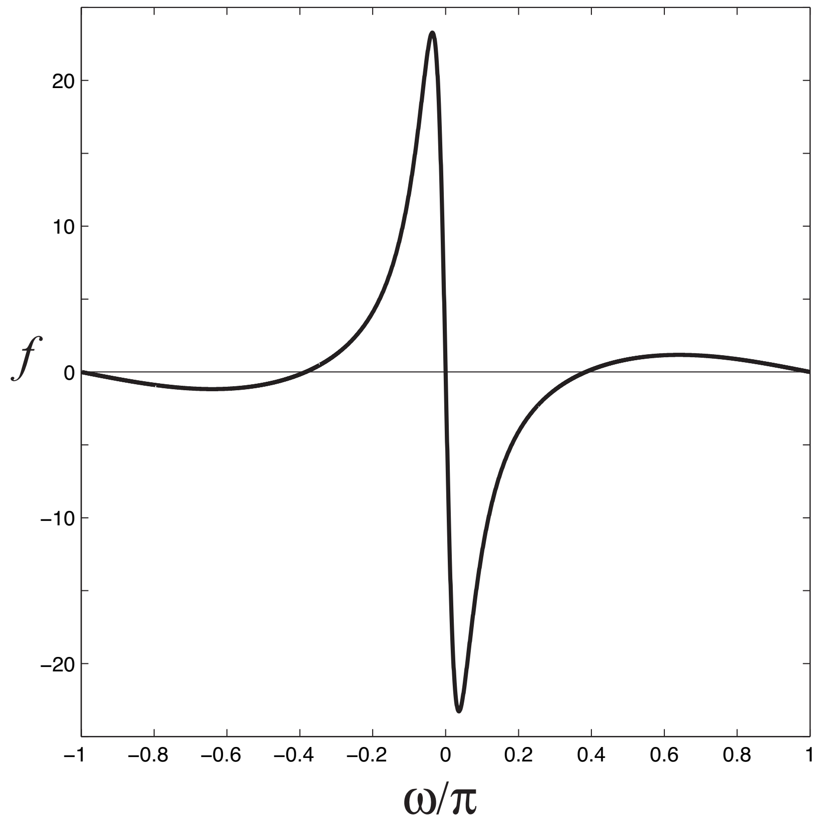
$$\Delta K = \Delta G - \mu \Delta R$$

Energy kick function

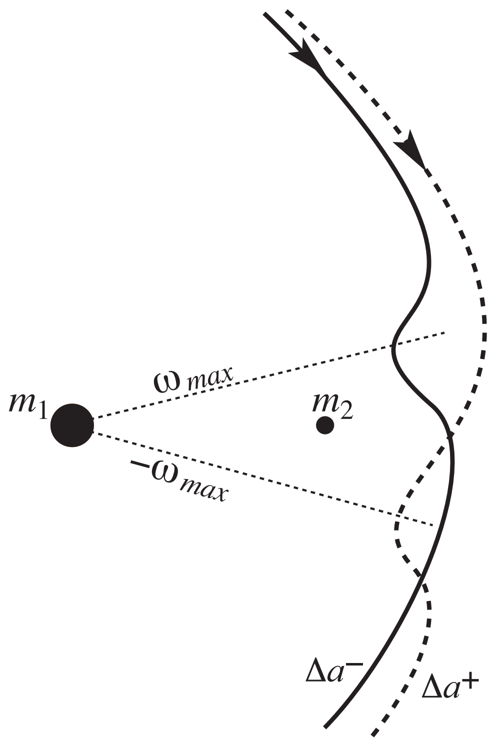
- Changes have form

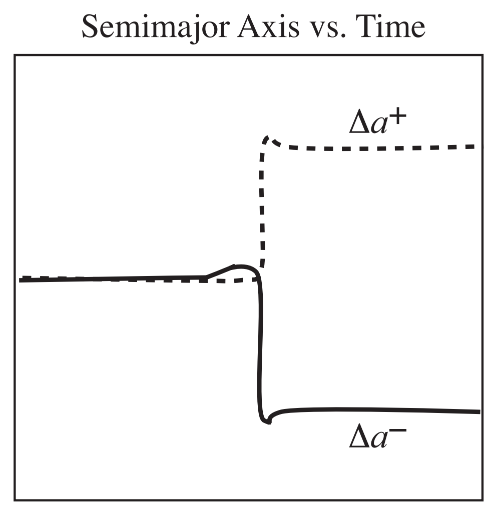
$$\Delta K = \mu f(\omega),$$

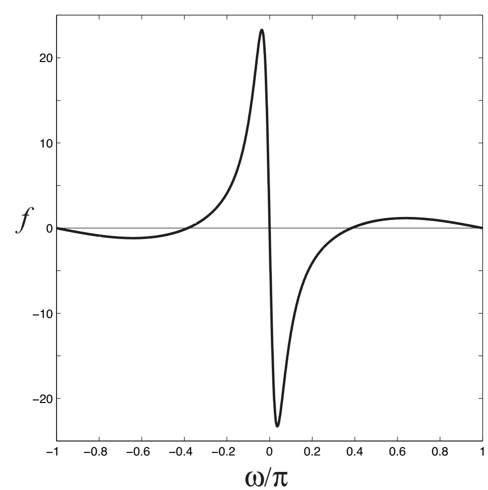
f is the **energy kick function** with parameters K, C_J



Maximum changes on either side of perturber







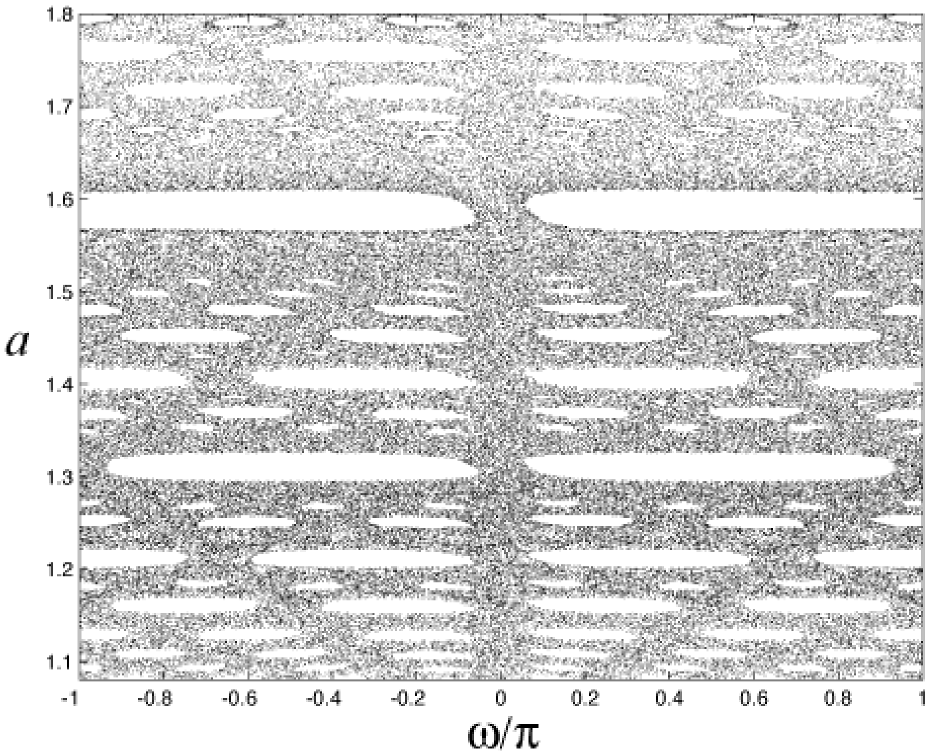
The periapsis kick map (Keplerian Map)

- Cumulative effect of **consecutive passes** by perturber
- Can construct an **update map** $(\omega_{n+1}, K_{n+1}) = F(\omega_n, K_n)$ on the cylinder $\Sigma = S^1 \times \mathbb{R}$, i.e., $F : \Sigma \to \Sigma$ where

$$\begin{pmatrix} \omega_{n+1} \\ K_{n+1} \end{pmatrix} = \begin{pmatrix} \omega_n - 2\pi(-2(K_n + \mu f(\omega_n)))^{-3/2} \\ K_n + \mu f(\omega_n) \end{pmatrix}$$

