Dynamical Systems and Space Mission Design

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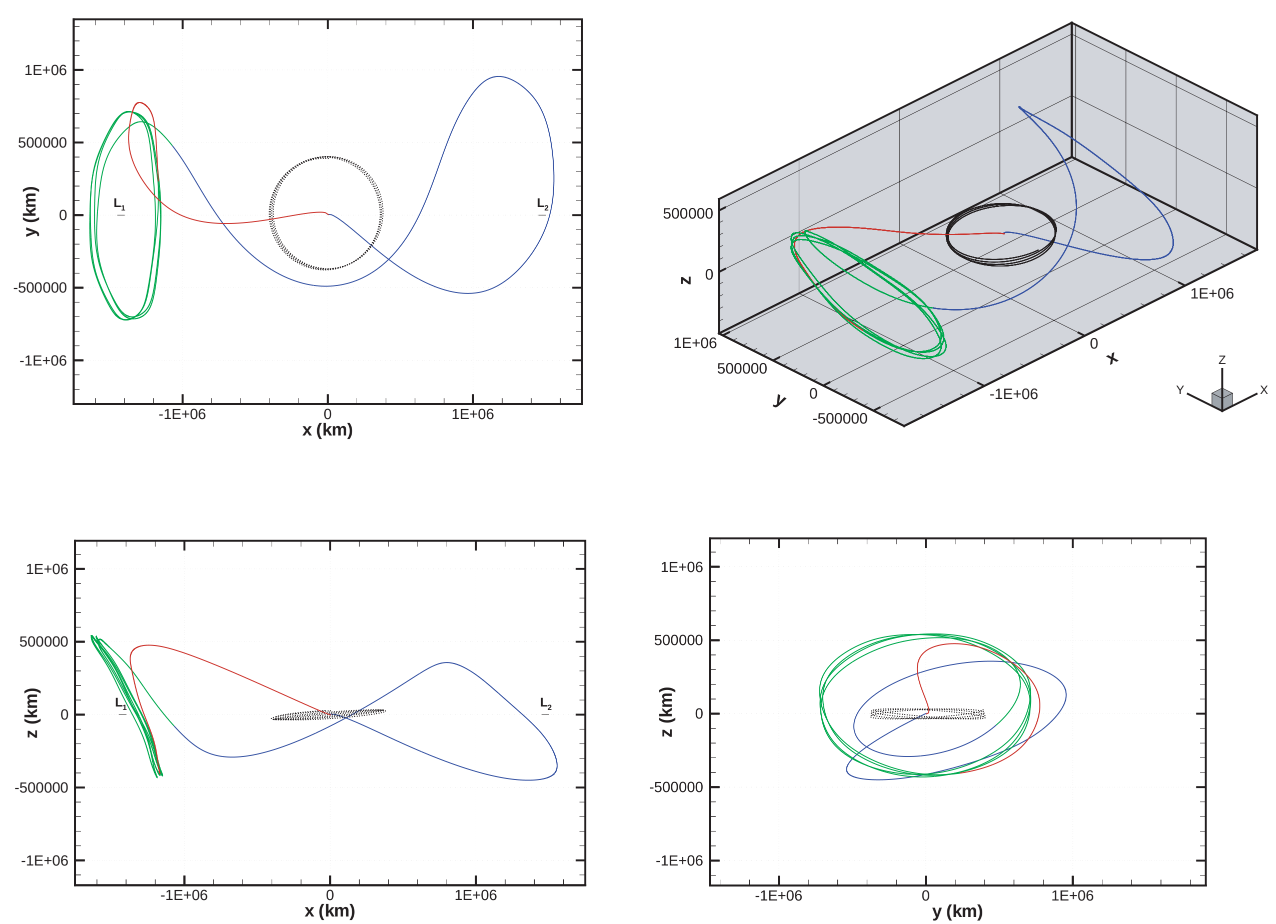
Halo Orbit and Its Computation

- From now on, we will focus on 3D CR3BP.
- We will put more emphasis on numerical computations, especially issues concerning halo orbit missions, such as Genesis Discovery Mission
- Outline of Lecture 5A and 5B:
  - Importance of halo orbits.
  - Finding periodic solutions of the linearized equations.
  - Highlights on 3rd order approximation of a halo orbit.
  - Using a textbook example to illustrate Lindstedt-Poincaré method.
  - Use L.P. method to find a 3rd order approximation of a halo orbit.
  - Finding a halo orbit numerically via differential correction.
  - Orbit structure near L_1 and L_2

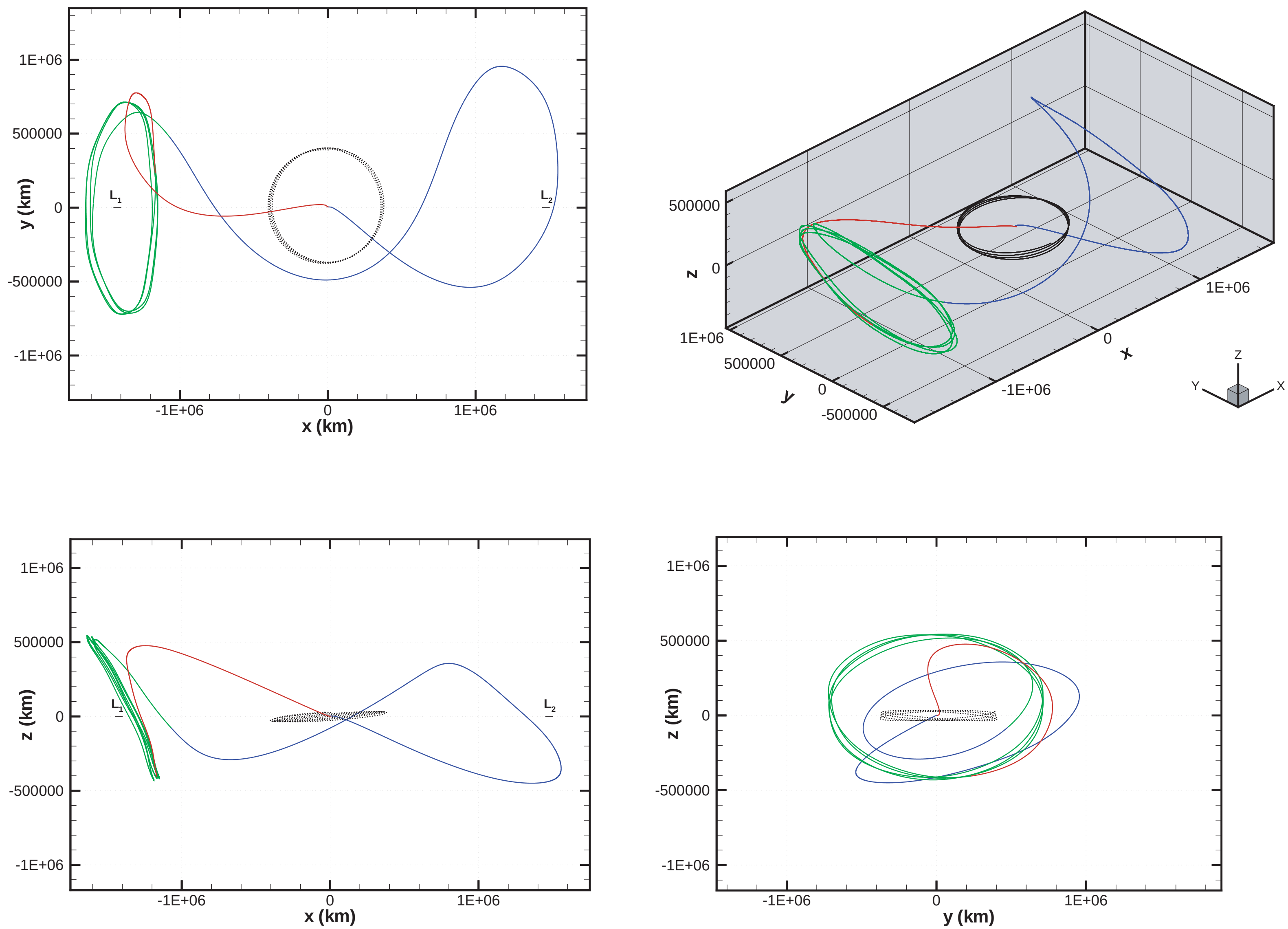
Importance of Halo Orbits: Genesis Discovery Mission

- Genesis spacecraft will
  - collect solar wind from a L_1 halo orbit for 2 years,
  - return those samples to Earth in 2003 for analysis.
- Will contribute to understanding of origin of Solar system.