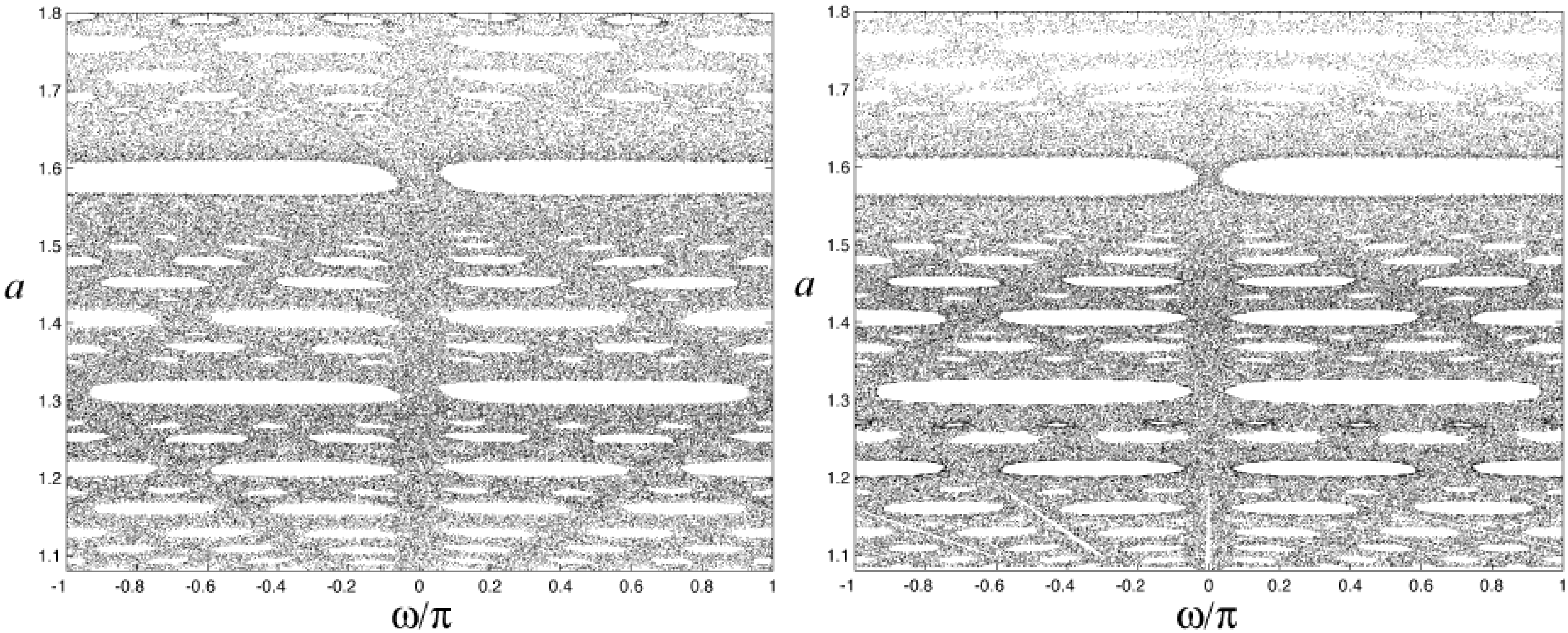
- **Area-preserving (symplectic twist) map**
- Example: particle in Jupiter-Callisto system $\mu = 5 \times 10^{-5}$

Verification of Keplerian map: phase portrait



Keplerian map

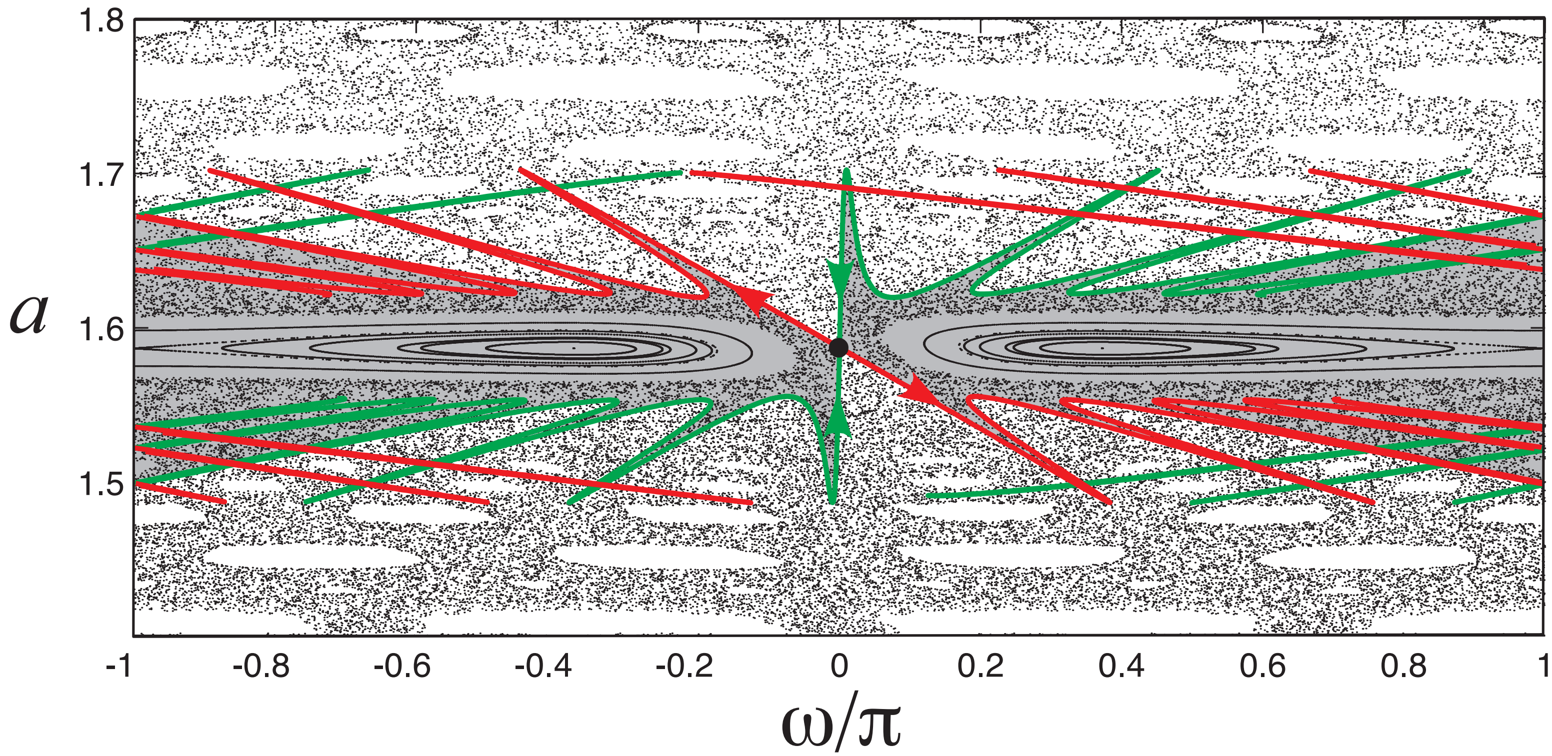
Verification of Keplerian map: phase portrait



Keplerian map numerical integration of ODEs

- Keplerian map = fast orbit propagator
- preserves phase space features
  — but breaks left-right symmetry present in original system
  — can be removed using another method (Hamilton-Jacobi)

Dynamics of Keplerian map

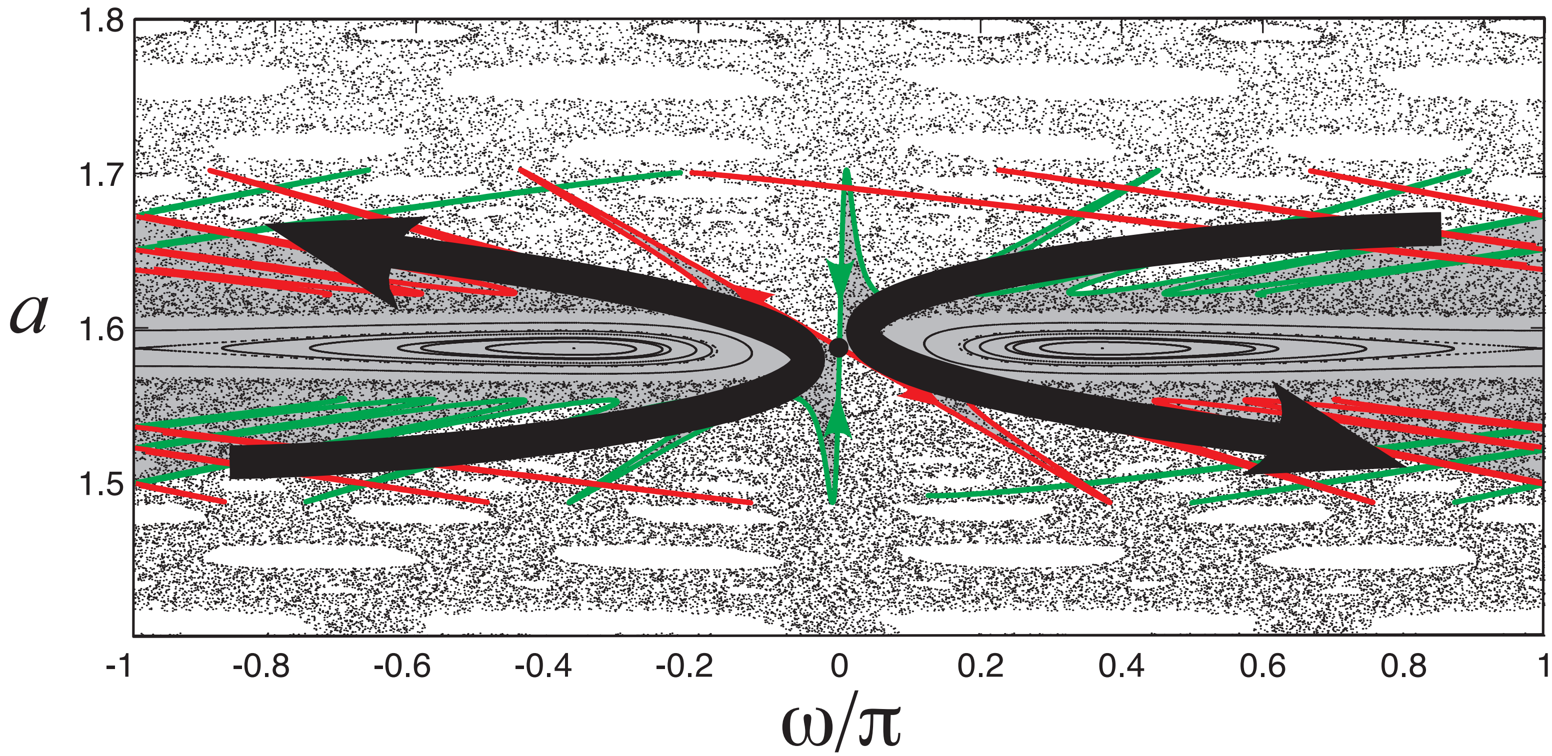


Resonance zone[1]

- Structured motion around resonance zones

[1] in the terminology of MacKay, Meiss, and Percival [1987]

Dynamics of Keplerian map



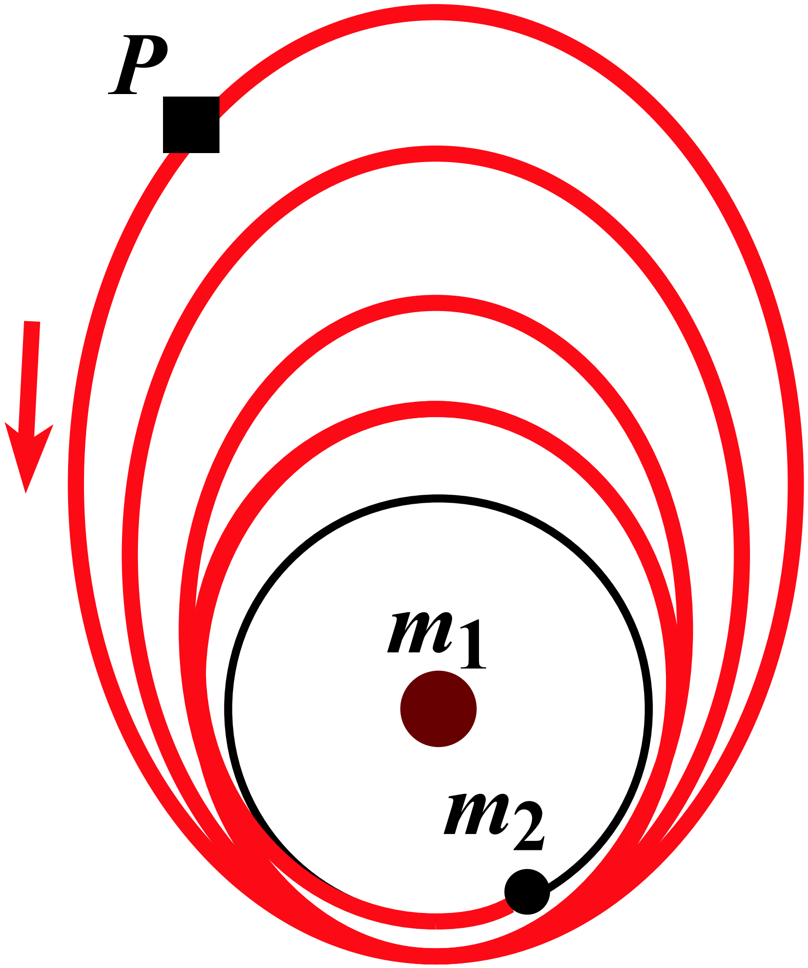
Resonance zone[2]

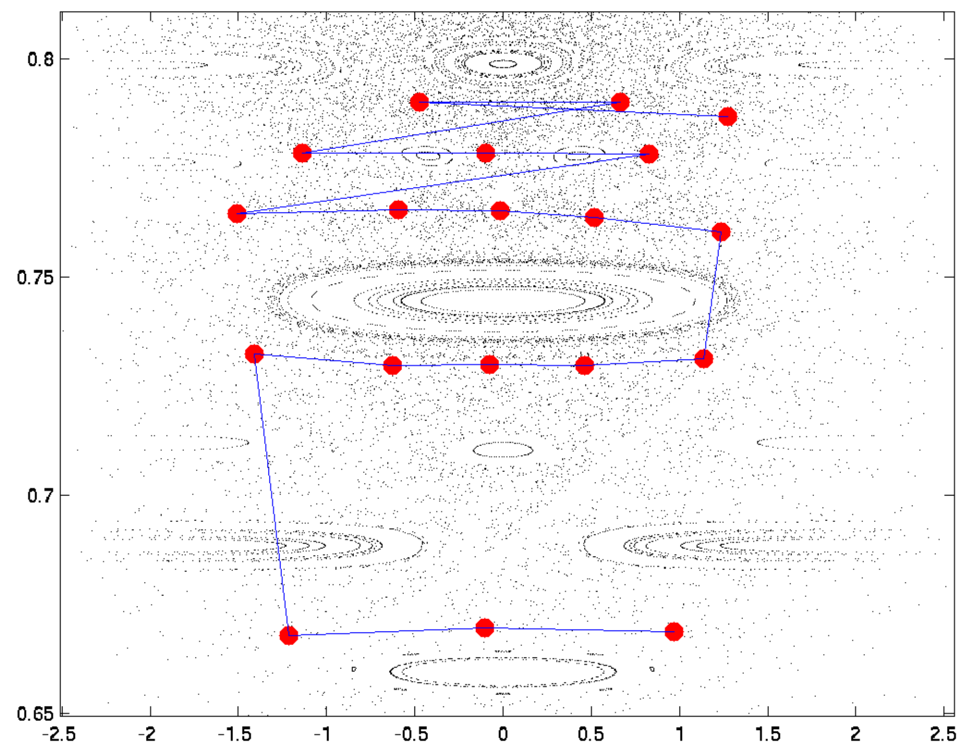
- Structured motion around resonance zones

[2] in the terminology of MacKay, Meiss, and Percival [1987]

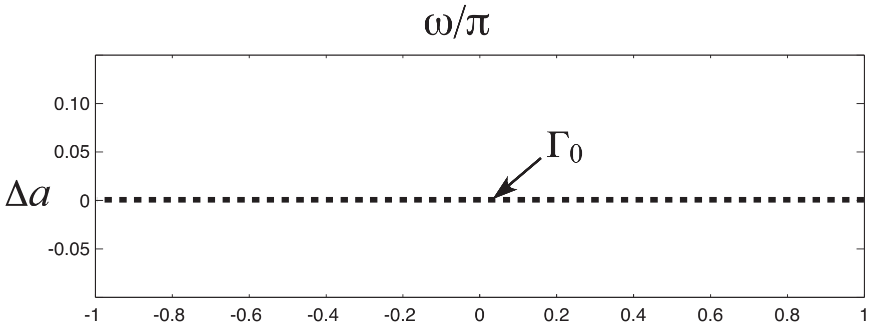
Large orbit changes via multiple resonance zones