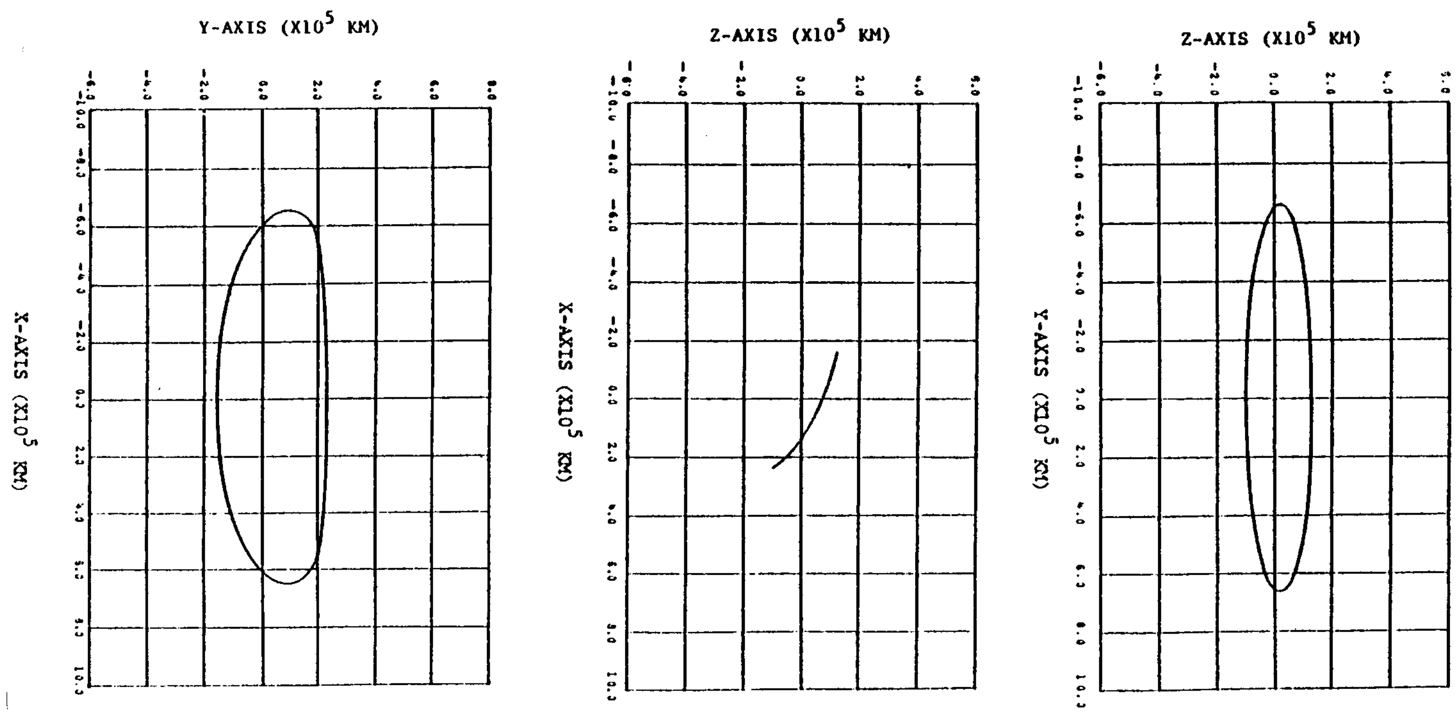
Important of Halo Orbits: Genesis Discovery Mission

- A L_1 halo orbit (1.5 million km from Earth) provides uninterrupted access to solar wind beyond Earth's magnetoshphere.



Importance of Halo Orbits: ISEE-3 Mission

- Since halo orbit is ideal for studying solar effects on Earth, NASA has had and will continue to have great interest in these missions.
- The first halo orbit mission, ISEE-3, was launched in 1978.
- ISEE-3 spacecraft monitored solar wind and other solar-induced phenomena, such as solar radio bursts and solar flares, about a hour prior to disturbance of space enviroment near Earth.



Importance of Halo Orbits: Terrestial Planet Finder

- JPL has begun studies of a TPF misson at L_2 involving 4 free flying **optical elements** and a **combiner spacecraft**.
- **Interferometry**: achieve **high resolution** by distributing small optical elements along a **lengthy baseline** or pattern.
- Look into using a L_2 **halo** orbit and its nearby **quasi-halo** orbits for **formation** flight.

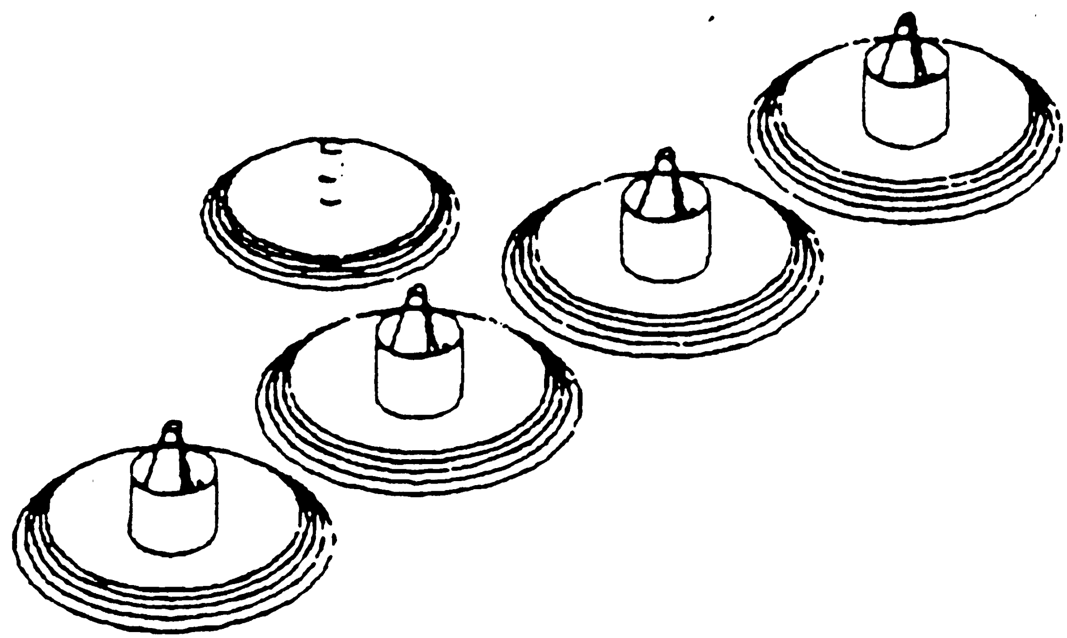
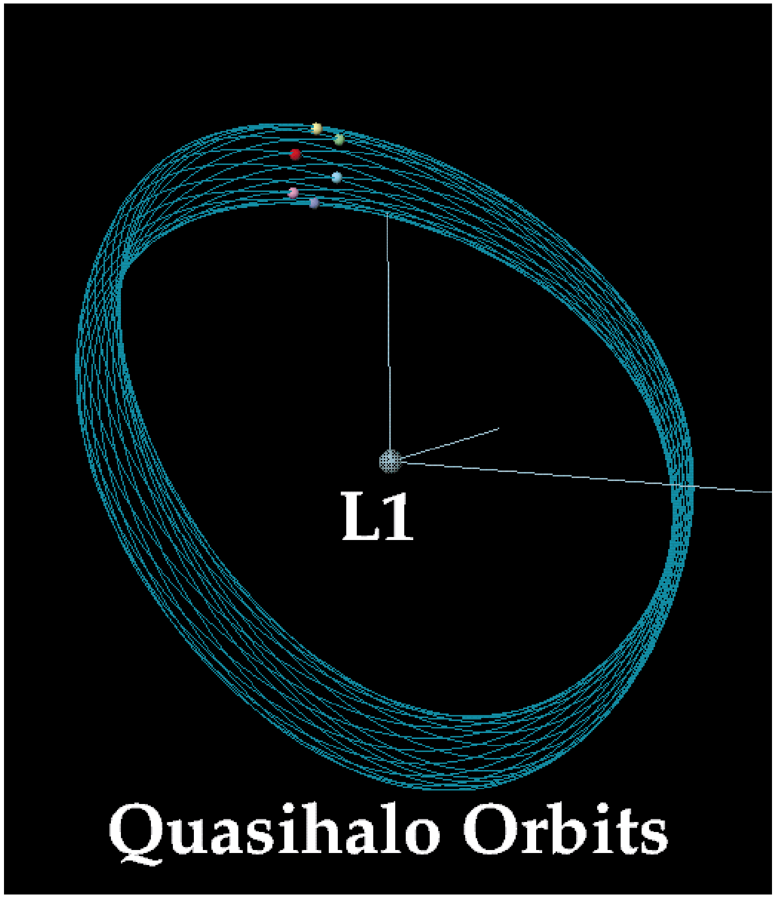