- **multiple flybys** for **orbit reduction or expansion**





Large orbit changes, $\Gamma_n = F^n(\Gamma_0)$



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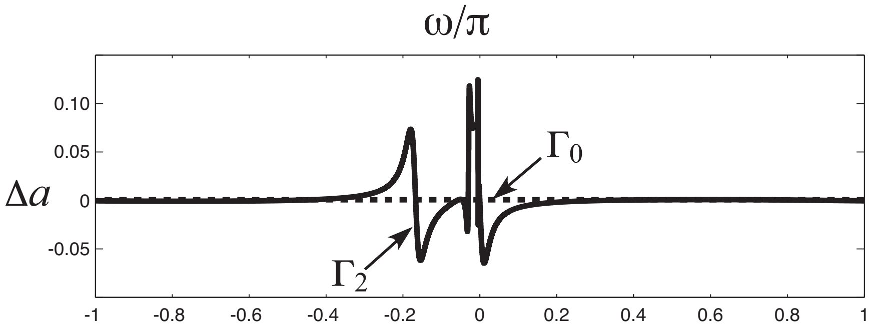

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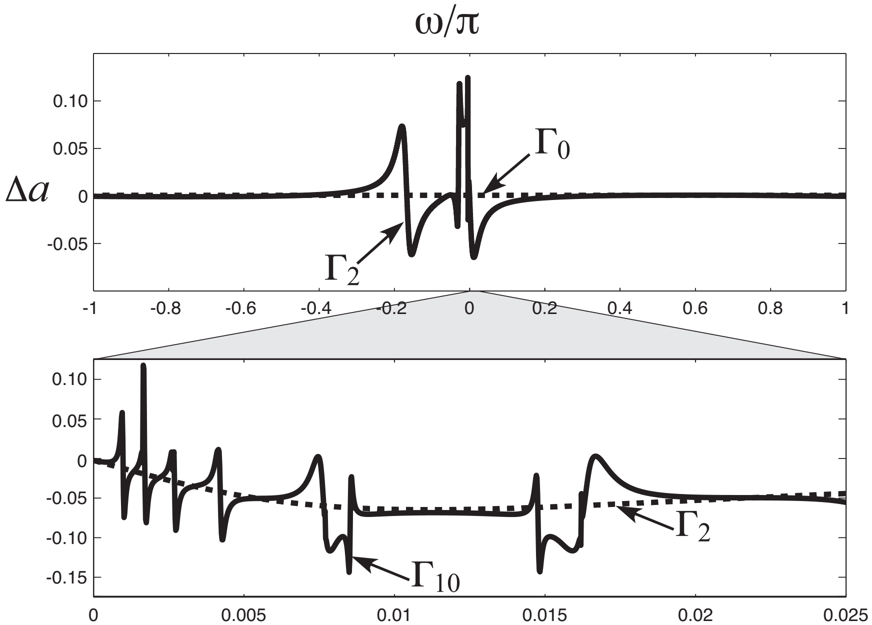

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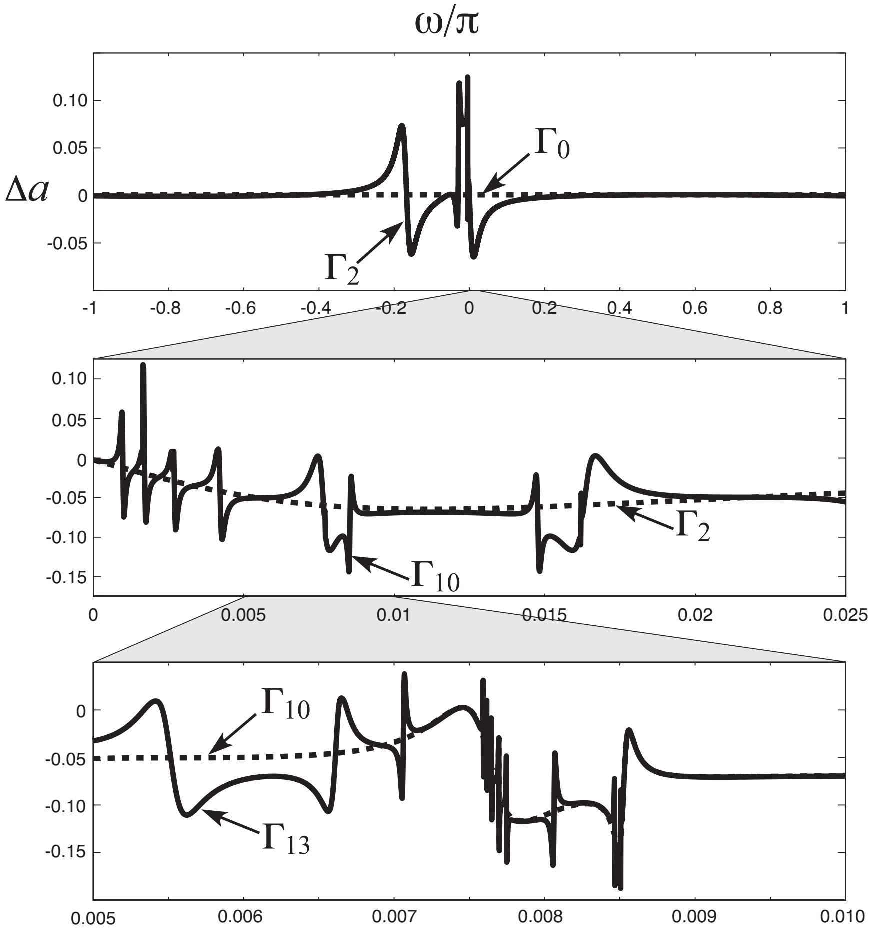

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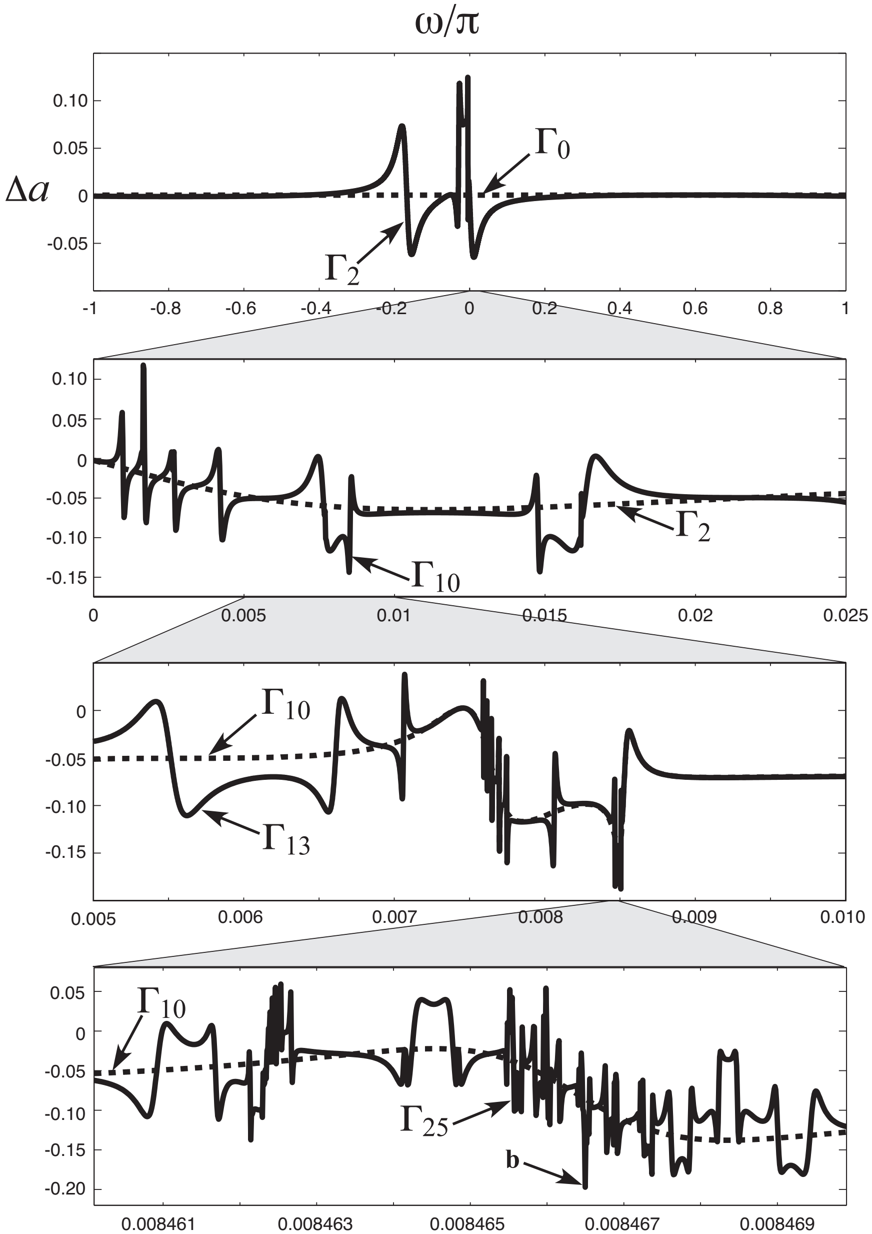

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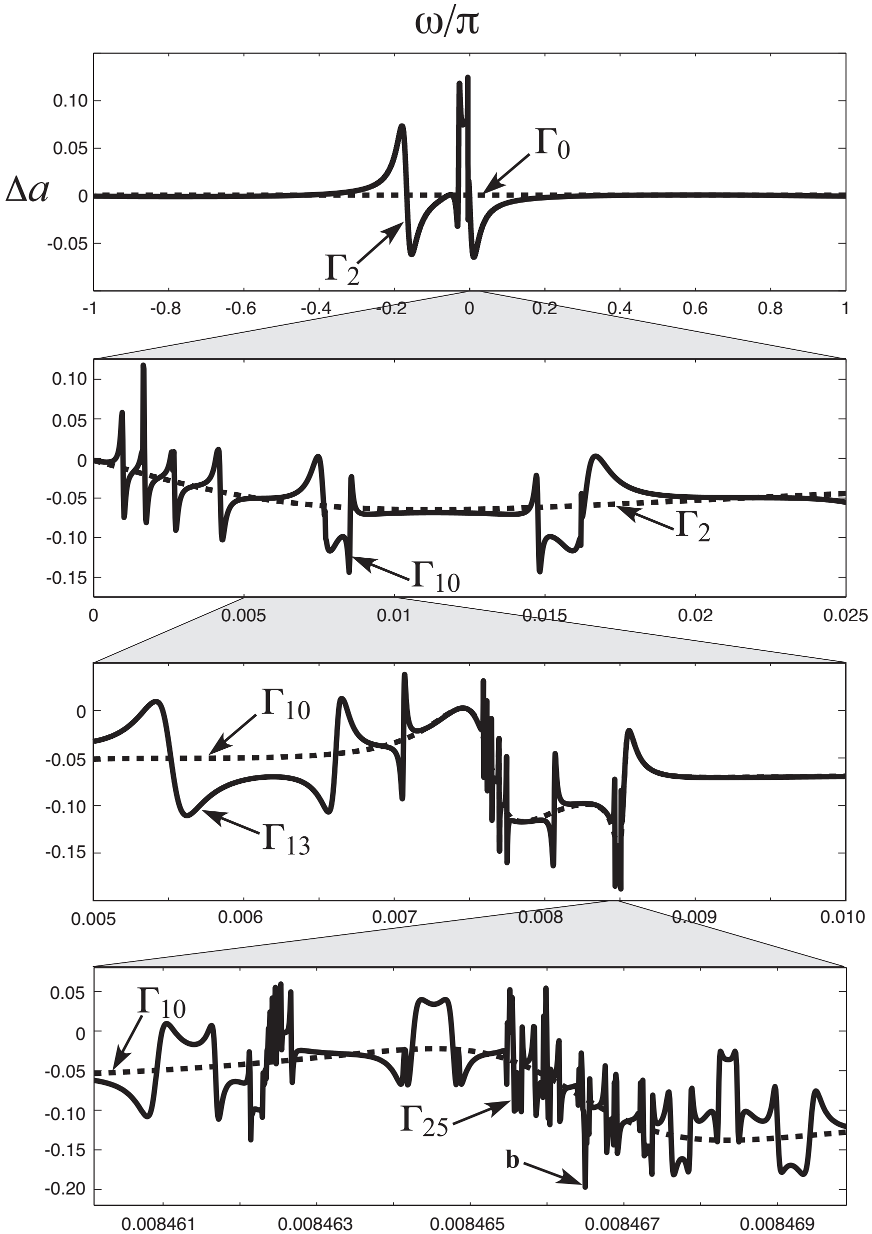
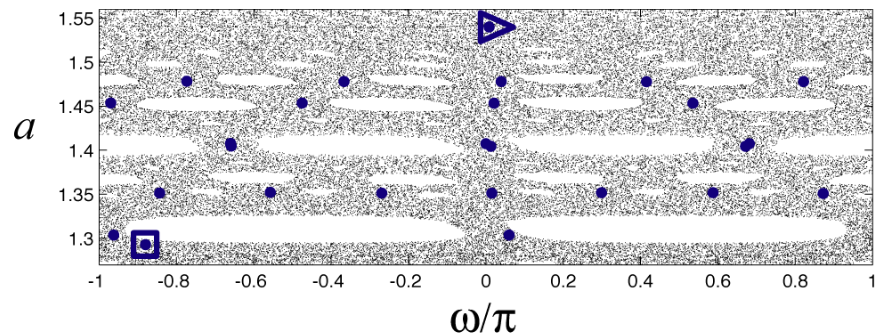
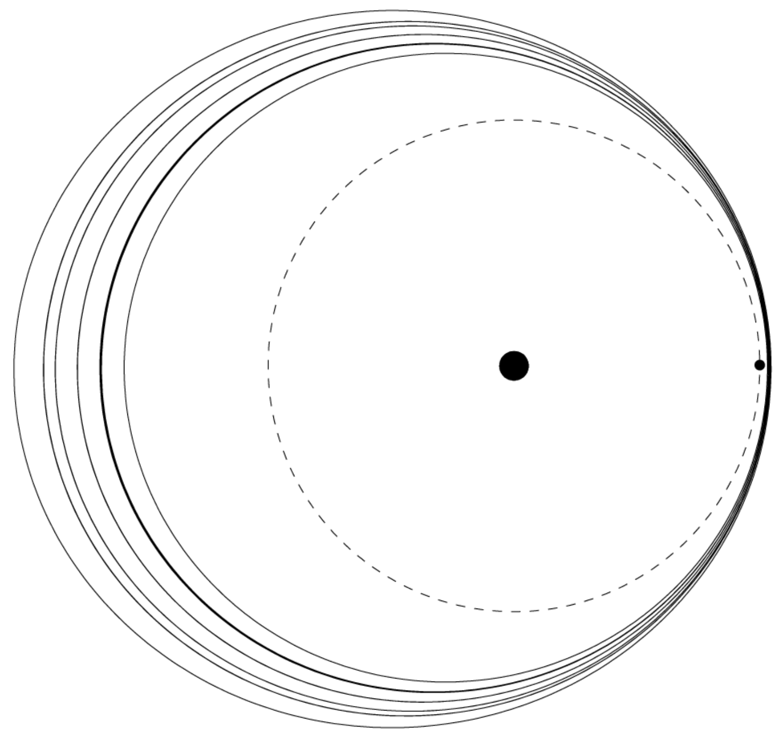
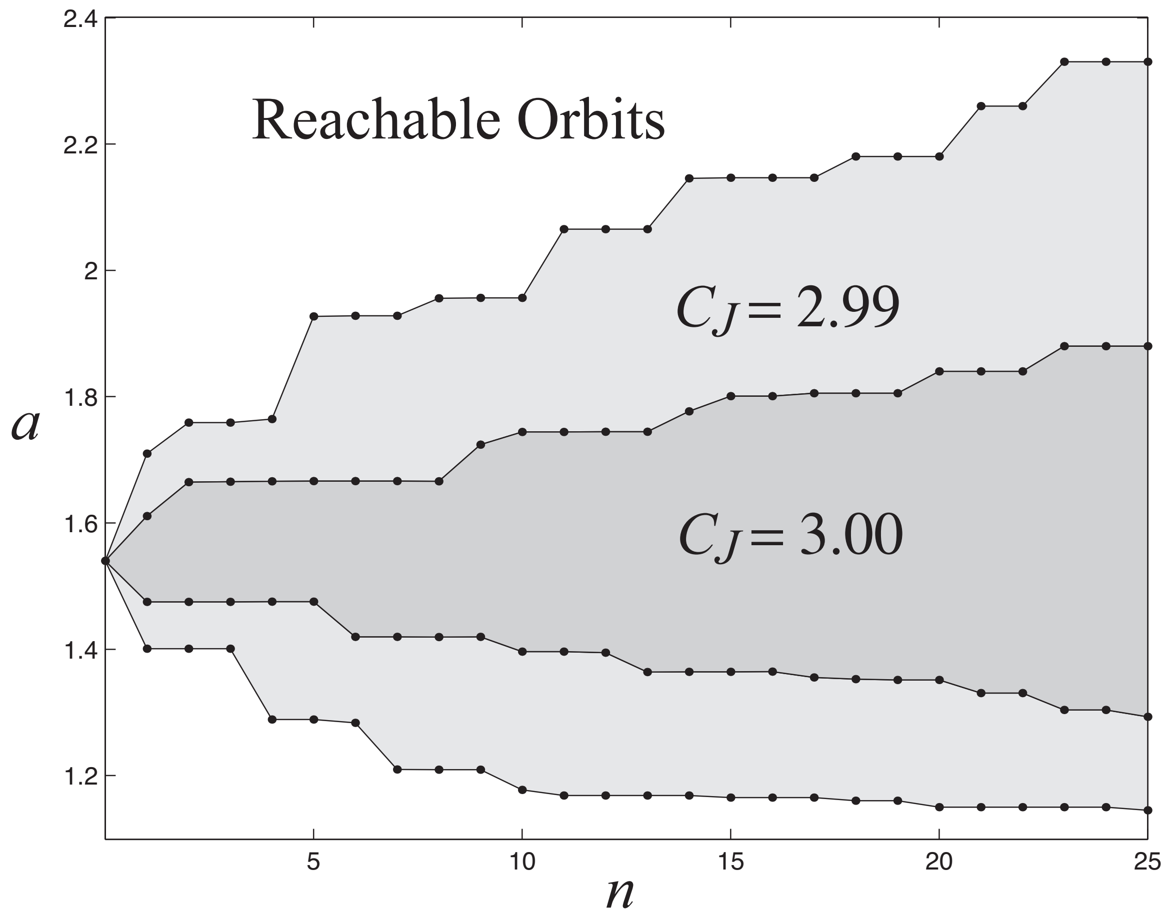