Figure 1. Terrestrial Planet Finder

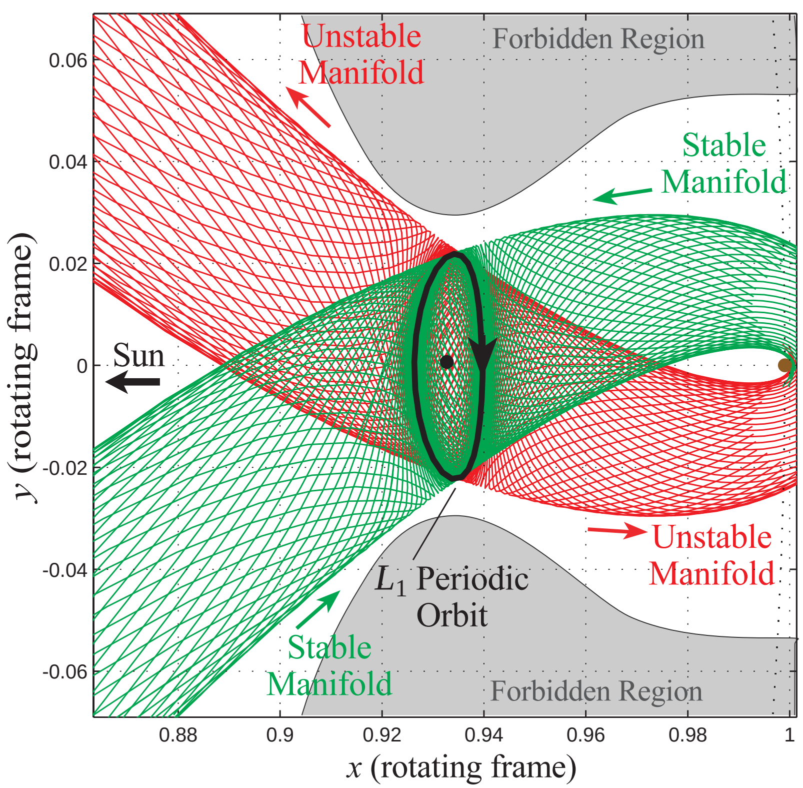
Importance of Halo Orbits: Terrestial Planet Finder

- The L_2 option offer several advantages:
  - Additional spacecraft can be launched into formation later.
  - The L_2 offers a larger payload capacity.
  - Communications are more efficient at L_2.
  - Observations and mission operations are simpler at L_2.



Importance of Halo Orbits: 3D Dynamical Channels

- In 3D dynamical channels theory,
  invariant manifolds of a solid torus of quasi-halo orbits could play similar role as invariant manifold tubes of a Lyapunov orbit.
- Halo, quasi-halos and their invariant manifolds could be key in
  - understanding material transport throughout Solar system,
  - constructing 3D orbits with desired characteristics.

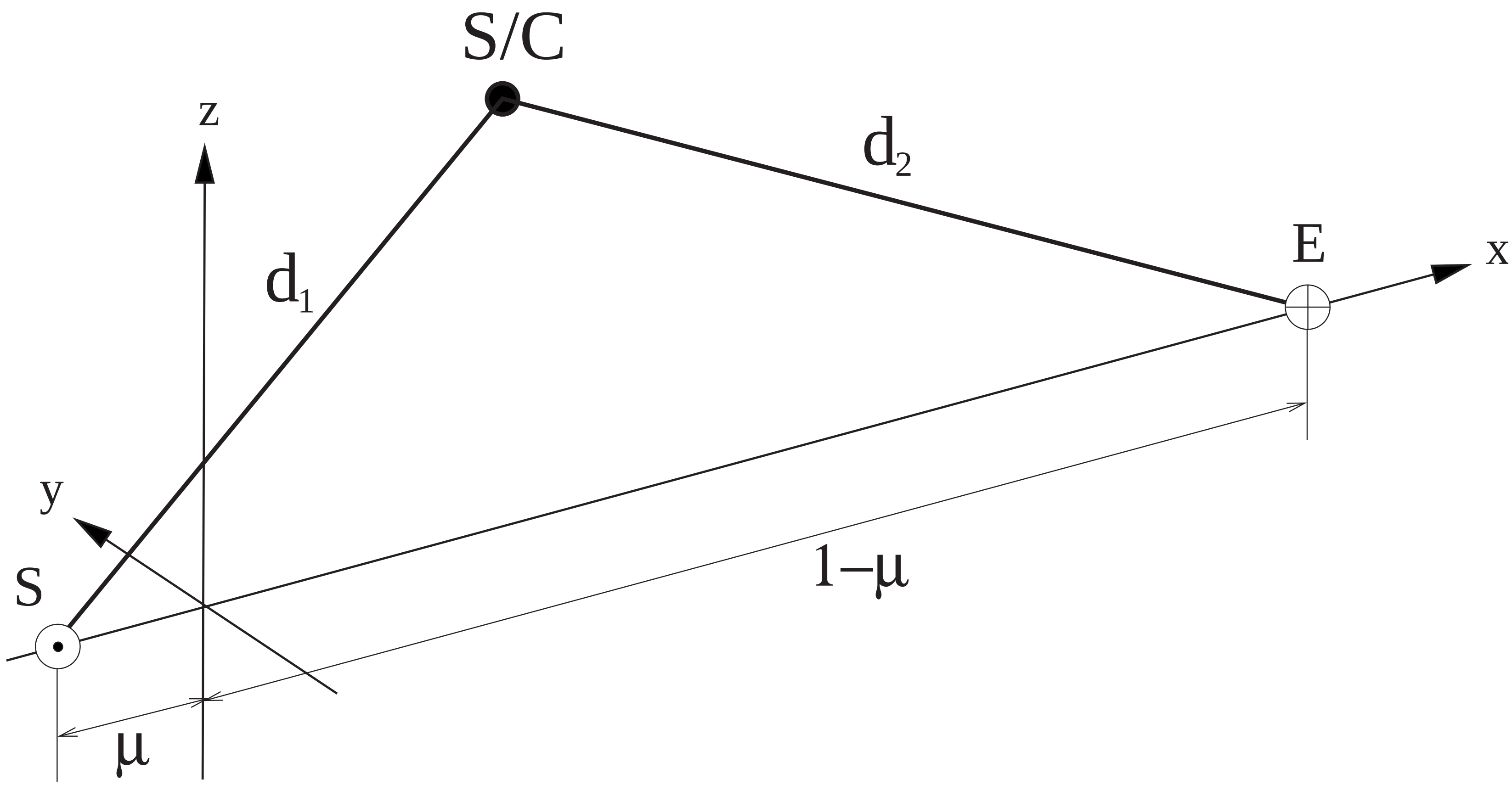


3D Equations of Motion

- Recall equations of CR3BP:

$$\ddot{X} - 2\dot{Y} = \Omega_X \qquad \ddot{Y} + 2\dot{X} = \Omega_Y \qquad \ddot{Z} = \Omega_Z$$

where $\Omega = (X^2 + Y^2)/2 + (1-\mu)d_1^{-1} + \mu d_2^{-1}$.



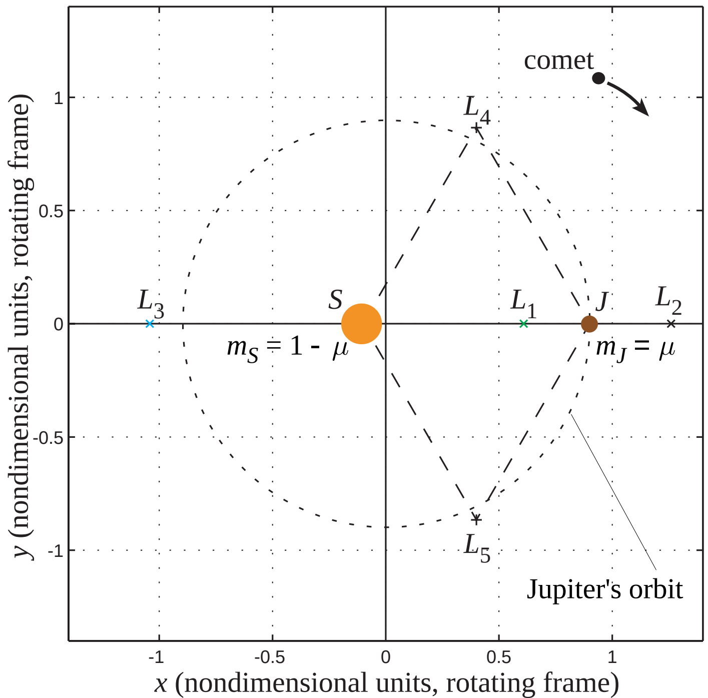
3D Equations of Motion

- Equations for satellite moving in vicinity of L_1 can be obtained by translating the origin to the location of L_1:

$$x = (X - 1 + \mu + \gamma)/\gamma, \qquad y = Y/\gamma, \qquad z = Z/\gamma,$$

where $\gamma = d(m_2, L_1)$

- In new coordinate sytem, variables x, y, z are scale so that the distance between L_1 and small primary is 1.
- New independent variable is introduced such that $s = \gamma^{3/2} t$.



3D Equations of Motion

- CR3BP equations can be developed using Legendre polynomial P_n

$$\ddot{x} - 2\dot{y} - (1 + 2c_2)x = \frac{\partial}{\partial x} \sum_{n \geq 3} c_n \rho^n P_n\left(\frac{x}{\rho}\right)$$

$$\ddot{y} + 2\dot{x} + (c_2 - 1)y = \frac{\partial}{\partial y} \sum_{n \geq 3} c_n \rho^n P_n\left(\frac{x}{\rho}\right)$$

$$\ddot{z} + c_2 z = \frac{\partial}{\partial z} \sum_{n \geq 3} c_n \rho^n P_n\left(\frac{x}{\rho}\right)$$

where $\rho = x^2 + y^2 + z^2$, and $c_n = \gamma^{-3}(\mu + (-1)^n(1-\mu)(\frac{\gamma}{1-\gamma})^{n+1})$.

  - Useful if successive approximation solution procedure is carried to high order via algebraic manipulation software programs.

Analytic and Numerical Methods: Overview

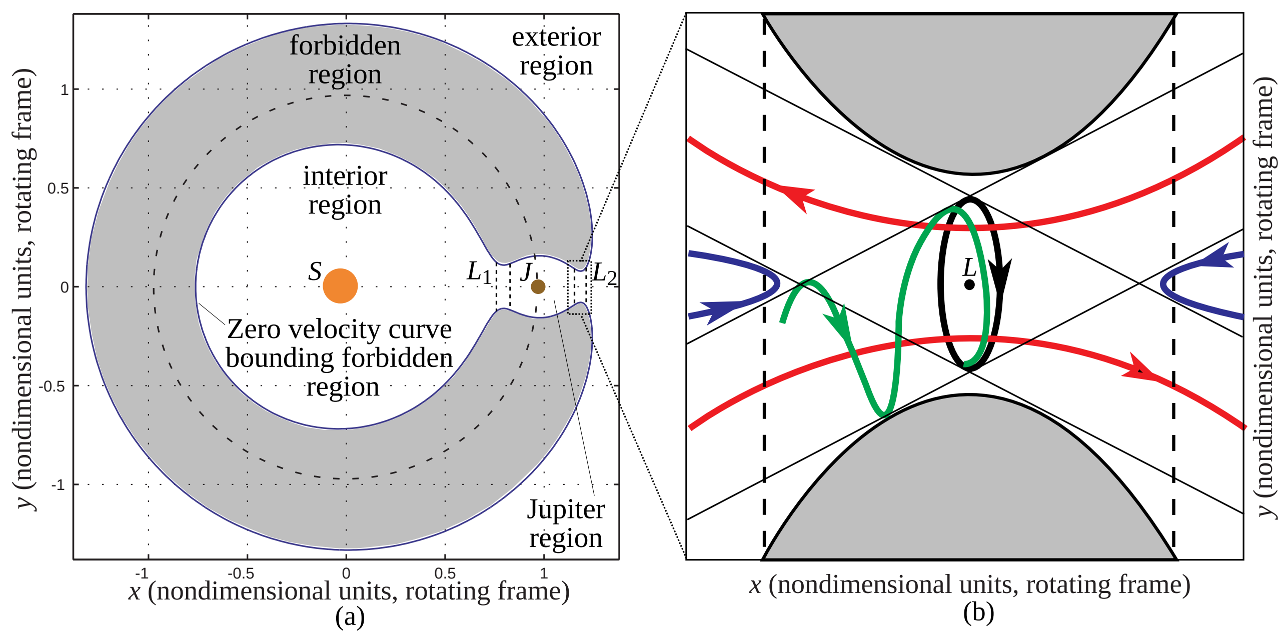
- Lack of general solution motivated researchers to develop semi-analytical method.
- ISEE-3 halo was designed in this way.
  See Farquhar and Kamel [1973], and Richardson [1980].
- **Linear analysis** suggested
  existence of periodic (and quasi-periodic) orbits near L_1.
- **3rd order** approximation, using **Lindstedt-Poincaré** method, provided further insight about these orbits.
- **Differential corrector** produced the desired orbit
  using 3rd order solution as **initial guess**.