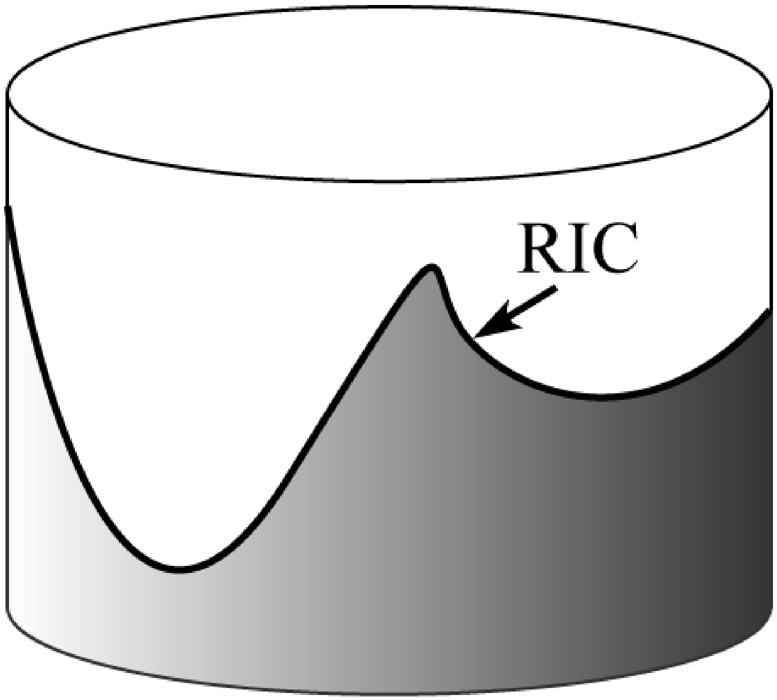
example trajectory

Reachable orbits and diffusion

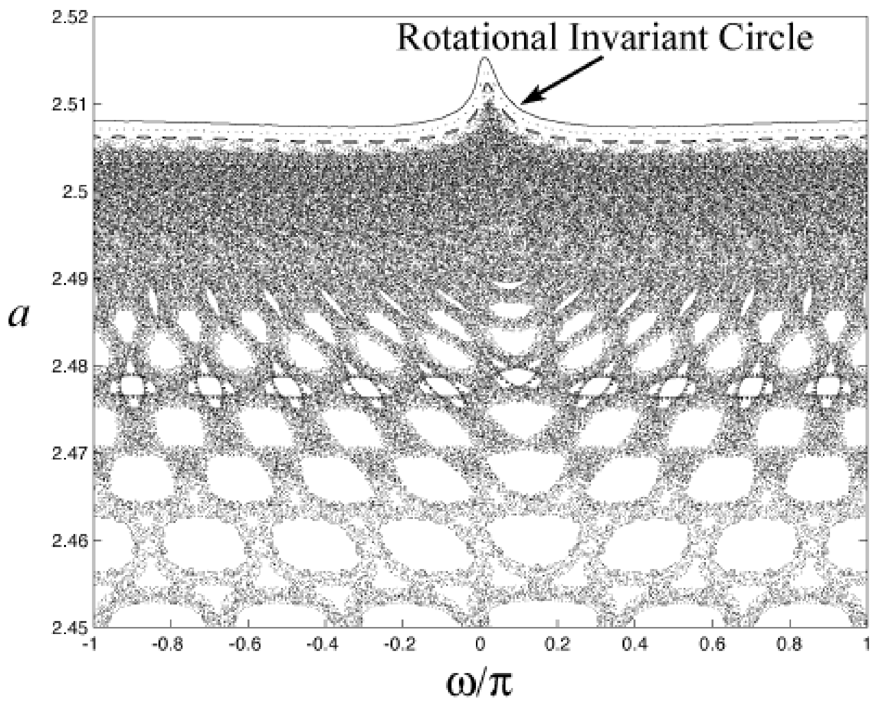


- Diffusion in semimajor axis
- ... increases as C_J decreases (larger kicks)