Periodic Solutions of Linearized Equations

- Periodic nature of solution can be seen in linearized equations:

$$\begin{aligned} \ddot{x} - 2\dot{y} - (1 + 2c_2)x &= 0 \\ \ddot{y} + 2\dot{x} + (c_2 - 1)y &= 0 \\ \ddot{z} + c_2 z &= 0 \end{aligned}$$

- The z-axis solution is simple harmonic, does not depend on x or y.
- Motion in xy-plane is coupled, has $(\pm\alpha, \pm i\lambda)$ as eigenvalues.
- General solutions are unbounded, but there is a periodic solution.



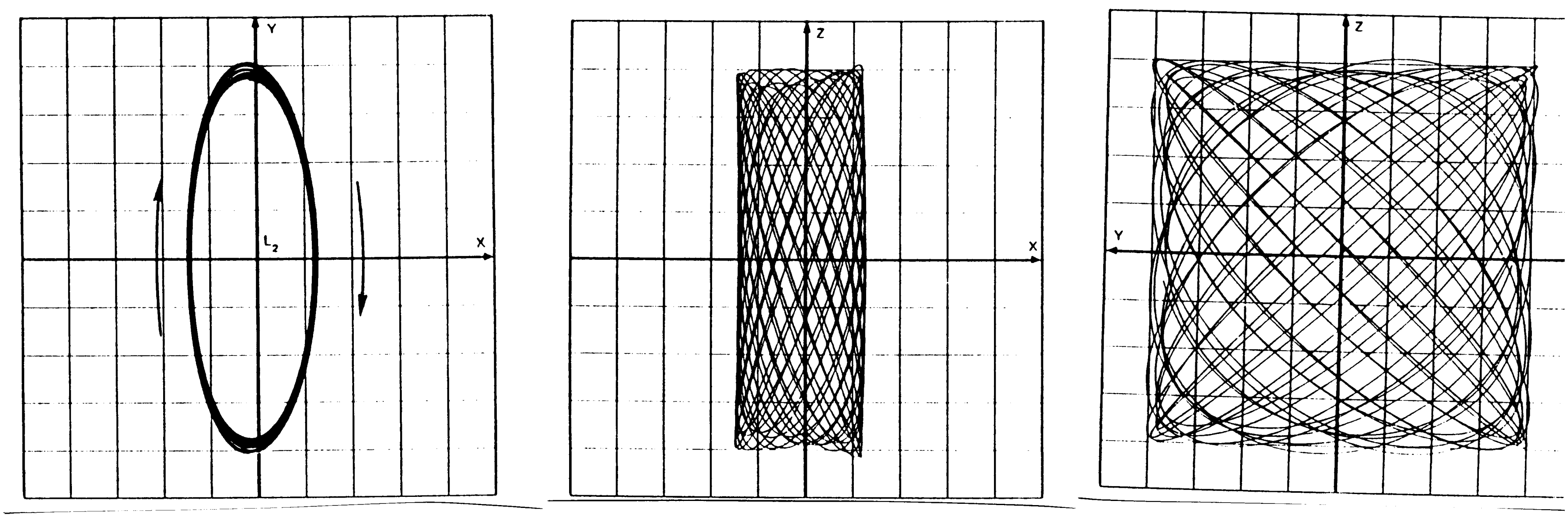
Periodic Solutions of Linearized Equations

- Linearized equations has a bounded solution (**Lissajous** orbit)

$$
\begin{aligned}
x &= -A_x \cos(\lambda t + \phi) \\
y &= k A_x \sin(\lambda t + \phi) \\
z &= A_z \sin(\nu t + \psi)
\end{aligned}
$$

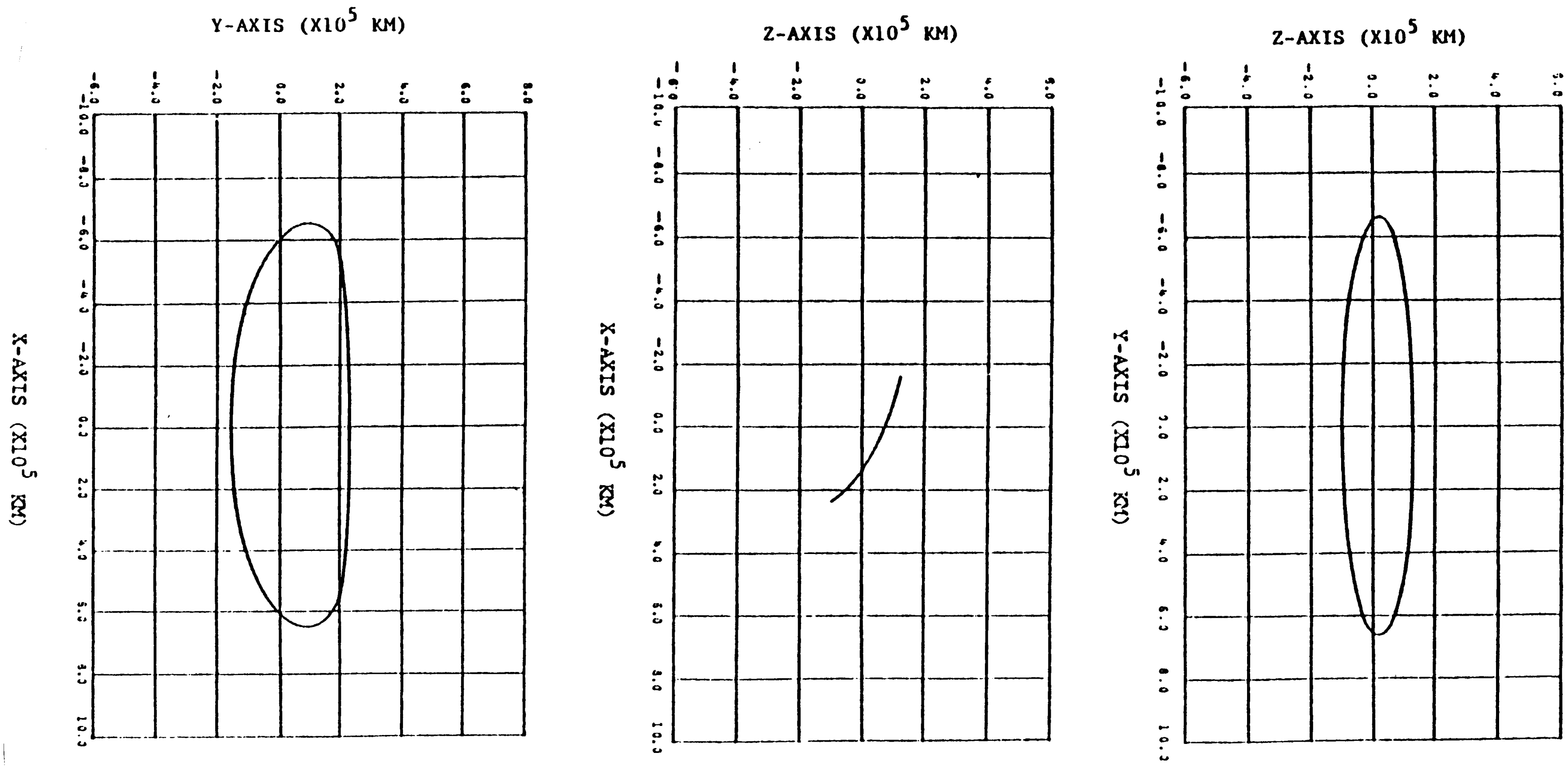
with $k = (\lambda^2 + 1 + 2c_2)/2\lambda$. ($\lambda = 2.086, \nu = 2.015, k = 3.229$.)

- Amplitudes, A_x and A_z, of in-plane and out-of-plane motion characterize the size of orbit.



Periodic Solutions of Linearized Equations

- If frequencies are equal ($\lambda = \nu$), **halo** orbit is produced.
- But $\lambda = \nu$ only when amplitudes A_x and A_z are large enough that **nonlinear** contributions become significant.
- For ISEE3 halo, $A_z = 110,000$ km, $A_x = 206,000$ km and $A_y = kA_x = 665,000$ km.



Halo Orbits in 3rd Order Approximation

- Halo orbit is obtained only when amplitudes A_x and A_z are large enough that **nonlinear** contributions make $\lambda = \nu$.
- Lindstedt-Poincaré procedure has been used to find periodic solution for a 3rd order approximation of PCR3BP system.

$$\begin{aligned}
\ddot{x} - 2\dot{y} - (1+2c_2)x &= \frac{3}{2}c_3(2x^2 - y^2 - z^2) \\
&\quad +2c_4x(2x^2 - 3y^2 - 3z^2) + o(4), \\
\ddot{y} + 2\dot{x} + (c_2 - 1)y &= -3c_3xy - \frac{3}{2}c_4y(4x^2 - y^2 - z^2) + o(4), \\
\ddot{z} + c_2 z &= -3c_3xz - \frac{3}{2}c_4z(4x^2 - y^2 - z^2) + o(4).
\end{aligned}$$

- Notice that for periodic solution, x, y, z are $o(A_z)$ with $A_z << 1$ in normalized unit.

Halo Orbits in 3rd Order Approximation

- Lindstedt-Poincaré method:
  - It is a successive approximation procedure.
  - Periodic solution of linearized equation (with $\lambda = \nu$) will form the first approximation.
  - Richardson used this method to find the 3rd order solution.

$$
\begin{aligned}
x &= a_{21}A_x^2 + a_{22}A_z^2 - A_x \cos\tau_1 \\
&\quad + (a_{23}A_x^2 - a_{24}A_z^2)\cos 2\tau_1 + (a_{31}A_x^3 - a_{32}A_xA_z^2)\cos 3\tau_1, \\
y &= kA_x \sin\tau_1 \\
&\quad + (b_{21}A_x^2 - b_{22}A_z^2)\sin 2\tau_1 + (b_{31}A_x^3 - b_{32}A_xA_z^2)\sin 3\tau_1, \\
z &= \delta_m A_z \cos\tau_1 \\
&\quad + \delta_m d_{21}A_xA_z(\cos 2\tau_1 - 3) + \delta_m(d_{32}A_zA_x^2 - d_{31}A_z^3)\cos 3\tau_1.
\end{aligned}
$$

where $\tau_1 = \lambda\tau + \phi$ and $\delta_m = 2 - m, m = 1, 3$.

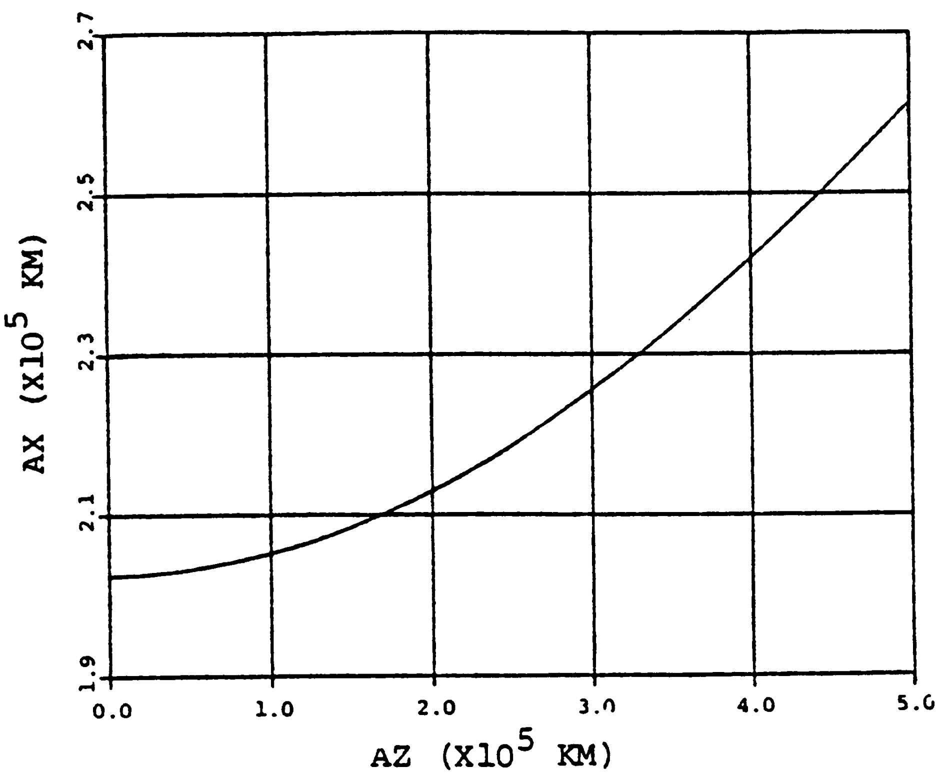
  - Details will be given later. Here, we will provide some highlights.

Halo Orbit Amplitude Constraint Relationship

- For halo orbits, we have amplitude constraint relationship

$$l_1 A_x^2 + l_2 A_z^2 + \Delta = 0.$$

  - For halo orbits about L_1 in Sun-Earth system, $l_1 = -1.59650314$, $l_2 = 1.740900800$ and $\Delta = 0.29221444425$.
  - Halo orbit can be characterized completely by A_z. ISEE-3 halo orbit had $A_z = 110,000$ km.



Halo Orbit Amplitude Constraint Relationship

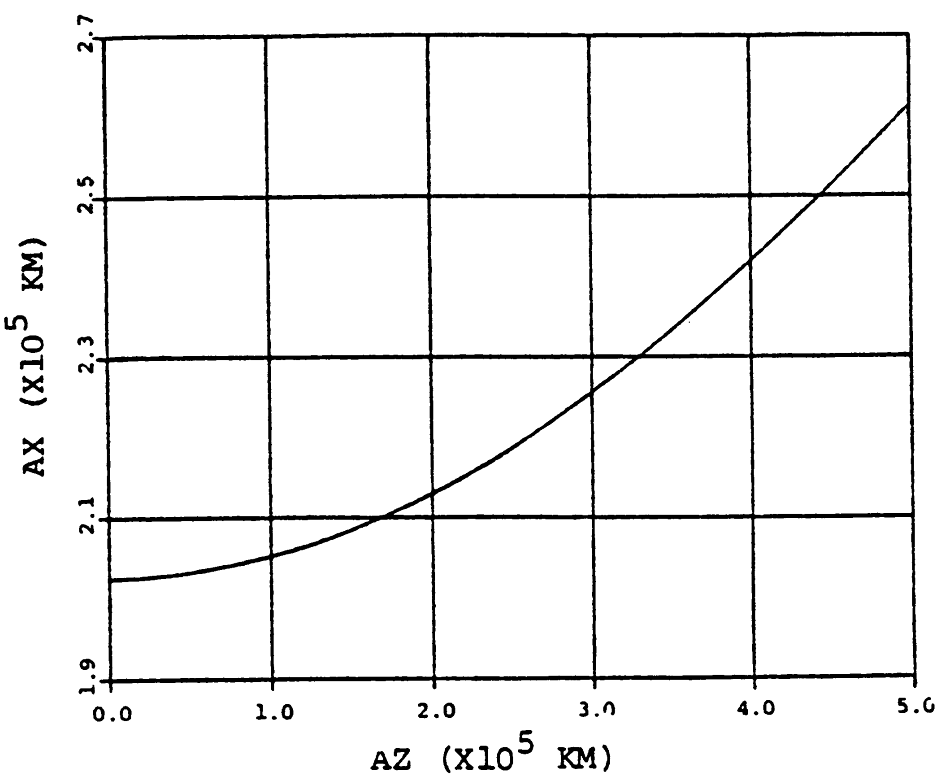
- For halo orbits, we have amplitude constraint relationship

$$l_1 A_x^2 + l_2 A_z^2 + \Delta = 0.$$

  - Minimim value for A_x to have a halo orbit $(A_z > 0)$ is

$$\sqrt{|\Delta / l_1|},$$

  which is about $200,000$ km (14% of normalized distance).