Reachable orbits: upper boundary for small μ

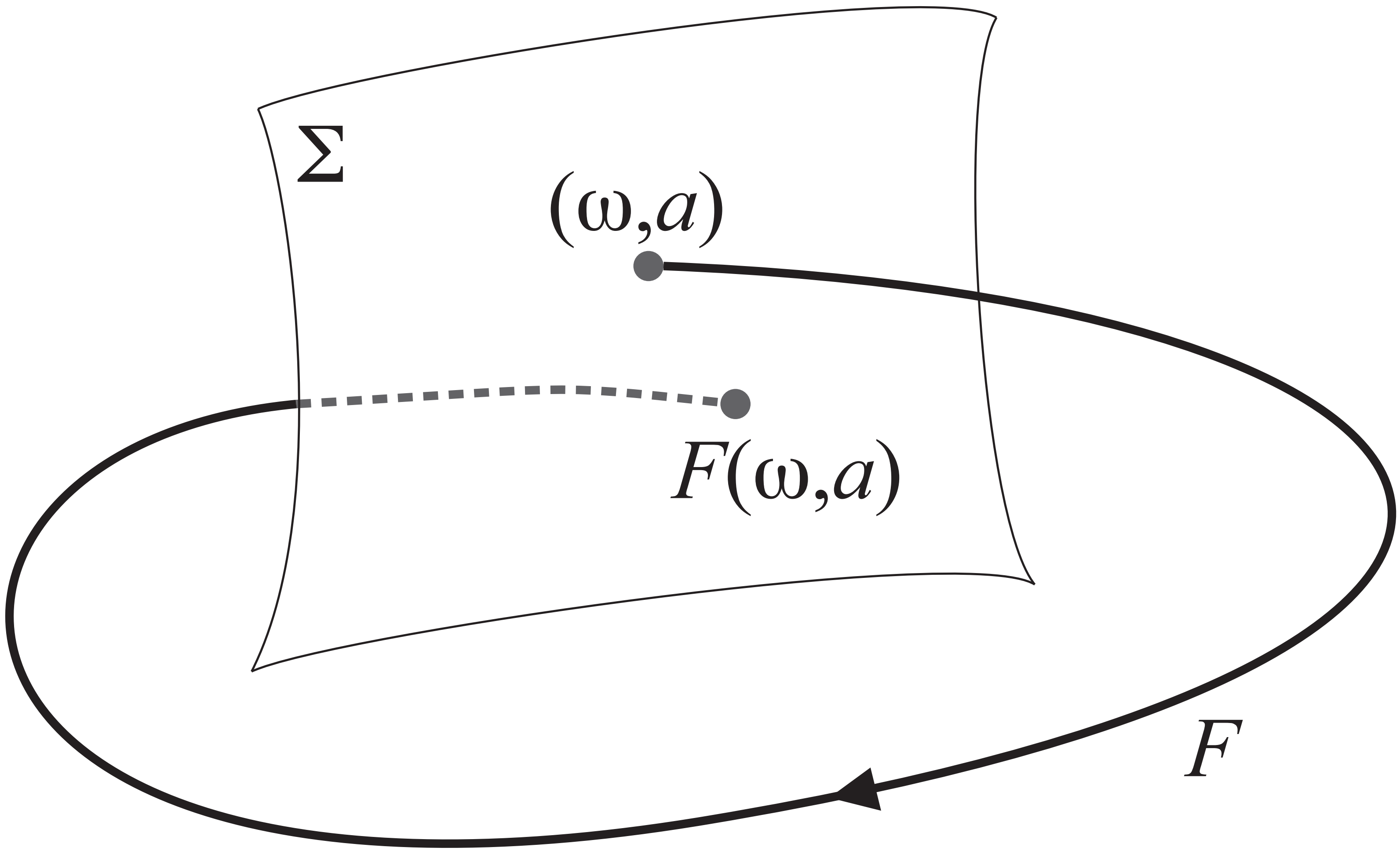


A rotational invariant circle (RIC)



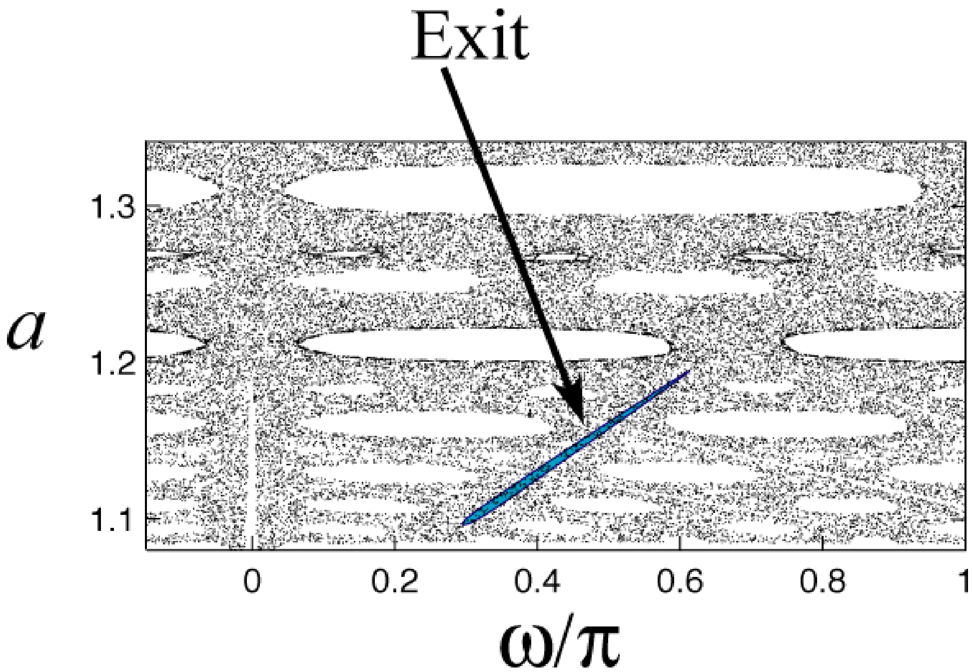
RIC found in Keplerian map for $\mu = 5 \times 10^{-6}$

Identify Keplerian map as Poincaré return map

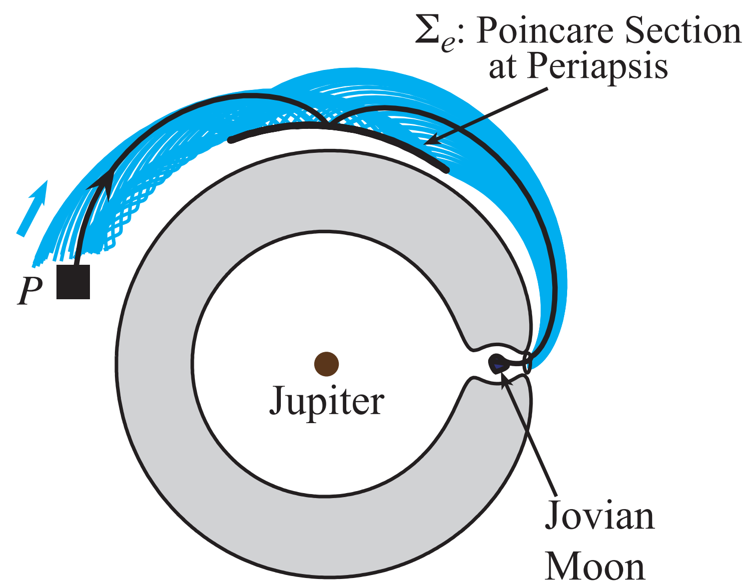


- **Poincaré map at periapsis** in orbital element space
- $F : \Sigma \to \Sigma$ where $\Sigma = \{l = 0 \mid C_J = \text{constant}\}$