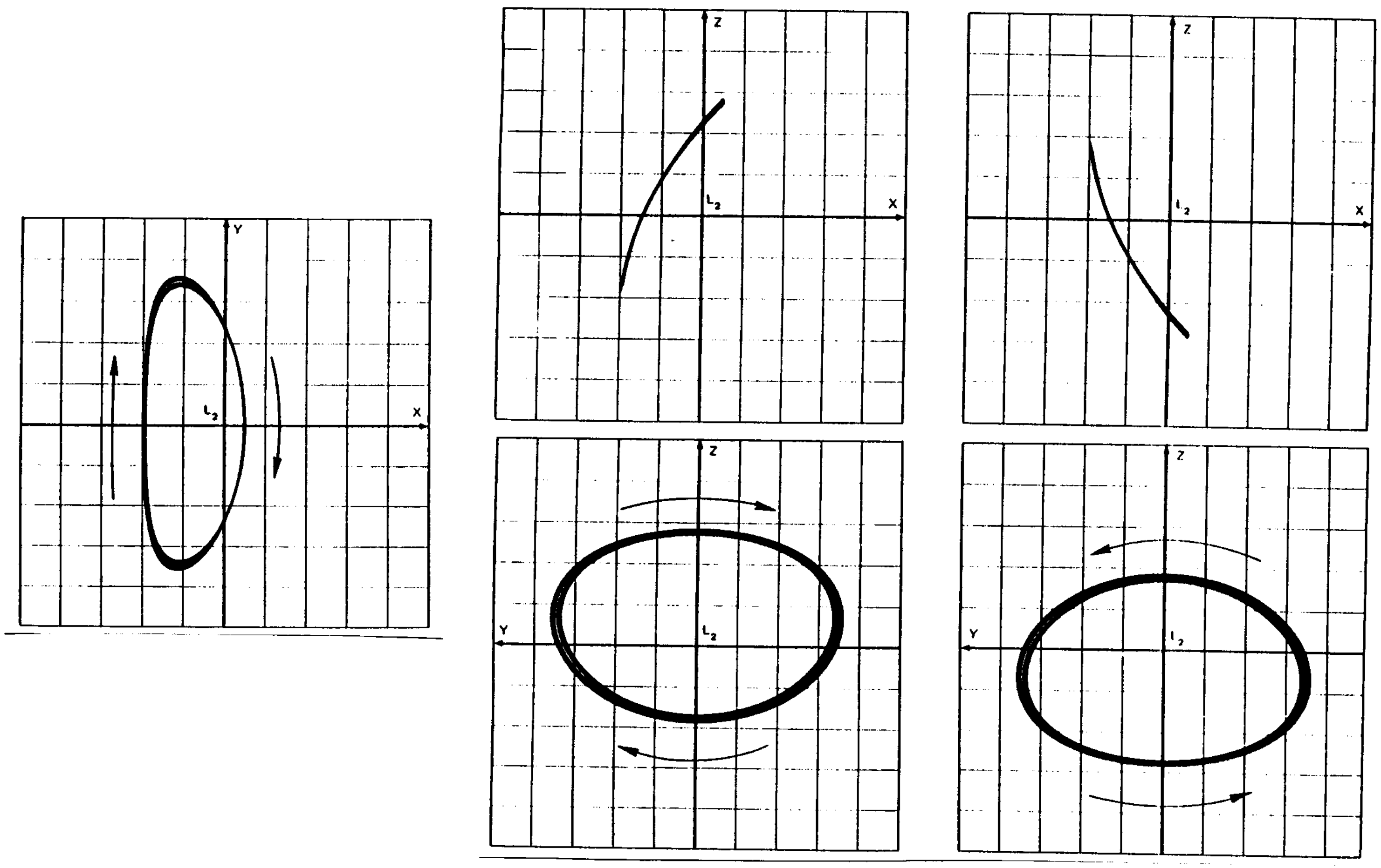


Halo Orbit Phase-angle Relationship

- Bifurcation manifests through phase-angle relationship

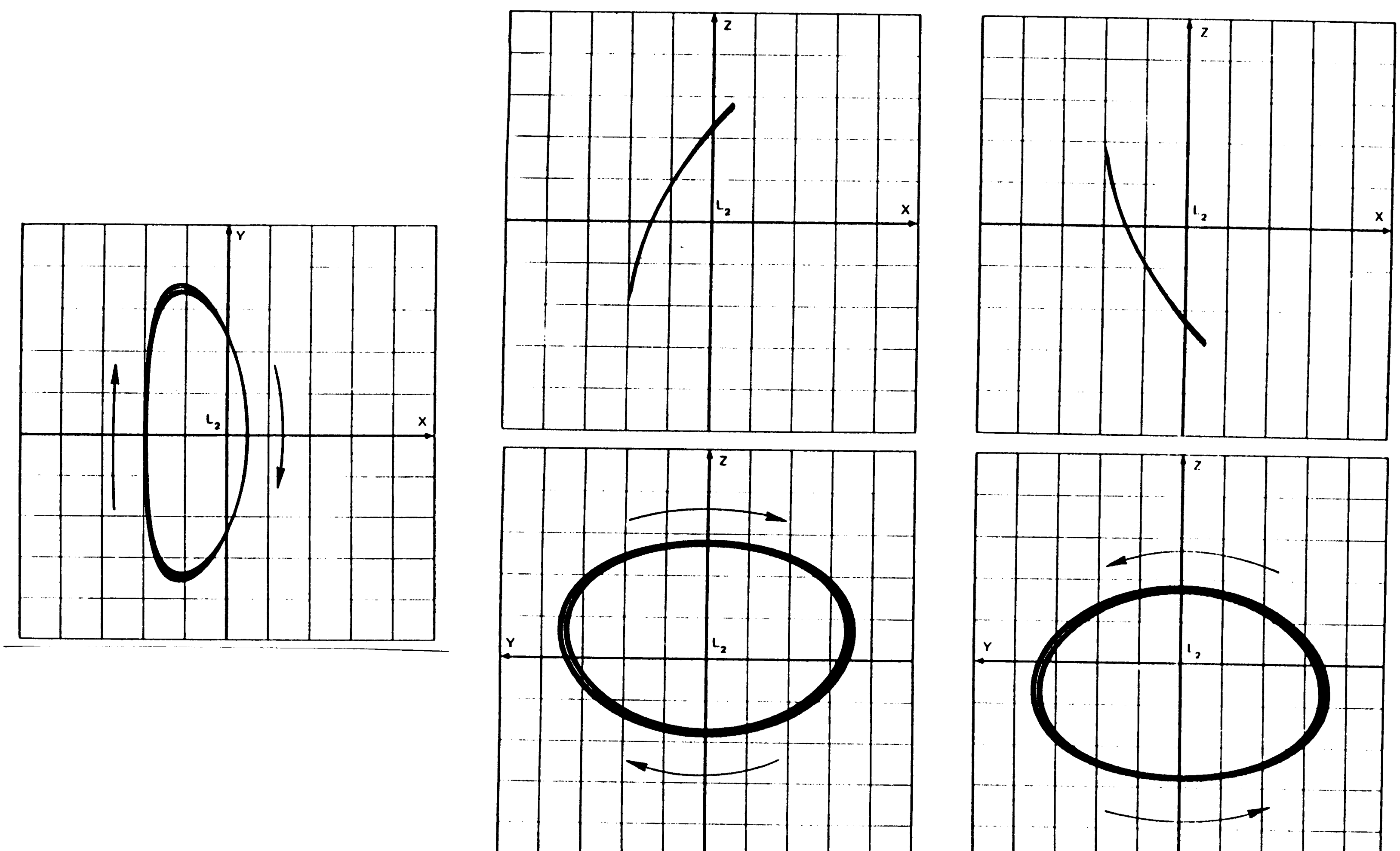
$$\psi - \phi = m\pi/2, \qquad m = 1, 3.$$

  - 2 solution branches are obtained according to whether $m = 1$ or $m = 3$.