Relationship to capture around perturber

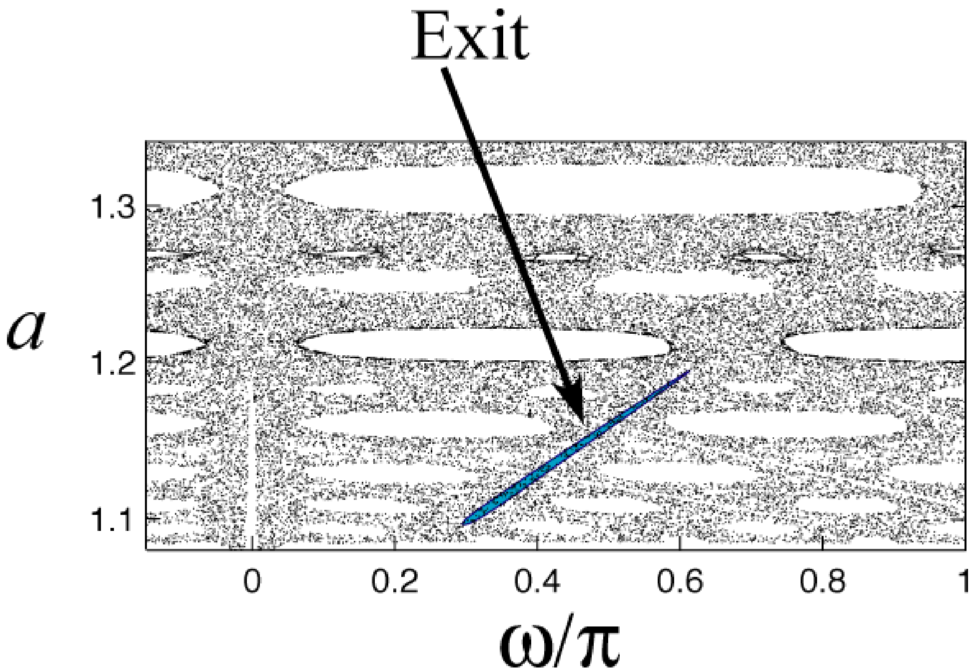


exit from jovicentric to moon region



- **Exit**: where tube of capture orbits intersects Σ

Relationship to capture around perturber



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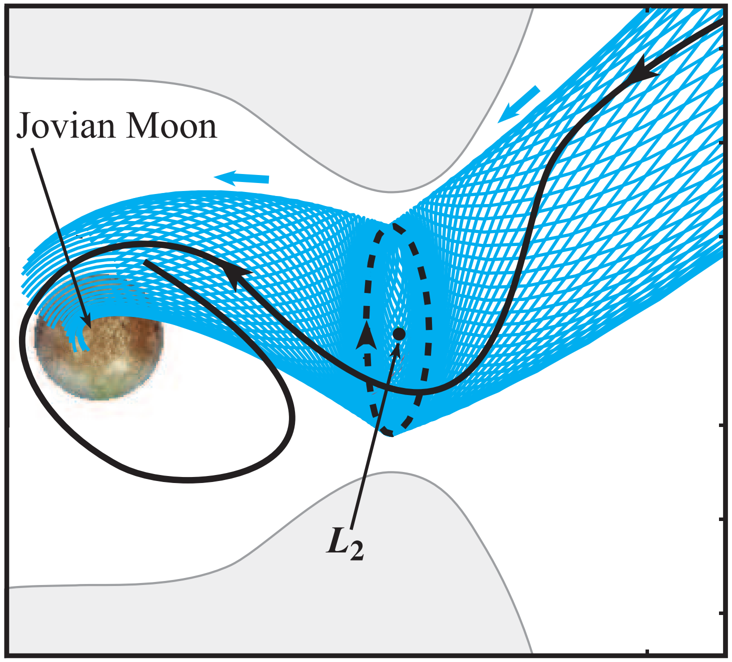
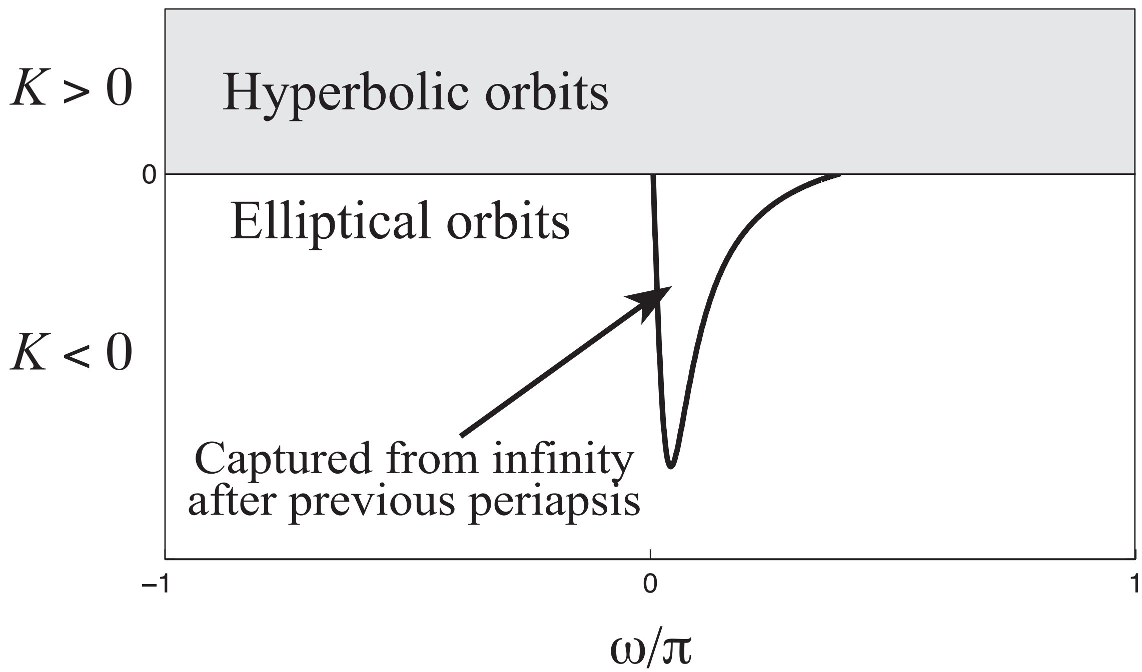
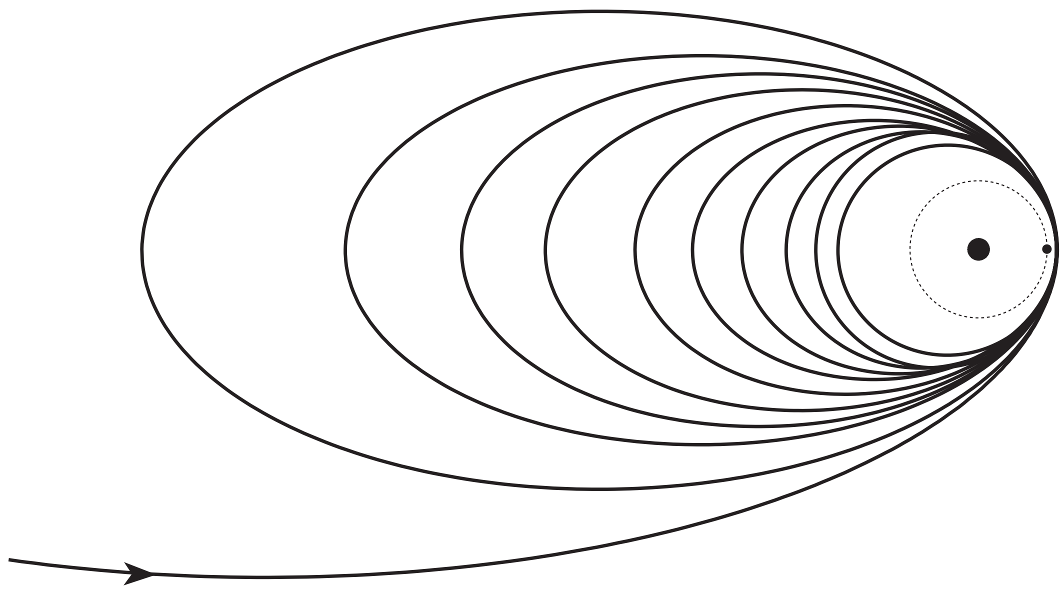


- **Exit**: where tube of capture orbits intersects Σ
- Orbits reaching exit are **ballistically captured**, passing by L_2

Relationship to capture from infinity





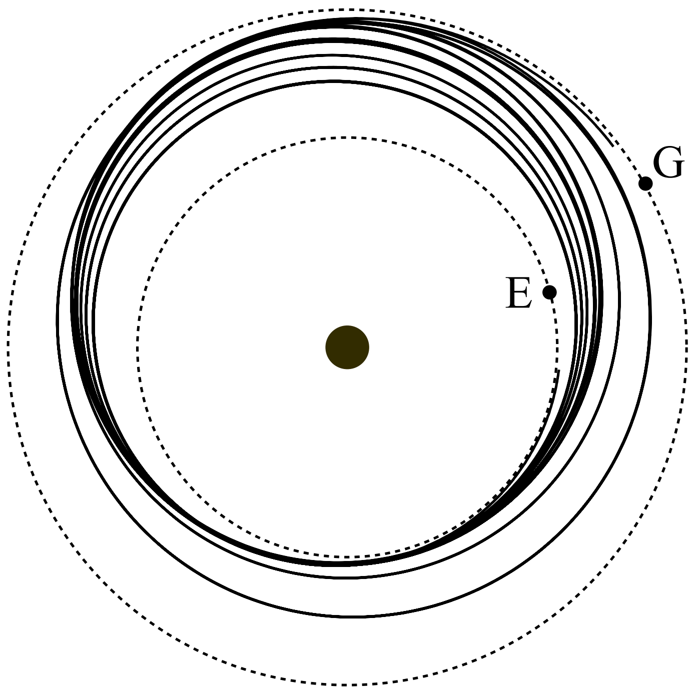
Summary and conclusions

- Consecutive gravity assists
- Reduced to simple lower-dimensional map
  - nice analytical form
  - many phase space features preserved
- Dynamically connected to
  - capture to and escape from perturber
  - capture to and escape from infinity
- Applicable to some astronomical phenomena
- Preliminary optimal trajectory design

Final word

- Extensions:

  - out of plane motion (**4D map**)
  - control in the presence of uncertainty
  - eccentric orbits for the perturbers
  - multiple perturbers
    transfer from one body to another



- Consider other problems with localized perturbations?
  – chemistry, vortex dynamics, ...

Reference:
Ross & Scheeres, *SIAM J. Applied Dynamical Systems*, 2007.
more at: **www.shaneross.com**