Halo Orbit Phase-angle Relationship

- Bifurcation manifests through phase-angle relationship:
  - For $m = 1$, $A_z > 0$. Northern halo.
  - For $m = 3$, $A_z < 0$. Southern halo.
  - Northern & southern halos are mirror images across xy-plane.



Lindstedt-Poincaré Method: Duffing Equation

- To illustrate L.P. method, let us study Duffing equation:
  - First non-linear approximation of pendulum equation (with $\lambda = 1$)

$$\ddot{q} + q + \epsilon q^3 = 0.$$

  - For $\epsilon = 0$, it has a periodic solution

$$q = a\cos t$$

    if we assume the initial condition $q(0) = a, \dot{q}(0) = 0$.
  - For $\epsilon \neq 0$, suppose we would like to look for a periodic solution of the form

$$q = \sum_{n=0}^{\infty} \epsilon^n q_n(t) = q_0(t) + \epsilon q_1(t) + \epsilon^2 q_2(t) + \cdots$$

Lindstedt-Poincaré Method: Duffing Equation

- Finding a periodic solution for Duffing equation:
  - By substituting and equating terms having same power of ϵ, we have a system of successive differential equations:

$$\begin{aligned}\ddot{q}_0 + q_0 &= 0,\\ \ddot{q}_1 + q_1 &= -q_0^3,\\ \ddot{q}_2 + q_2 &= -3q_0^2 q_1,\end{aligned}$$

    and etc.
  - Then $q_0 = a\cos t$, for $q_0(0) = a, \dot{q}(0) = 0$.
  - Since

$$\ddot{q}_1 + q_1 = -q_0^3 = -a^3\cos^3 t = -\frac{1}{4}a^3(\cos 3t + 3\cos t),$$

    the solution has a **secular** term

$$q_1 = -\frac{3}{8}a^3 t\sin t + \frac{1}{32}a^3(\cos 3t - \cos t).$$

Lindstedt-Poincaré Method: Duffing Equation

- Due to presence of **secure** terms, naive method such as expansions of solution in a power series of ϵ would not work.
- To avoid **secure** terms, Lindstedt-Poincaré method
  - Notices that **non-linearity** alters **frequency** λ (corr. to linearized system) to $\lambda\omega(\epsilon)$ ($\lambda = 1$ in our case).
  - Introduce a new independent variable $\tau = \omega(\epsilon)t$:
$$t = \tau\omega^{-1} = \tau(1 + \epsilon\omega_1 + \epsilon^2\omega_2 + \cdots).$$
  - Expand the periodic solution in a power series of ϵ:
$$q = \sum_{n=0}^{\infty} \epsilon^n q_n(\tau) = q_0(\tau) + \epsilon q_1(\tau) + \epsilon^2 q_2(\tau) + \cdots$$
  - Rewrites Duffing equation using τ as independent variable:
$$q'' + (1 + \epsilon\omega_1 + \epsilon^2\omega_2 + \cdots)^2(q + \epsilon q^3) = 0.$$

Lindstedt-Poincaré Method: Duffing Equation

- By substituing q (in power series expansion) into Duffing equation (in new independent variable τ) and equating terms with same power of ϵ, we obtain equations for successive approximations:

$$
\begin{aligned}
q_0'' + q_0 &= 0, \\
q_1'' + q_1 &= -q_0^3 - 2\omega_1 q_0, \\
q_2'' + q_2 &= -3q_0^2 q_1 - 2\omega_1(q_1 + q_0^3) + (\omega_1^2 + 2\omega_2)q_0,
\end{aligned}
$$

  and etc.

- Potential **secular** terms can be gotten rid of by imposing suitable values on ω_n.
  - The general solution of 1st equation can be written as

$$q_0 = a\cos(\tau + \tau_0).$$

    where a and τ_0 are integration constants.

Lindstedt-Poincaré Method: Duffing Equation

- Potential **secular** terms can be gotten rid of by imposing suitable values on ω_n.
  - By substituting $q_0 = a\cos(\tau + \tau_0)$ into 2nd equation, we get

$$\begin{aligned} q_1'' + q_1 &= -a^3\cos^3(\tau+\tau_0) - 2\omega_1 a\cos(\tau+\tau_0) \\ &= -\frac{1}{4}a^3\cos 3(\tau+\tau_0) - (\frac{3}{4}a^2 + 2\omega_1)a\cos(\tau+\tau_0). \end{aligned}$$

  - In previous naive method, we had $\omega_1 \equiv 0$ and a **secular** term caused by $\cos t$ term.
  - Now if we set $\omega_1 = -3a^2/8$, we can get rid of $\cos(\tau + \tau_0)$ term and the ensuing **secular** term.
  - Then

$$q_1 = \frac{1}{32}a^3\cos 3(\tau+\tau_0).$$

Lindstedt-Poincaré Method: Duffing Equation

- Potential **secular** terms can be gotten rid of by imposing suitable values on ω_n.
  - Similarly, by substituting $q_1 = \frac{1}{32}a^3 \cos 3(\tau + \tau_0)$ into 3rd equation, we get

$$q_2'' + q_2 = \left(\frac{51}{128}a^4 - 2\omega_2\right)a\cos(\tau + \tau_0) + (\text{terms not giving secular terms}).$$

  - Setting $\omega_2 = 51a^4/256$, we obtain q_2 free of **secular** terms, and so on.

Lindstedt-Poincaré Method: Duffing Equation

- Therefore, to 1st order of ϵ, we have **periodic** solution

$$\begin{aligned} q &= a\cos(\tau + \tau_0) + \frac{1}{32}\epsilon\cos 3(\tau + \tau_0) + o(\epsilon^2) \\ &= a\cos(\omega t + \tau_0) + \frac{1}{32}\epsilon\cos 3(\omega t + \tau_0) + o(\epsilon^2). \end{aligned}$$

- Notice that

$$\begin{aligned} \omega &= (1 + \epsilon\omega_1 + \epsilon^2\omega_2 + \cdots)^{-1} \\ &= \{1 - \epsilon\omega_1 - \frac{1}{2}\epsilon^2(2\omega_2 - \omega_1^2) + \cdots\} \\ &= (1 - \frac{3}{8}\epsilon a^2 - \frac{15}{256}\epsilon^2 a^4 + o(\epsilon^3). \end{aligned}$$

- **Lindstedt method consists in successive adjustments of frequencies.**

Halo Orbit and Its Computation

- We have covered
  - Importance of halo orbits.
  - Finding periodic solutions of the linearized equations.
  - Highlights on 3rd order approximation of a halo orbit.
  - Using a textbook example to illustrate Lindstedt-Poincaré method.
- In Lecture5B, we will cover
  - Use L.P. method to find a 3rd order approximation of a halo orbit.
  - Finding a halo orbit numerically via differential correction.
  - Orbit structure near L_1 and L_2